Automorphisms of the semigroup C^*-algebra for the free product of semigroups of rational numbers

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Motivation. The inductive sequences of the Toeplitz algebras

The Toeplitz algebra is the universal C^*-algebra:

$$\mathcal{T} = C^*\{U \mid U^*U = 1\} \cong C_r^*(\mathbb{Z}^+) \subset B(l^2(\mathbb{Z}^+)).$$

Let $P = (p_1, p_2, ...)$ be an infinite sequence of arbitrary integers. For every $k \in \mathbb{N}$ there exists a unique $*$-endomorphism $\varphi_k : \mathcal{T} \longrightarrow \mathcal{T} : U \mapsto U^{p_k}$. So we have the inductive sequence

$$\mathcal{T} \xrightarrow{\varphi_1} \mathcal{T} \xrightarrow{\varphi_2} \mathcal{T} \xrightarrow{\varphi_3} \dots \qquad \varinjlim\{\mathcal{T}, \varphi_k\}.$$

The necessary and sufficient conditions for the limit $*$-endomorphisms of $\varinjlim\{\mathcal{T}, \varphi_k\}$ to be the automorphisms were obtained in the paper [1].

[1] R. N. Gumerov, Limit automorphisms of the C^*-algebras generated by isometric representations for semigroups of rationals, Siberian Math. J., **59**:1, 73–84 (2018).

The aim of the report

The report is concerned with the inductive sequences of the Toeplitz-Cuntz algebras and morphisms between two copies of a such sequence.

$$
\begin{array}{ccccc}
\mathcal{TO}_n & \xrightarrow{\varphi_1} & \mathcal{TO}_n & \xrightarrow{\varphi_2} & \cdots \\
\downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \\
\mathcal{TO}_n & \xrightarrow{\varphi_1} & \mathcal{TO}_n & \xrightarrow{\varphi_2} & \cdots
\end{array}
\qquad\qquad
\begin{array}{c}
\varinjlim\{\mathcal{TO}_n, \varphi_k\} \\
\Big\downarrow{\scriptstyle \varinjlim\{\theta_k\}} \\
\varinjlim\{\mathcal{TO}_n, \varphi_k\}.
\end{array}
$$

The aim of the report is to give the necessary and sufficient conditions for the limit $*$-endomorphisms to be the automorphisms of the C^*-algebra which is the inductive limit.

Definition 1

The *Toeplitz-Cuntz algebra* $\mathcal{TO}_n$, $n \geq 2$, is the universal C^*-algebra on generators $U_1, U_2, \ldots, U_n$ subject to the following relations:

i) $U_k^* U_k = 1$ for $k = 1, 2, \ldots, n$;

ii) $U_k^* U_l = 0$ whenever $k \neq l$;

iii) $U_1 U_1^* + U_2 U_2^* + \cdots + U_n U_n^* < 1$.

Cuntz [2] proved that the C^*-algebra generated by any set of n bounded linear operators on a Hilbert space satisfying the relations *i*)-*iii*) is canonically isomorphic to $\mathcal{TO}_n$.

[2] J. Cuntz, K-theory for certain C^*-algebras, Ann. Math. **113** 181–197 (1981).

Inductive sequences of C^*-algebras and their limits

Definition 2

An inductive (direct) sequence is a collection $\{\mathfrak{A}_k, \varphi_k\}$ consisting of C^*-algebras $\mathfrak{A}_k$ and $*$-homomorphisms (connecting morphisms) $\varphi_k : \mathfrak{A}_k \longrightarrow \mathfrak{A}_{k+1}$ written as the diagram

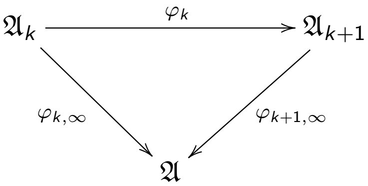
$$\mathfrak{A}_1 \xrightarrow{\varphi_1} \mathfrak{A}_2 \xrightarrow{\varphi_2} \mathfrak{A}_3 \xrightarrow{\varphi_3} \ldots$$

Definition 3

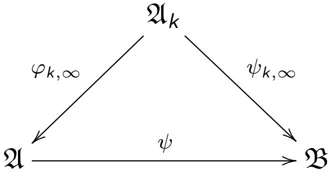
An inductive (direct) limit of the inductive sequence $\{\mathfrak{A}_k, \varphi_k\}$ is a pair $(\mathfrak{A}, \{\varphi_{k,\infty}\})$ consisting of a C^*-algebra $\mathfrak{A}$ and a sequence of $*$-homomorphisms $\{\varphi_{k,\infty} : \mathfrak{A}_k \longrightarrow \mathfrak{A}\}$ with the two properties listed further.

The characteristic properties of the inductive limit

1) for every $k \in \mathbb{N}$ we have $\varphi_{k,\infty} = \varphi_{k+1,\infty} \circ \varphi_k$, i.e.



2) for every C^*-algebra $\mathfrak{B}$ and a sequence of morphisms $\psi_{k,\infty} : \mathfrak{A}_k \longrightarrow \mathfrak{B}$ satisfying $\psi_{k,\infty} = \psi_{k+1,\infty} \circ \varphi_k$, $k \in \mathbb{N}$, there exists a unique $*$-homomorphism ψ making the diagram



commutative for each $k \in \mathbb{N}$.

Inductive sequences of Toeplitz-Cuntz algebras

We construct a sequence of copies of the same algebra $\mathcal{TO}_n$.

Consider an n-tuple of infinite sequences of arbitrary prime integers

$$P_1 = (p_{11}, p_{21}, ...), ..., P_n = (p_{1n}, p_{2n}, ...).$$

For every $k \in \mathbb{N}$ there exists a unique $*$-endomorphism $\varphi_k : \mathcal{TO}_n \longrightarrow \mathcal{TO}_n$, defined as follows:

$$\varphi_k(U_1) = U_1^{p_{k1}}, ..., \varphi_k(U_n) = U_n^{p_{kn}}.$$

So we have the inductive sequence

$$\mathcal{TO}_n \xrightarrow{\varphi_1} \mathcal{TO}_n \xrightarrow{\varphi_2} \mathcal{TO}_n \xrightarrow{\varphi_3} \ldots.$$

The inductive limit of this sequence is denoted by $\varinjlim\{\mathcal{TO}_n, \varphi_k\}$.

The reduced semigroup C^*-algebra

Let S be a cancellative semigroup with the unit. Consider the Hilbert space

$$l^2(S) := \{f : S \to \mathbb{C} \mid \sum_{a \in S} |f(a)|^2 < +\infty\}.$$

Denote the canonical orthonormal basis of $l^2(S)$ by $\{e_a \mid a \in S\}$, where $e_a(b) = 1$ if $a = b$ and $e_a(b) = 0$ if $a \neq b$.

Definition 4

The *reduced semigroup C^*-algebra* $C_r^*(S)$ is the C^*-subalgebra in $B(l^2(S))$ generated by the set of isometries $\{T_a \mid a \in S\}$. Here the operator T_a is defined as follows:

$$T_a(e_b) = e_{ab}, \ a, b \in S.$$

Free product of semigroups

Let $\{S_1, S_2, \ldots, S_n\}$ be a family of disjoint semigroups and $a_1, a_2, a_3, \ldots \in \bigsqcup_{k=1}^{n} S_k$.

The free product of semigroups is the set

$$S_1 * \ldots * S_n := \{a_1 a_2 \ldots a_l \mid l \in \mathbb{N} \text{ and } \forall i, k (1 \leq i \leq l-1, 1 \leq k \leq n)$$

$$a_i \in S_k \Rightarrow a_{i+1} \notin S_k\}$$

equipped with the semigroup operation $a_1 \ldots a_l * b_1 \ldots b_m =$

$$\begin{cases} a_1 \ldots a_l b_1 \ldots b_m, & \text{if } a_l \in S_i, b_1 \in S_j, i \neq j; \\ a_1 \ldots a_{l-1}(a_l \cdot b_1) b_2 \ldots b_m, & \text{if } a_l, b_1 \in S_i \text{ for some } i, \end{cases}$$

where $a_1 \ldots a_l, b_1 \ldots b_m \in S_1 * \ldots * S_n$, $l, m \in \mathbb{N}$.

Semigroups of rational numbers

We have the n-tuple of infinite sequences of prime integers

$$P_1 = (p_{11}, p_{21}, ...), ..., P_n = (p_{1n}, p_{2n}, ...).$$

Take the additive semigroups of positive rational numbers

$$\mathbb{Q}^+_{P_k} = \left\{ \frac{m}{p_{1k} \cdots p_{sk}} \mid m \in \mathbb{N}, s \in \mathbb{N} \right\}, \ 1 \leq k \leq n.$$

Define semigroups $\mathbb{Q}^+_{P_k} \times \{k\}$, $1 \leq k \leq n$, with the binary operation given by $(a, k) + (b, k) = (a + b, k), \quad a, b \in \mathbb{Q}^+_{P_k}$.

Finally consider the free product of the semigroups

$$S := (\mathbb{Q}^+_{P_1} \times \{1\}) * \ldots * (\mathbb{Q}^+_{P_n} \times \{n\}) \sqcup \{0\}.$$

Here we add the neutral element 0.

The semigroup C^*-algebra $C_r^*(S)$ is the inductive limit for the inductive sequence of the Toeplitz-Cuntz algebras

Theorem 1 [3].

Let $\{\mathcal{TO}_n, \varphi_k\}$ be the inductive sequence of the Toeplitz-Cuntz algebras

$$\mathcal{TO}_n \xrightarrow{\varphi_1} \mathcal{TO}_n \xrightarrow{\varphi_2} \mathcal{TO}_n \xrightarrow{\varphi_3} \ldots,$$

where $\varphi_k : \mathcal{TO}_n \longrightarrow \mathcal{TO}_n : U_i \longmapsto U_i^{p_{ki}}$, $k \in \mathbb{N}$, $1 \leq i \leq n$.
Let S be the free product of semigroups of rational numbers associated with the tuple $(P_1, P_2, \ldots P_n)$. Then there exists an isomorphism

$$\varinjlim\{\mathcal{TO}_n, \varphi_k\} \cong C_r^*(S).$$

[3] S.A. Grigoryan, R.N. Gumerov, E.V. Lipacheva, *Limits of inductive sequences of Toeplitz-Cuntz algebras*, Proc. Steklov Inst. Math. **313** 60–69 (2021).

Let $L = (l_1, \ldots, l_n)$ be a multi-index consisting of positive integers.

Construct the following diagram

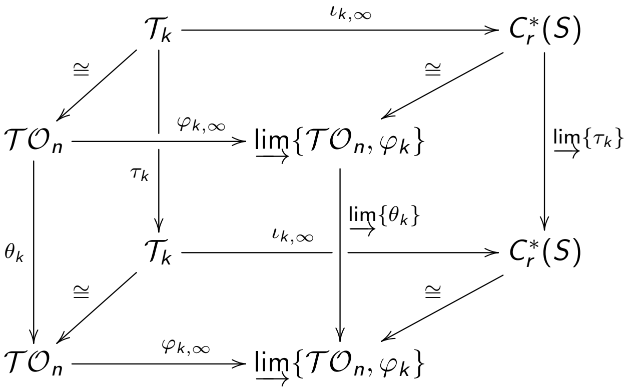
$$\begin{array}{ccccccc} \mathcal{TO}_n & \xrightarrow{\varphi_1} & \mathcal{TO}_n & \xrightarrow{\varphi_2} & \cdots & & \varinjlim\{\mathcal{TO}_n, \varphi_k\} \\ \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & & & \downarrow{\scriptstyle \varinjlim\{\theta_k\}} \\ \mathcal{TO}_n & \xrightarrow{\varphi_1} & \mathcal{TO}_n & \xrightarrow{\varphi_2} & \cdots & & \varinjlim\{\mathcal{TO}_n, \varphi_k\}, \end{array}$$

where $\theta_k : \mathcal{TO}_n \longrightarrow \mathcal{TO}_n$, defined as follows:

$$\theta_k(U_1) = U_1^{l_1}, ..., \theta_k(U_n) = U_n^{l_n},$$

$k \in \mathbb{N}$, and $\varinjlim\{\theta_k\}$ is the limit $*$-endomorphism.

The idea



Here $\mathcal{T}_k \subset B(l^2(S))$ and $C_r^*(S) = \overline{\bigcup_{k=1}^{+\infty} \mathcal{T}_k}$.

The criterion

Theorem 2.

Let $\{\mathcal{TO}_n, \varphi_k\}$ be the inductive sequence of the Toeplitz-Cuntz algebras associated with a tuple $(P_1, P_2, \ldots P_n)$.
Let $L = (l_1, \ldots, l_n)$ be a multi-index of positive integers.
Then the limit $*$-endomorphism

$$\varinjlim\{\tau_k\} : C_r^*(S) \longrightarrow C_r^*(S)$$

is an automorphism of the C^*-algebra if and only if, for each $i = 1, \ldots, n$, either $l_i = 1$ or every prime divisor of the integer l_i occurs infinitely often in the sequence P_i.

Corollary.

The limit $*$-endomorphism $\varinjlim\{\theta_k\}$ is an automorphism if and only if $\varinjlim\{\tau_k\}$ is an automorphism.

Example

Let $n = 2$. Take two sequences of prime numbers

$$P_1 = (2, 2, 2, \ldots), \;\; P_2 = (3, 3, 3, \ldots)$$

and the semigroup $S = (\mathbb{Q}^+_{P_1} \times \{1\}) * (\mathbb{Q}^+_{P_2} \times \{2\}) \sqcup \{0\}$.

Then the limit $*$-endomorphism

$$\varinjlim\{\tau_k\} : C^*_r(S) \longrightarrow C^*_r(S)$$

is an automorphism of the C^*-algebra if and only if $L = (2^s, 3^t)$ for some non-negative integers s and t.

For example,
for $L = (4, 9)$ the limit $*$-endomorphism is an automorphism;
for $L = (6, 6)$ the limit $*$-endomorphism is not an automorphism.

Thanks for your attention!