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# Remark on boundary conditions for the KPZ equation

**Rassadin A.E.**
**Higher school of economics in Nizhny Novgorod**

# The Kardar-Parisi-Zhang (KPZ) equation

$$\frac{\partial h}{\partial t} = \frac{c}{2} \cdot (\nabla h)^2 + \nu \cdot \Delta h$$

***Kardar M, Parisi G and Zhang Y C 1986 Phys. Rev. Lett. 56 889.***

$h(x, y, t)$ **Is (reduced) height of a surface**

$c$ **Is a rate of sputtering**

$\nu$ **Is a coefficient of a surface diffusion**

$\vec{J} = -\nu \cdot \nabla h$ **Is a flux of the surface diffusion**

The KPZ equation ought to be provided by initial condition:

$$h(x,y,0) = h_0(x,y)$$

As a rule the KPZ equation is solved on the whole plane:

$$(x,y) \in R^2$$

But in practice of microelectronics:

$$(x,y) \in D \subset R^2$$

The KPZ equation should be endowed by boundary conditions, for example:

$$\left.\frac{\partial h}{\partial n}\right|_{\partial D} = 0$$

The question is to compare solution of the Cauchy problem on the plane and solution of the mixed problem the KPZ equation with the following restriction:

$$diam(\sup ph_0) << diamD$$

## To clarify this question let one consider growing surfaces with cylindrical generatrix only

$$\frac{\partial h}{\partial t} = \frac{c}{2} \cdot \left( \frac{\partial h}{\partial x} \right)^2 + \nu \cdot \frac{\partial^2 h}{\partial x^2} \qquad h(x,0) = h_0\left( \frac{x}{l} \right)$$

**Let us rescale variables as follows:**

$$\frac{x}{l} \to x \qquad \frac{\nu \cdot t}{l^2} \to t \qquad \frac{c \cdot h}{\nu} \to h$$

$$\frac{\partial h}{\partial t} = \frac{1}{2} \cdot \left( \frac{\partial h}{\partial x} \right)^2 + \frac{\partial^2 h}{\partial x^2} \qquad h(x,0) = h_0(x)$$

**Neumann boundary conditions:**

$$\left. \frac{\partial h}{\partial x} \right|_{x=-L} = \left. \frac{\partial h}{\partial x} \right|_{x=+L} = 0$$

**Let one introduce new auxiliary function φ(x,t):** $h = 2 \cdot \ln \varphi$

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2} \qquad \varphi(x,0) = \exp\left[\frac{h_0(x)}{2}\right] \qquad x \in R$$

$$\varphi(x,t) = \frac{1}{2\sqrt{\pi \cdot t}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-\xi)^2}{4 \cdot t} + \frac{h_0(\xi)}{2}\right] \cdot d\xi$$

$$h_0(x) = 2 \cdot \ln[1 + m_0 \cdot \psi_0(x)] \qquad 0 \le \psi_0(x) \le 1 \qquad m_0 > -1$$

$$\varphi(x,t) = 1 + m_0 \cdot \psi(x,t) \qquad \sup p\,\psi_0 = [-1,1]$$

$$\psi(x,t) = \frac{1}{2\sqrt{\pi \cdot t}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-\xi)^2}{4 \cdot t}\right] \cdot \psi_0(\xi) \cdot d\xi$$

**It is convenient to choose:** $\psi_0(x) = \theta(1 - |x|)$

$\theta(z)$ **is the Heaviside step function**

$$\psi(x,t) = \frac{1}{2} \cdot \left[ erf\left( \frac{x+1}{2\sqrt{t}} \right) - erf\left( \frac{x-1}{2\sqrt{t}} \right) \right]$$

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\zeta^2) \cdot d\zeta$$

**is the Gauss error function**

$$\psi_0(x) = (1 - |x|) \cdot \theta(1 - |x|)$$

$$\psi(x,t) = \frac{1}{2} \cdot \left[ erf\left( \frac{x+1}{2\sqrt{t}} \right) - erf\left( \frac{x-1}{2\sqrt{t}} \right) \right] +$$
$$+ \frac{x}{2} \cdot \left[ erf\left( \frac{x+1}{2\sqrt{t}} \right) - 2 \cdot erf\left( \frac{x}{2\sqrt{t}} \right) + erf\left( \frac{x-1}{2\sqrt{t}} \right) \right] +$$
$$+ \sqrt{\frac{t}{\pi}} \cdot \left[ \exp\left( -\frac{(x+1)^2}{4 \cdot t} \right) - 2 \cdot \exp\left( -\frac{x^2}{4 \cdot t} \right) + \exp\left( -\frac{(x-1)^2}{4 \cdot t} \right) \right]$$

$$h(x,t) = 2 \cdot \ln[1 + m_0 \cdot \psi(x,t)]$$

$$x \in [-L, L] \qquad L >> 1 \qquad \left.\frac{\partial \varphi}{\partial x}\right|_{x=-L} = \left.\frac{\partial \varphi}{\partial x}\right|_{x=+L} = 0$$

$$\varphi(x,t) = 1 + m_0 \cdot \psi(x,t)$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} \qquad \psi(x,0) = \psi_0(x) \qquad \left.\frac{\partial \psi}{\partial x}\right|_{x=-L} = \left.\frac{\partial \psi}{\partial x}\right|_{x=+L} = 0$$

**Separation of variables:** $\psi(x,t) = \exp(-\lambda \cdot t) \cdot X(x)$

**Sturm-Liouville problem:**

$$-X''(x) = \lambda \cdot X(x)$$

$$X'(-L) = X'(+L) = 0$$

$$L^2([-L, L]) = V^+ \oplus V^-$$

$$\frac{1}{\sqrt{2L}} \qquad \frac{1}{\sqrt{L}}\cdot\cos\frac{\pi\cdot n\cdot x}{L} \qquad \frac{1}{\sqrt{L}}\cdot\sin\frac{\pi\cdot(2n-1)\cdot x}{2L}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\lambda_0 = 0 \qquad \lambda_n^+ = \frac{\pi^2 n^2}{L^2} \qquad \lambda_n^- = \frac{\pi^2(2n-1)^2}{4L^2}$$

$$\psi_0(x) = \theta(1-|x|)$$

$$\psi(x,t) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\exp\left(-\frac{\pi^2 n^2 t}{L^2}\right)\cdot\frac{\sin \pi n/L}{\pi n/L}\cdot\cos\frac{\pi\cdot n\cdot x}{L}$$

$$h(x,t) = 2\cdot\ln[1+m_0\cdot\psi(x,t)]$$

$$\psi_0(x) = (1 - |x|) \cdot \theta(1 - |x|)$$

$$\psi(x,t) = \frac{1}{2L} + \frac{1}{L}\sum_{n=1}^{\infty} \exp\left(-\frac{\pi^2 n^2 t}{L^2}\right) \cdot \left(\frac{\sin \pi n/2L}{\pi n/2L}\right)^2 \cdot \cos\frac{\pi \cdot n \cdot x}{L}$$

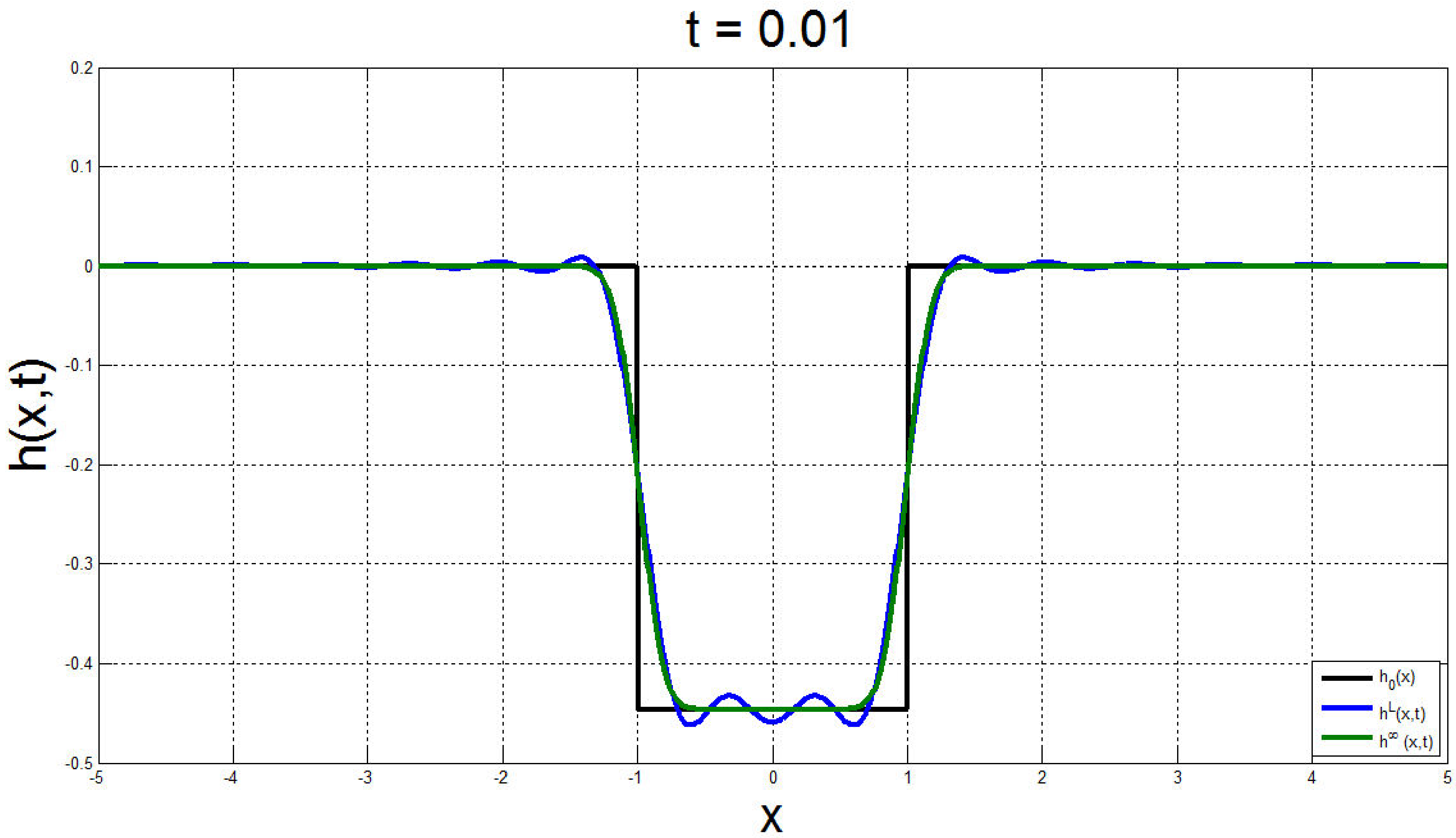
$$h(x,t) = 2 \cdot \ln[1 + m_0 \cdot \psi(x,t)]$$

**Results of computer experiment under values:**

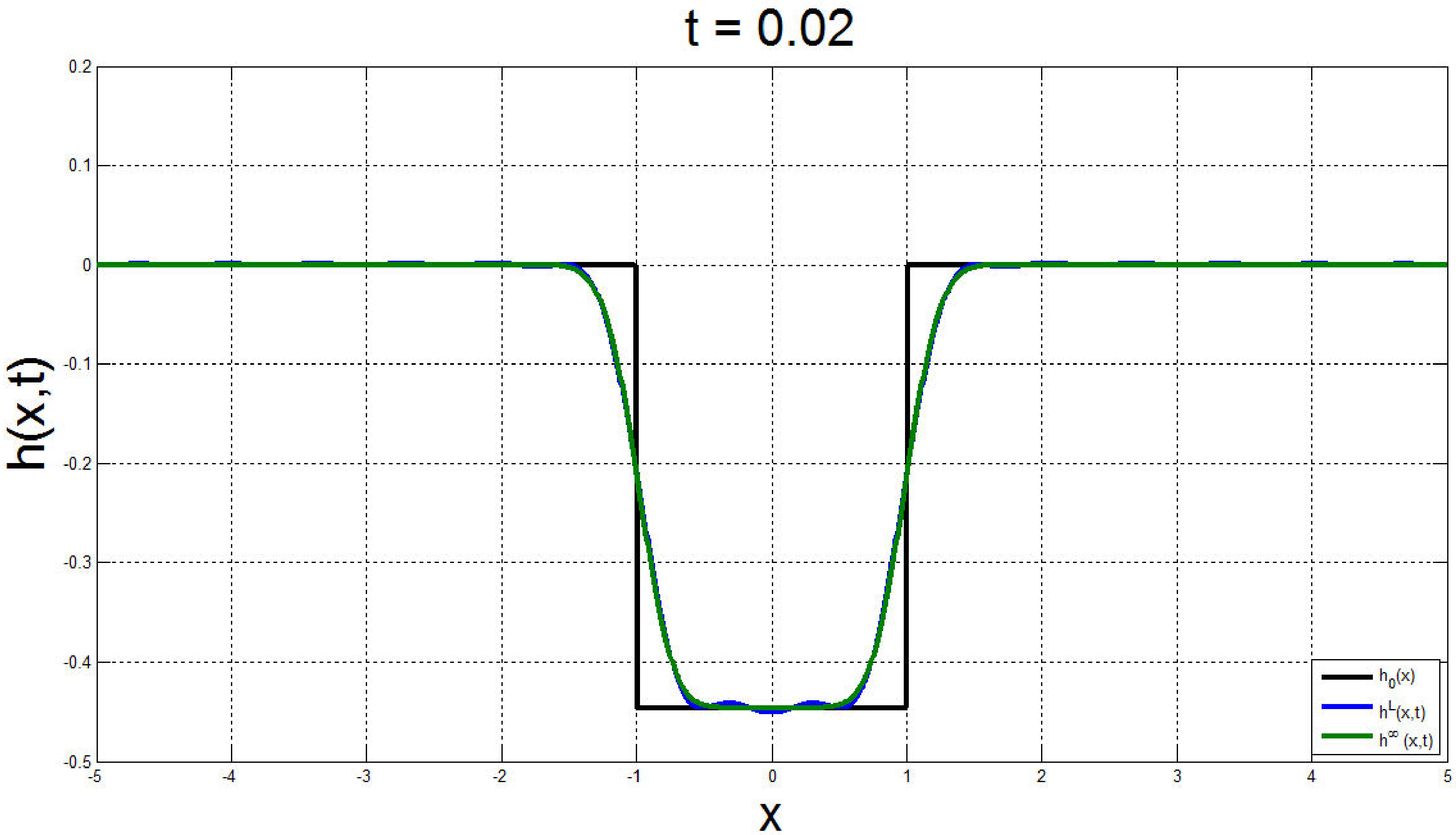
$$N = 30$$

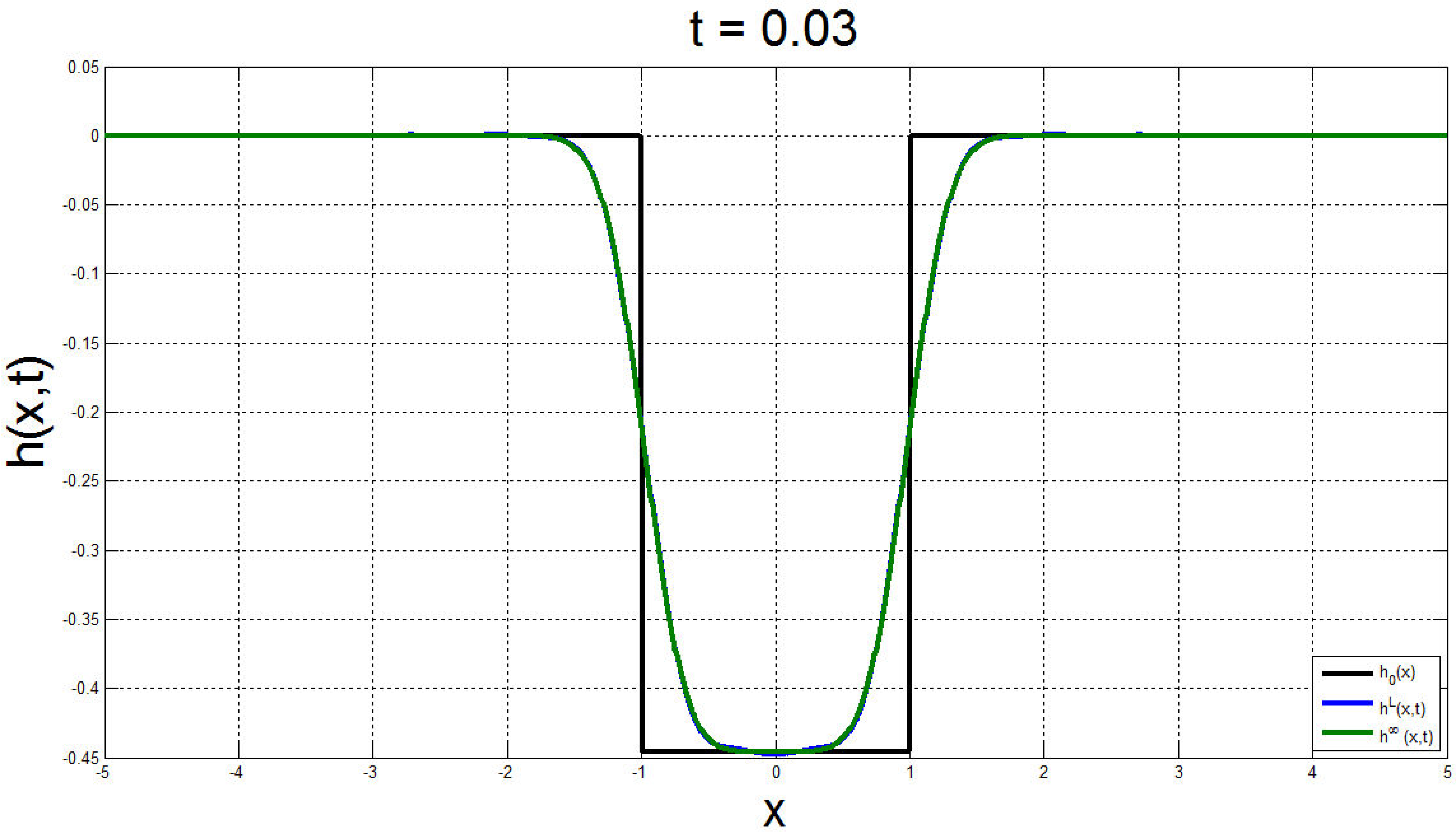
$$L = 10$$

$$m_0 = -0.2$$

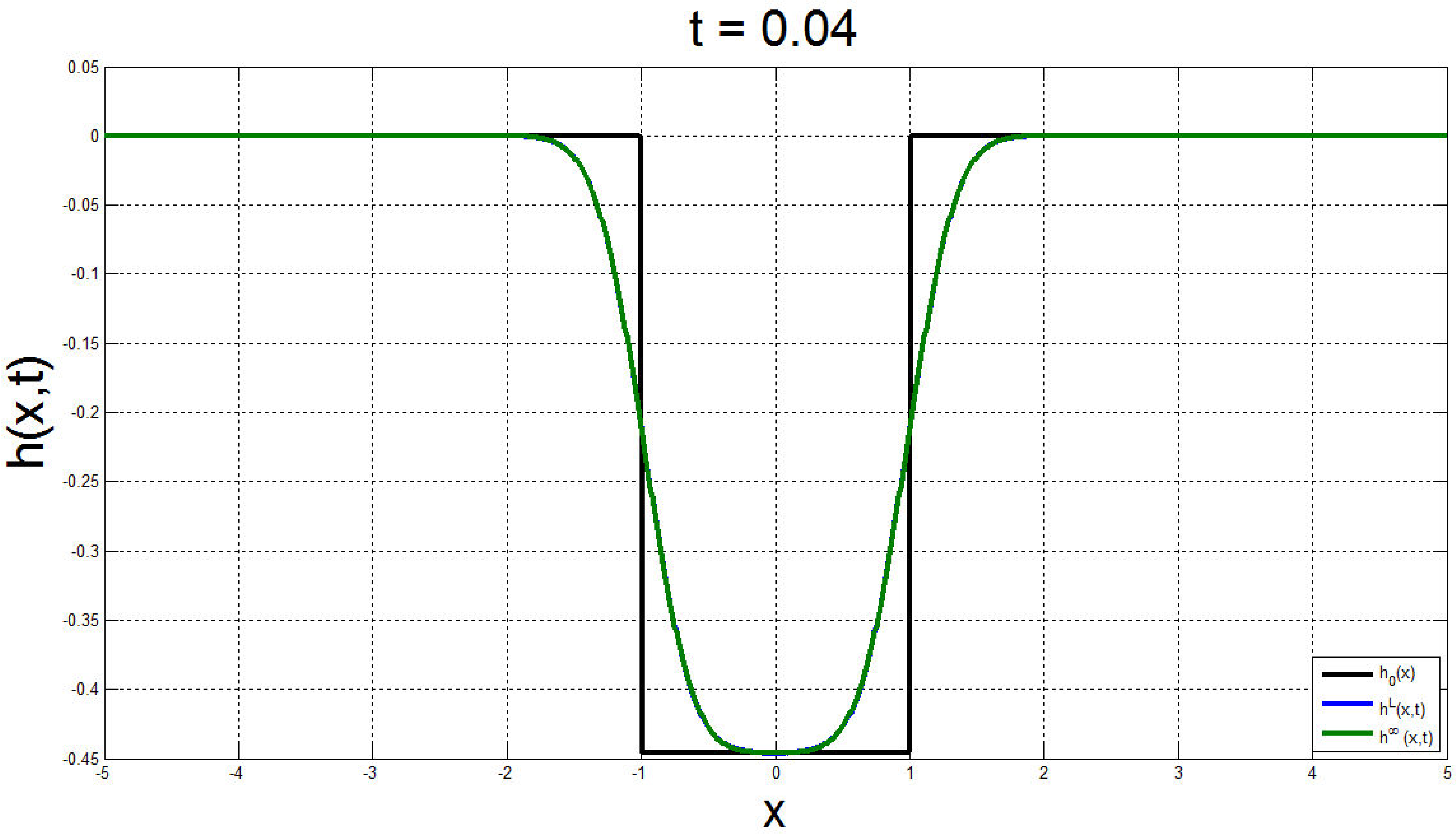


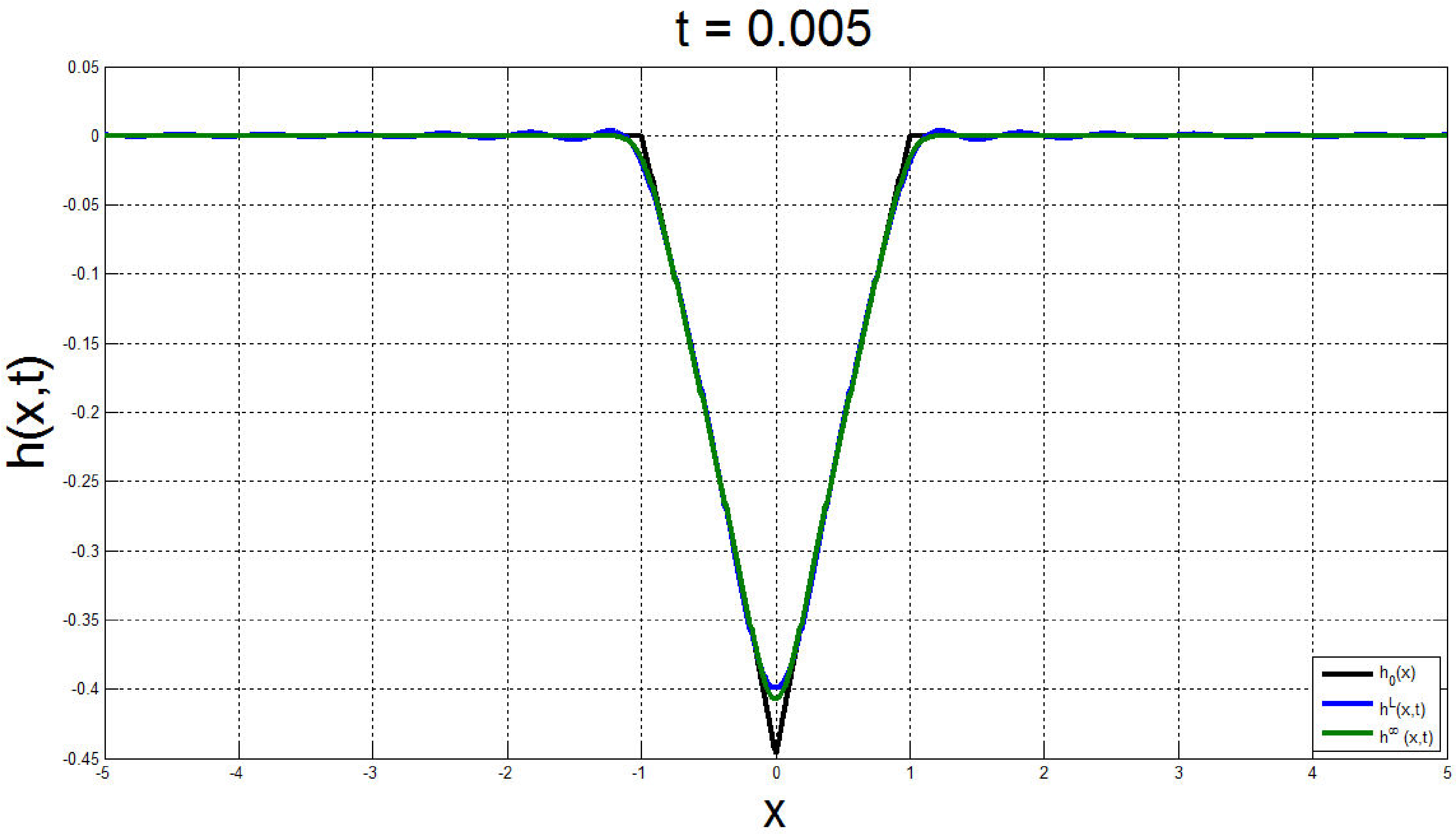
t = 0.01
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
h(x,t)
-5
-4
-3
-2
-1
0
1
2
3
4
5
x
h0(x)
hL(x,t)
h∞ (x,t)

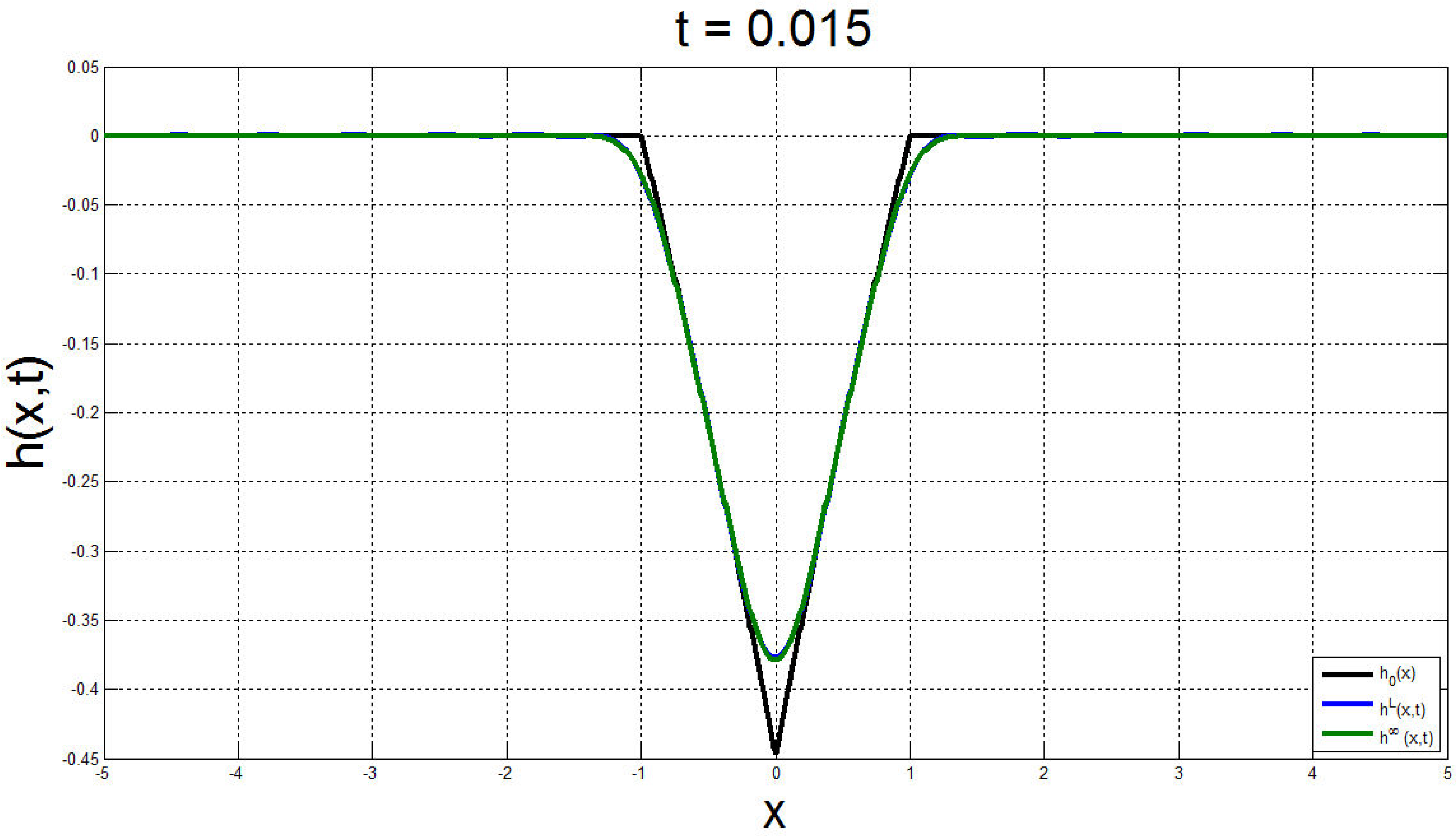

t = 0.02
h(x,t)
x
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-5
-4
-3
-2
-1
0
1
2
3
4
5
$h_0(x)$
$h^L(x,t)$
$h^\infty(x,t)$



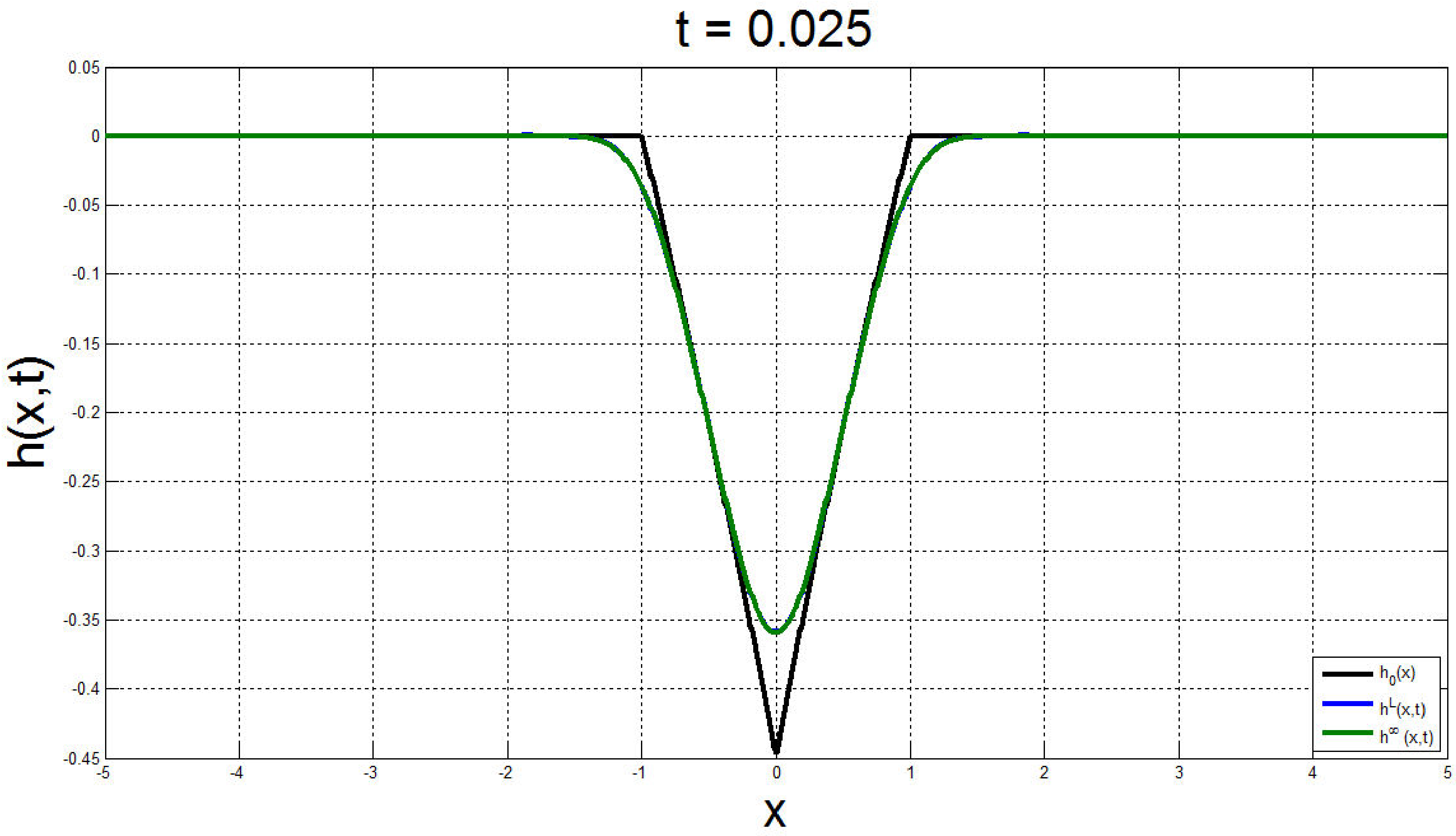
t = 0.03
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
h(x,t)
-5
-4
-3
-2
-1
0
1
2
3
4
5
x
h0(x)
hL(x,t)
h∞ (x,t)


t = 0.04
h(x,t)
x
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-5
-4
-3
-2
-1
0
1
2
3
4
5
h0(x)
hL(x,t)
h∞ (x,t)


t = 0.005
h(x,t)
x
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-5
-4
-3
-2
-1
0
1
2
3
4
5
$h_0(x)$
$h^L(x,t)$
$h^\infty(x,t)$



t = 0.015
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
h(x,t)
-5
-4
-3
-2
-1
0
1
2
3
4
5
x
h0(x)
hL(x,t)
h∞ (x,t)


t = 0.025
h(x,t)
x
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-5
-4
-3
-2
-1
0
1
2
3
4
5
$h_0(x)$
$h^L(x,t)$
$h^\infty (x,t)$

# Conclusion

**The KPZ equation forgets about the presence of boundary conditions quite quickly.**

**THANK YOU FOR YOUR ATTENTION!**