Convex structure of generalized states, effects and ultraproducts

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Introduction

In the report abstract convex structures of states and generalized states defined on the event structures will be considered. The concept of operations on generalized states and related effects and ultraproducts of the corresponding convex structures will also be considered [1, 2].
Of particular interest are the so-called «pure» operations, that is, those that translate pure states into pure states.

Convex Structures

Definition 1 (see, for example, S. Gudder, [1])

A set of states S is said to be a convex structure if it has the following two properties:
(1) for any positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying the condition $\sum_{i=1}^{n} \lambda_i = 1$ and any states $s_1, s_2, \ldots, s_n$ there is a unique element $\langle \lambda_1, \lambda_2, \ldots, \lambda_n; s_1, s_2, \ldots, s_n \rangle \in S$;
(2) $\langle \lambda_1, \lambda_2, \ldots, \lambda_n; s, s, \ldots, s \rangle = s$.

This state $\langle \lambda_1, \lambda_2, \ldots, \lambda_n; s_1, s_2, \ldots, s_n \rangle$ is called a mixture of states $s_1, s_2, \ldots, s_n$. For mixtures of two states, we use the notation $\langle \lambda, 1 - \lambda; s, t \rangle = \langle \lambda; s, t \rangle$.
A state $s \in S$ is said to be pure if it cannot be written as $s = \langle \lambda; t_1, t_2 \rangle$ for some $t_1 \neq t_2$.
In other words, pure states are the extreme points of a convex state set.

Convex Structures

Let us introduce the notion of distance in the convex structure S. The closeness of states s and t can be measured by comparing mixtures $\langle \lambda; s_1, s \rangle$ and $\langle \lambda; t_1, t \rangle$ with other states.

Definition 2

We define the distance function of close states $\sigma(s,t)$ as follows: if there exist $s_1, t_1 \in S$ such that the condition $\langle \lambda; s_1, s \rangle = \langle \lambda; t_1, t \rangle$ holds, then

$$\sigma(s,t) = \inf \{ 0 < \lambda \leq 1 : \langle \lambda; s_1, s \rangle = \langle \lambda; t_1, t \rangle \} ;$$

otherwise, $\sigma(s,t) = 1/2$.

In general, this function is not a metric.

Definition 3

A convex structure S is said to be a σ-convex structure if the following two properties hold:
(1) If $s_n \in S$ and $\lim_{n,m\to\infty} \sigma(s_n, s_m) = 0$, then there exists a unique $s \in S$ such that $\lim_{n\to\infty} \sigma(s_n, s) = 0$;
(2) If $\lambda_i > 0, \sum \lambda_i = 1, t_1, t_2, \ldots \in S$ and $s_n = \left\langle \lambda_1, \ldots, \lambda_n, 1 - \sum_{i=n+1}^{\infty} \lambda_i; t_1, \ldots, t_{n+1} \right\rangle$, then $\lim_{n,m\to\infty} \sigma(s_n, s_m) = 0$.

Thus, we can consider infinite (countable) mixtures of states.

Definition 4

A map $f : S \to \mathbb{R}$ is said to be an affine functional if we have

$$f(\langle \lambda_1, \lambda_2, \ldots, \lambda_n; s_1, s_2, \ldots, s_n \rangle) = \sum_{i=1}^{n} \lambda_i f(s_i)$$

for any sets $\lambda_1, \lambda_2, \ldots, \lambda_n$; $s_1, s_2, \ldots, s_n$, $\lambda_i > 0$, $\sum \lambda_i = 1$.

Note that the set of affine functionals S^* is a linear space with respect to pointwise operations.

Total Structures

We define "$\mathbf{0}$" and "$\mathbf{e}$" affine functionals: $\mathbf{0}(s) = 0, \mathbf{e}(s) = 1$ for every $s \in S$. Also we define a partial order relation on S^*:
$f \leq g \Leftrightarrow f(s) \leq g(s)$ for every $s \in S$.

Definition 5

The functional $f \in S^*$ is called an effect if $\mathbf{0} \leq f \leq \mathbf{e}$.
The set of effects will be denoted by $E(S)$. It forms a convex subset of the linear space S^*.
A total convex structure is a σ-convex structure with the property:
$f(s) = f(t)$ for every effect f implies that $s = t$.

It is well known, (see S.Gudder, [1]), that then (S, σ) is a complete metric space. Let's call this complete metric space a state space.

Total Structures

Since $E(S)$ forms a convex subset of space S^*, then it has extreme points, which are called a propositions. The set of propositions $\mathcal{P}(S) \in S^*$ inherits the order of S^* and so is a poset with least element $\mathbf{0}$ and greatest element $\mathbf{e}$.

We now endow S^* with the weak *-topology. This is the natural topology for S^* since in this topology a sequence of effects f_n converges to an effect f if and only iff $f_n(s) \to f(s)$ for every state s.

Generalized Convex Structures

Let S be a convex structure, we define $S_+ = \{(\alpha, s) : \alpha \geq 0, s \in S\}$. We define $(\alpha, s) = (\beta, t)$ if $\alpha = \beta \neq 0$ and $s = t$ and $(0, s) = (0, t) = 0$ for all $s, t \in S$. If S is a set of states we call S_+ the set of generalized states. Let's define a convex structure on S_+:

$$\langle \lambda_1, \lambda_2, \ldots, \lambda_n; (\alpha_1, s_1), \ldots, (\alpha_n, s_n) \rangle = \\ = \left(\sum_{i=1}^{n} \lambda_i \alpha_i, \left\langle \frac{\lambda_1 \alpha_1}{\sum_{i=1}^{n} \lambda_i \alpha_i}, \ldots, \frac{\lambda_n \alpha_n}{\sum_{i=1}^{n} \lambda_i \alpha_i}; s_1, \ldots, s_n \right\rangle \right)$$

We identify an element of the form $(1, s) \in S_+$ with the element $s \in S$.

Generalized Convex Structures

Let S_+^* mean the set of affine functionals on S_+. It is known ([1]) that if $f \in S^*$, then there is a unique extension $\hat{f} \in S_+^*$ and if $\hat{f} \in S_+^*$, then $\hat{f}((\alpha, s)) = \alpha f(s)$ for all $(a, s) \in S_+$.
In particular, there is a unique extension of the unit functional $\hat{\mathbf{e}} \in S_+^*$ and $\hat{\mathbf{e}}((\alpha, s)) = \alpha$.

Definition 6

We define an operation as a map $F \in Af(S_+)$ (that is the map $F : S_+ \to S_+$ is affine) satisfying

$$\hat{\mathbf{e}}(F(w)) \leq \hat{\mathbf{e}}(w)$$

for all $w = (\alpha, s) \in S_+$.

If F is operation then for $(\alpha, t) \in S_+$, $F((\alpha, t)) = (\alpha', t')$, so there are two parts to an operation $\alpha \to \alpha'$, $t \to t'$. The part $t \to t'$ represents a «distortion» and $\alpha \to \alpha'$ a reduction strength due to a measurement. For $s \in S$, we interpret $\hat{e}(F(s))$ as the probability of transmission of the state s conditioned by the operation F.

If $F \in Af(S_+)$ we define the linear map $F^* : S_+^* \to S_+^*$ by

$$(F^* \hat{f})(w) = \hat{f}(F(w))$$

for every $\hat{f} \in S_+^*$, $w \in S_+$.
Associated with any operation F is its effect defined by

$$f = F^*(\hat{\mathbf{e}})|S.$$

Since $\hat{\mathbf{e}}(F(s)) = (F^*\hat{\mathbf{e}})(s) = f(s)$ for every $s \in S$, the effect f determines the probability of transmission but not the form of the transmitted state. Thus an operation contains more information than its effect.

Also, as in the case of ordinary states, the concept of an extreme points (pure states) in the space of generalized states S_+ is introduced.

Definition 7

An operation is called a pure operation if it transform pure states into pure states.

Example 1

Let H be a complex separable Hilbert space and let S be the set of density operators on H that is, S is the set of positive trace class operators of trace 1 on H. In this case S is a σ-convex structure. Consider the set of bounded self-adjoint operators A on H satisfying $0 \leq A \leq I$. Here $\mathbf{e} = I$. We define the effect

$$A(s) = tr(As).$$

It is known (E. Davies, [5]) that such effects describe all the effects on S, and an effect A is a proposition if and only if A is a projection operator.
We identify elements $w = (\alpha, s) \in S_+$ with operators αs. Thus S_+ can be thought of as the set of positive trace-class operators.
Then

$$\hat{\mathbf{e}}(w) = tr(w).$$

Let's approach the concept of operation from a slightly different angle.

Definition 8 ([5])

Let S_1 and S_2 are the state spaces on Hilbert spaces H_1 and H_2 (finite-dimensional or infinite-dimensional) respectively. An operation on S_1 is defined as a affine map $T : S_1 \to S_2$ which also satisfies

$$\mathbf{e}_2(T(s)) \leq \mathbf{e}_1(s) \quad (\,\mathbf{Tr}((T(s)) \leq \mathbf{Tr}(s)\,)$$

for all $s \in S_1$, where $\mathbf{e}_1$ is unit on S_1, $\mathbf{e}_2$ is unit on S_2.

Generalized Convex Structures

If T is a pure operation, then the structure of this operation is well known [5]. Exactly,

$$T(s) = BsB^*,$$

or

$$T(s) = \mathbf{Tr}(sB)|\psi >< \psi|,$$

where B is bounded and linear, $\psi \in H_2$.
In the latter case, the operation T is called degenerate, that is, the mapping T translates all states into one single state.

Ultrafilter

Definition 9

Given a set X, an ultrafilter on X is a set $\mathscr{U}$ consisting of subsets of X such that

1. The empty set is not an element of $\mathscr{U}$;
2. If A and B are subsets of X, A is a subset of B, and A is an element of $\mathscr{U}$, then B is also an element of $\mathscr{U}$;
3. If A and B are elements of $\mathscr{U}$, then the intersection $A \cap B$ is an element of $\mathscr{U}$;
4. If A is a subset of X, then either A or $X \setminus A$ is an element of $\mathscr{U}$.

We can see that the system $\mathscr{F} = \{A \subseteq X : A \ni x_0\}$, $x_0 \in X$ is an ultrafilter. The ultrafilter $\mathscr{F}$ is called trivial, we will not use such an ultrafilter. Everywhere further $\mathscr{U}$ is a nontrivial ultrafilter in the set $\mathbb{N}$ of natural integers.

Ultraproducts of sequence of sets

Definition 10

Let (X_n) be a sequence of nonempty sets. The ultraproduct (classic, set-theoretic) $(X_n)_{\mathscr{U}}$ is the quotient of the Cartesian product $\prod_{n=1}^{\infty} X_n$ by the equivalence relation:

$$(x)_n \sim (y)_n \Leftrightarrow \{n : x_n = y_n\} \in \mathscr{U}\,.$$

Ultraproducts of sequence of sets

Definition 11

Let $\mathscr{U}$ be a non-trivial ultrafilter on $\mathbb{N}$. If (x_n) is a sequence of points in a metric space (X, d) and $x \in X$. The point x is said to be the limit of x_n with respect to ultrafilter $\mathscr{U}$, denoted,

$$x = \lim_{\mathscr{U}} x_n,$$

if for every $\varepsilon > 0$ we have: $\{n : d(x_n, x) < \varepsilon\} \in \mathscr{U}$.

Clearly, if $\mathbf{K}$ is a compact Hausdorff space, $\mathscr{U}$ is any ultrafilter on the set $\mathbb{N}$, then any sequence $(x_n)_{n=1}^{\infty}$, $x_n \in \mathbf{K}$, has unique limit with respect to ultrafilter $\mathscr{U}$.

Ultraproduct for the Total Convex Structures

Definition 12

Let (S_n, σ_n) be a sequence of total convex structures, $\mathscr{U}$ be a non-trivial ultrafilter on $\mathbb{N}$. Let $\prod_{n=1}^{\infty} S_n$ be the Cartesian product of sequence (S_n).
We introduce an equivalence relation on $\prod_{n=1}^{\infty} S_n$, assuming

$$(s_n) \sim (t_n) \Leftrightarrow \lim_{\mathscr{U}} \sigma_n(s_n, t_n) = 0.$$

The quotient $\prod_{n=1}^{\infty} S_n$ by the this equivalence relation we will denote $(S_n)_{\mathscr{U}}$.

Ultraproduct for the Total Convex Structures

On ultraproducts $(S_n)_{\mathscr{U}}$ we will naturally introduce the metric $\sigma_{\mathscr{U}}$

$$\sigma_{\mathscr{U}}(s,t) = \lim_{\mathscr{U}} \sigma_n(s_n, t_n),\ t = (t_n)_{\mathscr{U}}, s = (s_n)_{\mathscr{U}},$$

and the effect

$$f_{\mathscr{U}}(s) = \lim_{\mathscr{U}} f_n(s_n),\ s = (s_n)_{\mathscr{U}}.$$

The pair $((S_n)_{\mathscr{U}}, \sigma_{\mathscr{U}})$ is called the ultraproduct of a sequence of total convex structures.

Theorem 1

Let $(S_n, \sigma_n)_{n \geq 1}$ be a sequence of total convex structures, $\mathscr{U}$ be a nontrivial ultrafilter in set of natural numbers $\mathbb{N}$. Then the ultraproduct $(S_{\mathscr{U}}, \sigma_{\mathscr{U}})$ is a total convex structure.

Ultraproduct for the Total Convex Structures

Denote by $\hat{\sigma}$ the metric in S_+. Since $(S_+, \hat{\sigma})$ is a total convex structure with respect to this metric, the definition of the ultraproduct of the sequence of an generalized states spaces coincides with the previous definition. Note that in this case $S_{\mathscr{U}+} = (\tilde{\mathbb{R}}_+)_{\mathscr{U}} \times S_{\mathscr{U}}$, where $(\tilde{\mathbb{R}}_+)_{\mathscr{U}} = \{(\alpha_n)_{\mathscr{U}} : \alpha_n \geq 0, \sup \alpha_n < \infty\}$.

Theorem 2

Let $(S_{n+}, \hat{\sigma}_n)_{n\geq 1}$ be a sequence of generalized state spaces, $\mathscr{U}$ be a nontrivial ultrafilter in set of natural numbers $\mathbb{N}$. Then the ultraproduct $(S_{\mathscr{U}+}, \hat{\sigma}_{\mathscr{U}})$ is a total convex structure.

Example 2

Let S_1 be the set of probability measures μ on $(\Omega_1, \Sigma_1) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, absolutely continuous with respect to the Lebesgue measure. Easy to see, [4], that such measures are H-quasi-invariant and ergodic with respect to sifts on elements of $H = \{x \in \mathbb{R} : |x| < \infty\}$,
S_2 be the set of H-quasi-invariant and ergodic probability product-measures $\mu^n = \prod_{k=1}^{n} \mu_k$, $\mu_k \in S_1$, on $(\Omega_2, \Sigma_2) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $n \in \mathbb{N}$, with $H = \{x_n \in \mathbb{R}^n : \sup \|x_n\| < \infty\}$. With the usual definition of convex combinations, S_1 and S_2 are a total convex structures.
Let E_1 be the set of random variables f on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying $0 \leq f \leq 1$.
Then

$$f(\mu) = \int f d\mu$$

determines the effect.
The functional $\mathbf{e}_1$ is represented by the function 1 which is identically 1.

Ultraproduct for the Total Convex Structures

Let E_2 be the set of random variables g on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ satisfying $g = (f_1 + f_2 + \cdots + f_n)/n,\ f_k \in E_1$.
Then

$$g(\mu^n) = \int g d\mu^n$$

is the effect on S_2.
It is well known (see, for example, S. Haliullin, D.H. Mushtari, [4], S. Haliullin [3]) that H-ergodic measures are the extreme points of a set of H-quasi-invariant measures, moreover, they are either mutually absolutely continuous or singular.
It is also well known that Gaussian measures in $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ belong to the class of H-ergodic measures.

Consider a pure operation T from S_1 to S_2:

$$T(\mu) = \mu^n,$$

where μ^n is power of Gaussian measure $\mu, (n \in \mathbb{N})$.
Denote by (Σ_1, S_1) and (Σ_2^n, S_2^n) the event structures (see, for example, S. Haliullin, [2]).
Let us further consider the following ultraproducts: ultrapover (Σ_1, S_1) and ultraproduct of sequence (Σ_2^n, S_2^n) by a nontrivial ultrafilter.
Then the measure $\mu_{\mathscr{U}}$ is extreme point en $(\Sigma_1, S_1)_{\mathscr{U}}$, but the measure $(\mu^n)_{\mathscr{U}}$ is not an extreme point en $(\Sigma_2^n, S_2^n)_{\mathscr{U}}$.

Thus, we have proved the following theorem:

Theorem 3

There are two sequences of state spaces such that the ultraproduct of pure operations is not a pure operation.

References

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Thank you for attention!