Renormalizations in digital representation of continuous observables

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Content

Digital representation

Let's consider the following series for the coordinate x on the lattice:

$$x = \sum_{s=-n_-}^{n_+-1} x_s q^s = \sum_{s=-n_-}^{n_+-1} d(s,x) q^s. \qquad (1)$$

Here, $x_s = d(s,x)$ is the digit number sin the digital representation of x.
On the line we can also write the following expression:

$$x = \sum_{s=-\infty}^{\infty} x_s q^s = \sum_{s=-\infty}^{\infty} d(s,x) q^s. \qquad (2)$$

Sum with prime on the line

Let's consider the base-q positional system with digits x_s. For such system the series:

$$\sum_{s=-\infty}^{\infty} x_s q^s, \tag{3}$$

generally speaking, does not converge.
We can define "the sum with the prime", for which the following expression occurs:

$$\sum_{s=0}^{\infty}{}' q^s = \frac{1}{1-q}, \tag{4}$$

which can be interpreted as using of formula for the converging geometrical progression beyond the borders of applicability.

Sum with prime on the line

Renormalization automatically induces the expression:

$$\sum_{s\in\mathbb{Z}}{}' q^s = 0, \tag{5}$$

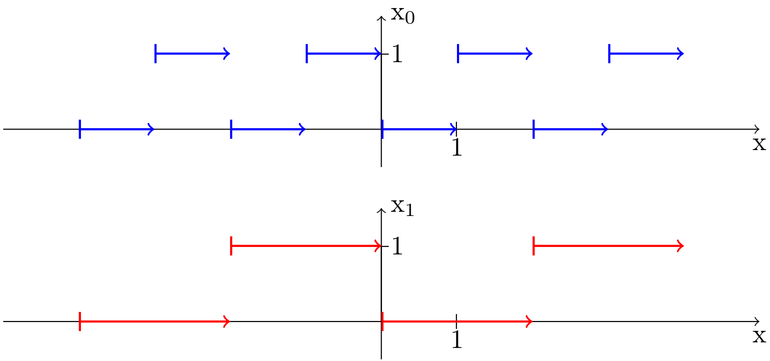
which, generally speaking, can be considered as the alternative definition of the renormalization.

Let us make a formal calculation:

$$x = \frac{qx - x}{q - 1} = \frac{1}{q - 1} \sum_{s\in\mathbb{Z}} (x_{s-1} - x_s) q^s. \tag{6}$$

Example 1: binary non-symmetrical system

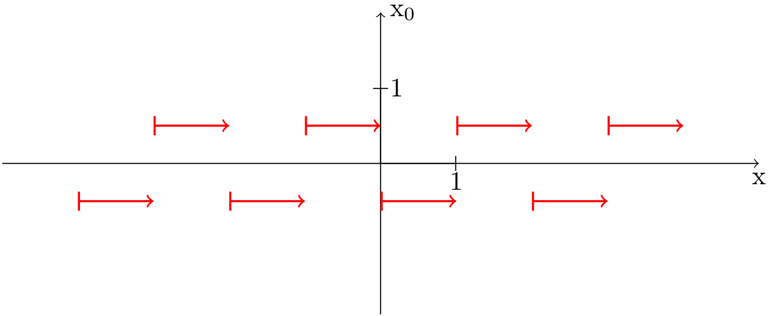
Let $q = 2$ and plot the binary digits number 0 and 1 in the system with digits $\{0, 1\}$.



We can see, that for the negative numbers in such a system the series diverges. This divergence can be "cured" by the renormalization:

$$x' = 2x - x = \sum_{s \in \mathbb{Z}} (x_{s-1} - x_s) 2^s. \tag{7}$$

Example 2: binary symmetrical system



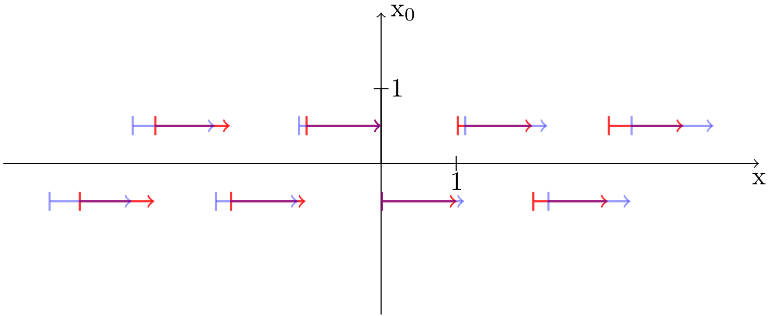
For such a system we have the divergence of both negative and positive numbers without the renormalization

$$x' = 2x - x = \sum_{s \in \mathbb{Z}} (x_{s-1} - x_s) 2^s. \tag{8}$$

“Non-integer” digits

On the line it is sometimes more convenient (for instance, when we need invariance in accordance to the unit scale) to work with “non-integer” values of digit number, which can be defined as following:

$$c(s, x) = c(0, q^{-s}x) = C(q^{-s}x) \qquad (9)$$



Digit number $\log_2(1.1)$ (blue lines) is obtained from digit number 0 (red lines) with scaling in 1.1 times over the x-axis

Integral representation

In the full analogy with the series, we can consider the formal integral:

$$\mathrm{x} \sim \int_{-\infty}^{\infty} \mathrm{c}(\mathrm{s}, \mathrm{x}) \mathrm{q}^{\mathrm{s}} \mathrm{ds} \tag{10}$$

It can be shown that in cases, when the integral converges, it gives the proper value. However, analogically to the considered previously series, such integral does not converge in the general case before the renormalization. The renormalization procedure is analogical to one, defined for the series.
Let us define:

$$\int_{-\infty}^{\infty'} \mathrm{q}^{\mathrm{s}} \mathrm{ds} = 0 \tag{11}$$

Renormalization for the integral

In analogy with the series we can write the general formula:

$$x = \frac{1}{q-1} \int_{-\infty}^{\infty} \left[c(s-1, x) - c(s, x)\right] q^{s} ds \tag{12}$$

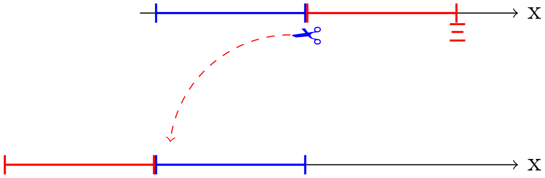
The alternative way to renormalize the integer – is to integrate by parts with the ommitting of the divergent summand:

$$\int_{-\infty}^{\infty}{}'' c(s, x) q^{s} ds = \frac{1}{\ln(q)} c(s, x) q^{s} \Big|_{-\infty}^{\infty} - \frac{1}{\ln(q)} \int_{-\infty}^{\infty} (c(s, x))'_{s} q^{s} ds = x. \tag{13}$$

The derivative $(c(s, x))'_{s} = \frac{d}{ds} c(s, x)$ is understood in terms of the generalized functions.

Renormalization on the lattice

As far as the lattice is finite, we can not obtain the divergent sums on it. Therefore, the purpose of the renormalization is no longer to avoid the divergence, but to change the representation of $\mathbb{Z}_N$ from $\{0, 1, \ldots, N-1\}$ to, for instance, $\{-k, -k+1, \ldots, -k+N-1\}$.



Example 1: binary case

In the binary case we can just turn from the series to the finite sum in the formula

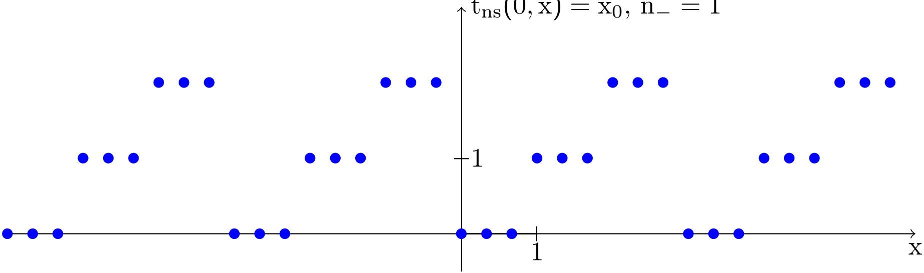
$$x = \sum_{s=-n_-}^{n_+-1}{}' x_s 2^s = \sum_{s=-n_-}^{n_+-1} (x_{s-1} - x_s) 2^s, \quad x_{-n_--1} = 0. \tag{14}$$

This renormalization can also be obtained by the redefinition of the last digit $x'_{n_+-1} = -x_{n_+-1}$:

$$\sum_{s=-n_-}^{n_+-1}{}'' x_s 2^s = \sum_{s=-n_-}^{n_+-2} x_s 2^s + x'_{n_+-1} 2^{n_+-1}. \tag{15}$$

The renormalization for binary systems is linear in accordance to the binary digits x_s.

Example 2: ternary non-symmetrical system



Let us switch the representation from the lattice $\{0, \Delta x, 2\Delta x, 3\Delta x, \ldots, (3^n - 1)\Delta x\}$ to the lattice $\{-3^{n-1}\Delta x, \ldots, -\Delta x, 0, \Delta x, 2\Delta x, \ldots, (3^n - 3^{n-1} - 1)\Delta x\}$, by substracting 3^{n_+} from the last 3^{n-1} nodes.

Example 2: ternary non-symmetrical system

Then we obtain::

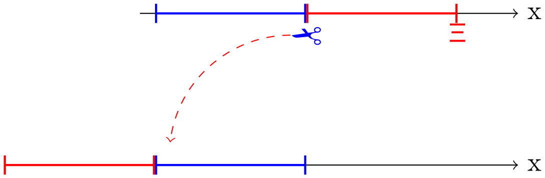
$$\begin{aligned} t''_{ns}(n_+ - 1, x) &= \begin{cases} 0, & t_{ns}(n_+ - 1, x) = 0, \\ 1, & t_{ns}(n_+ - 1, x) = 1, \\ -1, & t_{ns}(n_+ - 1, x) = 2 \end{cases} = \\ &= t_{ns}(n_+ - 1, x) - \frac{3}{2}(t_{ns}(n_+ - 1, x) - 1)t_{ns}(n_+ - 1, x). \end{aligned} \tag{16}$$

$$x'' = \sum_{s=-n_-}^{n_+-1}{}'' t_{ns}(s, x)3^s = \sum_{s=-n_-}^{n_+-2} t_{ns}(s, x)\, 3^s + t''_{ns}(n_+ - 1, x)\, 3^{n_+-1}. \tag{17}$$

Such renormalization appears to be non-linear to the ternary digits x_s.

Meaning of the renormalization on the lattice

We have seen, that the renormalization on the lattice is basically switching the representation of $\mathbb{Z}_N$.



In this context, the equivalence $x \sim x + \Xi \cdot n = x + \Delta x \cdot N \cdot n$, $n \in \mathbb{Z}$ − is also a renormalization.

Motivation

- In the work of Ivanov, Dudchenko and Naumov was shown, that the "quasienergy" on the lattice with the periodical boundary conditions is renormalized to 0.
- We hope, that the "lattice" renormalization, wich turns big positive numbers into small negative ones can be applicable to describe the Casimir effect in lattice models.

Thank you for your attention!

Our papers:

- M.G. Ivanov, “Binary Representation of Coordinate and Momentum in Quantum Mechanics“, Theoretical and Mathematical Physics, 196(1): 1002-1017 (2018).
- M. G. Ivanov, A. Yu. Polushkin, Ternary and binary representation of coordinate and momentum in quantum mechanics, AIP Conference Proceedings 2362, 040002 (2021); // https://doi.org/10.1063/5.0055033.
- M. G. Ivanov, A. Yu. Polushkin, Digital representation of continuous observables in Quantum Mechanics // arXiv:2301.09348 [quant-ph]
- M.G. Ivanov, V.A. Dudchenko, V.V. Naumov, “Number theory renormalization of the vacuum energy“, not published yet

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