Number-theory renormalization of vacuum energy

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Abstract

For QFT on a lattice of dimension $d \geqslant 3$, the vacuum energy (both bosonic and fermionic) is zero if the Hamiltonian is a function of the square of the momentum, and the calculation of the vacuum energy is performed in the ring of residue classes modulo N. This fact is related to a problem from number theory about the number of ways to represent a number as a sum of d squares in the ring of residue classes modulo N.

Content

Summation by parts and renormalization

$$x = \sum_{s=-\infty}^{+\infty} \mathbf{d}(s,x)\, q^s \quad \longrightarrow \quad x' = \frac{1}{q-1} \sum_{s=-\infty}^{+\infty} [\mathbf{d}(s-1,x) - \mathbf{d}(s,x)]\, q^s.$$

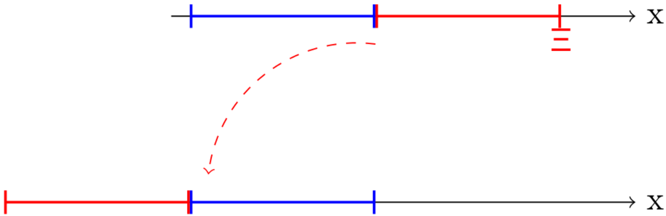
Here $\mathbf{d}(s,x)$ is digit in position s of чnumber x in q-base numeral system. The digit is a periodic function of x

$$\mathbf{d}(s,x) = \mathbf{d}(s, x + q^{s+1}).$$

This method works at infinite line for any q. For $q = 2$ the methods works at lattice.
There are the other renormalization methods, which work for arbitrary q. (M.G. Ivanov, A.Yu. Polushkin)

Renormalization is a change of representation of $\mathbb{Z}(N)$

1) Initial representation: $\mathbb{Z}(N) = \{0, 1, \ldots, N-1\}$.
Renormalized representation:
$\mathbb{Z}(N) = \{-k, -k+1, \ldots, 0, 1, \ldots, N-k-1\}$.



Look good for Casimir effect: large positive → small negative.

2) Use of the ring $\mathbb{Z}(N)$ is «renormalization» itself: $N = 0$.

The lattice $\mathbb{Z}^d(N)$

From the discrete Fourier transform.

- Field on the coordinate lattice $\rightarrow$ (quasi)momentum is determined on the inverse lattice.
- The representation $\mathbb{Z}(N)$ of (quasi)momentum components does not matter!

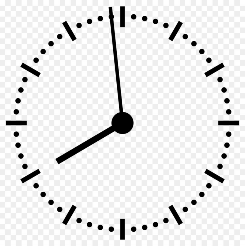
Energy in $\mathbb{Z}(N)$

If enrgy is a function of $\mathbf{p}^2 = p_1^2 + \cdots + p_d^2 \in \mathbb{Z}(N)$,

- it have to be independent from the reprezentation of $\mathbb{Z}(N)$,
- it is natural for (quasi)energy (like other p_α)) to be an element of the ring $\mathbb{Z}(N)$.
- The time is at the inverse lattice to the quasienergy lattice.

«Time is that which is measured by a clock.»
(H. Bondi, «Assumption and myth in physical theory»).

In our model the clock is such that the time is given by a finite number of q-ric digits. After the clock counts down the maximum possible time for a given number of digits, the countdown begins from the beginning. The next time after the $t = N - 1$ is $t = 0$.



The relativistic particle: $E(\mathbf{p}^2) = \sqrt{\mathbf{p}^2 + m^2}$

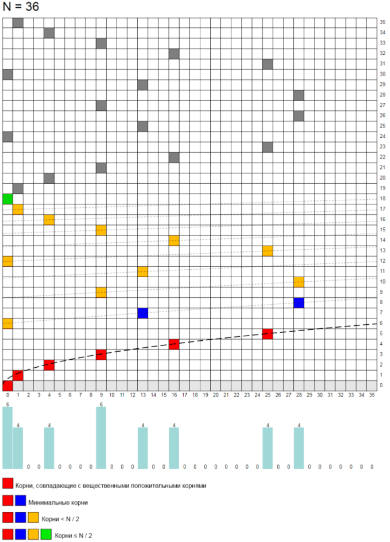
$$\hat{H}_f = \sum_{\mathbf{p}\in\mathbb{Z}^d(N),\mathbf{p}^2\in D} E(\mathbf{p}^2)\,\left(\hat{a}^\dagger_{\mathbf{p}+}\hat{a}_{\mathbf{p}+} - \hat{a}^\dagger_{\mathbf{p}-}\hat{a}_{\mathbf{p}-}\right),$$

$$\mathbf{p}^2 = \sum_{k=1}^{d} p_k^2 \in \mathbb{Z}(N), \quad E : D \to \mathbb{Z}(N), \quad D \subset \mathbb{Z}(N),$$

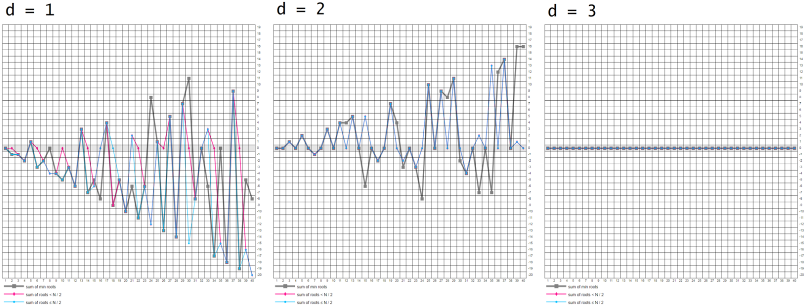
$$[\hat{a}_{\mathbf{p_1}\sigma_1}, \hat{a}^\dagger_{\mathbf{p_2}\sigma_2}]_+ = \delta_{\mathbf{p_1}\mathbf{p_2}}\,\delta_{\sigma_1\sigma_2}\,\hat{1}.$$

$$\mathcal{E}_{vac\ f} = -\mathcal{E} = -\sum_{\mathbf{p}\in\mathbb{Z}^d(N),\mathbf{p}^2\in D} E(\mathbf{p}^2) \in \mathbb{Z}(N). \qquad (1)$$

«The positive branch of the squre root» in $\mathbb{Z}(N)$?



Vacuum energy (N) (V.V. Naumov)



The method of square root calculation does not matter. The dimension of lattice is important!

Hypothesis (M.G. Ivanov, V.V. Naumov).
Probably it is multiplicities:
$\mathbf{p}^2$ takes each value $0 \pmod N$ times.

Theorem

$$c(k) = \Big(\text{the number of nodes } \mathbf{p} \text{ of } \mathbb{Z}^d(N) \text{ such that } \mathbf{p}^2 \equiv k \pmod N\Big).$$

Theorem. For an arbitrary N with $d \geqslant 3$, and for $N = 2^n$ with $d \geqslant 2$

$$\forall k \in \mathbb{Z}(N) \quad c(k) \equiv 0 \pmod N. \tag{2}$$

Proof:
V.A. Dudchenko: $N = p$ (odd primes) and $N = p^n$ by generating functions,
V.V. Naumov: composite N (and 2^n), by splitting of the lattice to sublattices.

Corollary

For any $E(\mathbf{p}^2)$, $E : D \to \mathbb{Z}(N)$, $D \subset \mathbb{Z}(N)$ vacuum energy (bosonic or fermionic) is 0.

$$\mathcal{E}_{vac\ b} = -\mathcal{E}_{vac\ f} = \mathcal{E} = \sum_{\mathbf{p}\in\mathbb{Z}^d(N),\mathbf{p}^2\in D} E(\mathbf{p}^2) = \sum_{k\in D} c(k)\, E(k) \equiv 0 \pmod N.$$

Hypothesis (V.V. Naumov)

1) For arbitrary integers $m \geqslant 0$ and $N \geqslant 1$, the number of ways in which an arbitrary element $X \in \mathbb{Z}(N)$ is represented as the sum of d terms of degree m

$$X \equiv x_1^m + \cdots + x_d^m \pmod N$$

is always a multiple of N, if $d \geqslant m+1$.
2) If $d < m+1$, then there is such a N and $X \in \mathbb{Z}(N)$, that the number of ways in which X is represented as the sum of d terms of degree m is not divisible by N.
The hypothesis is obvious for $m = 0$ and $m = 1$. For $m = 2$ the hypothesis follows from the Theorem.
The first statement of the hypothesis is verified numerically for all cases

- $m \leqslant 8$, $N \leqslant 1000$;
- $m \leqslant 35$, $N \leqslant 300$;
- $m \leqslant 100$, $N \leqslant 37$.

The second statement of the hypothesis is verified for all $m \leqslant 100$.

Literature-1

- J.C. Collins, *Renormalization*. Cambridge University Press, 1984.
- V.S. Vladimirov, I.V. Volovich, and E.I. Zelenov, *p-Adic Analysis and Mathematical Physics [in Russian]*, Nauka, Moscow (1994); English transl., World Scientific, Singapore (1994).
- Volovich, I.V. Number theory as the ultimate physical theory. P-Adic Num Ultrametr Anal Appl 2, 77–87 (2010).
- M.G. Ivanov, "Binary Representation of Coordinate and Momentum in Quantum Mechanics", *Theoretical and Mathematical Physics*, **196**(1): 1002-1017 (2018).
- M.G. Ivanov, A.Yu. Polushkin, "Ternary and binary representation of coordinate and momentum in quantum mechanics", *AIP Conference Proceedings* **2362**, 040002 (2021)
- M.G. Ivanov, A.Yu. Polushkin, "Digital representation of continuous observables in Quantum Mechanics", arXiv:2301.09348 [quant-ph]

Literature-2

- V.V. Dotsenko. *Arithmetic of quadratic forms [in Russian]*. Moscow: MCCME, 2015.
- N.A. Vavilov. ‘The computer as a new reality of mathematics. II. The Waring problem. [in Russian]“ *Computer tools in education* (2020).
- I. M. Vinogradov. *Elements of Number Theory [in Russian]*. Nauka, Moscow (1965); English transl. Dover Publications (2016).
- I.R. Shafarevich, Z.I. Borevich. *Number Theory [in Russian]*. Nauka, Moscow (1985); English transl. Elsevier Science (1986).

Спасибо за внимание!

Thank you for your attention!