Szegő-Weinberger type inequalities for symmetric domains with holes

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Based on the work

[Anoop, B., Drábek, SIAM J. Math. Anal., 2022]

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Consider the classical eigenvalue problem

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{EP})$$

where Ω is a Lipschitz bounded domain in $\mathbb{R}^N$, $N \geq 2$.

The spectrum of $(\mathcal{EP})$ consists of a discrete sequence of eigenvalues

$$0 = \mu_1(\Omega) < \mu_2(\Omega) \leq \mu_3(\Omega) \leq \dots$$

Any eigenfunction of $(\mathcal{EP})$, except of the first one, is sign-changing and has zero mean. Moreover, any kth eigenfunction has at most k nodal domains.

The second eigenvalue $\mu_2(\Omega)$ can be defined as

$$\mu_2(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} : u \in H^1(\Omega) \setminus \{0\}, \ \int_\Omega u \, dx = 0 \right\}.$$

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The Szegő-Weinberger inequality states that

$$\mu_2(\Omega) \leq \mu_2(B), \qquad (1)$$

where B is an open N-ball of the same measure as Ω, and equality holds if and only if $\Omega = B$.

Remark

The Szegő-Weinberger inequality is the Neumann counterpart of the Faber-Krahn inequality for the first Dirichlet eigenvalue: $\lambda_1(\Omega) \geq \lambda_1(B)$.

The Szegő-Weinberger inequality was

- Conjectured by Kornhauser & Stakgold [1952] for $N = 2$;
- Proved by Szegő [1954] for $N = 2$ when Ω is simply-connected;
- Proved by Weinberger [1956] for any $N \geq 2$ without topological restrictions on Ω.

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A generalization of the Szegő-Weinberger inequality for $\mu_3(\Omega)$:

$$\mu_3(\Omega) \leq 2^{\frac{2}{N}} \mu_2(B) \equiv 2^{\frac{2}{N}} \mu_3(B), \tag{2}$$

has been proved by

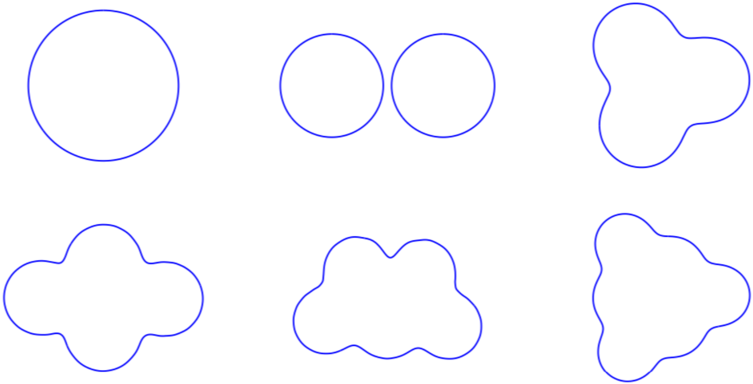
- GIROUARD, NADIRASHVILI, POLTEROVICH [2009] for $N = 2$
- BUCUR, HENROT [2019] for $N \geq 2$

Equality holds in (2) if Ω is a union of two disjoint equimeasurable balls.

Remark

The inequality (2) is the Neumann counterpart of the Hong-Krahn-Szegő inequality for the second Dirichlet eigenvalue: $\lambda_2(\Omega) \geq 2^{\frac{2}{N}} \lambda_2(B)$.

Maximizing sets for $\mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7$ obtained numerically by ANTUNES, OUDET [2017]:



In the planar case $N = 2$, various improvements of the Szegő-Weinberger inequality are known under additional assumptions on the symmetry of Ω.

Definition

$\Omega \subset \mathbb{R}^2$ is q-symmetric (or symmetric of order q) if Ω is invariant under the rotation by angle $2\pi/q$. (In other words, $R^{2\pi/q}\Omega = \Omega$, where $R^{2\pi/q}$ is the rotation by angle $2\pi/q$.)

- Hersch [1965]: for any simply-connected q-symmetric Ω with $q \geq 3$:

$$\mu_3(\Omega) \leq \mu_2(B) \; (= \mu_3(B)). \tag{3}$$

In fact, it was later shown by Ashbaugh & Benguria [1993] that $\mu_2(\Omega) = \mu_3(\Omega)$ for such class of domains.

- Ashbaugh & Benguria [1993]: (3) holds for any 4-symmetric Ω (without topological restrictions).

- Hersch [1965]: for any simply-connected 4-symmetric Ω:

$$\mu_4(\Omega) \leq \mu_4(B). \tag{4}$$

Our main result is a generalization of the inequalities (1), (3), (4) in two directions:

- to the higher-dimensional case
- to domains with "holes"

Definition

$\Omega \subset \mathbb{R}^N$ is q-symmetric if $R_{i,j}^{2\pi/q}\Omega = \Omega$ for any $1 \leq i < j \leq N$, where $R_{i,j}^{2\pi/q}$ denotes the rotation by angle $2\pi/q$ in the coordinate plane (x_i, x_j).

Proposition

Let $N \geq 3$. If Ω is q-symmetric with $q \neq 1, 2, 4$, then Ω is radially symmetric.

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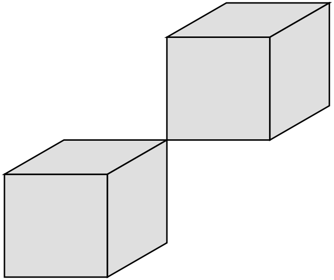
Definition

$\Omega \subset \mathbb{R}^N$ is *centrally symmetric* provided $x \in \Omega$ if and only if $-x \in \Omega$.

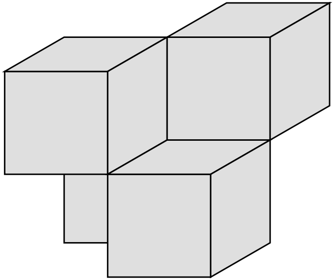
Remark

In the planar case $N = 2$, the 2-symmetry is equivalent to the central symmetry. When $N \geq 4$ is an *even* dimension, the 2-symmetry is a stronger notion than the central symmetry. When $N \geq 3$ is an *odd* dimension, these two notions are independent.

Explicit examples in $\mathbb{R}^3$:



Central symmetry but not 2-symmetry



2-symmetry but not central symmetry

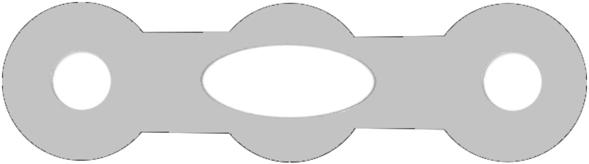
Let us now characterise a class of domains with “holes”.

Assumption

$\Omega = \Omega_{\text{out}} \setminus \overline{\Omega}_{\text{in}}$ is a domain in $\mathbb{R}^N$, where the domain Ω_{in} is compactly contained in the domain Ω_{out}. If Ω_{in} is nonempty, then we additionally assume $0 \in \Omega_{\text{in}}$.

Remark

Such Ω might possess other “holes” except Ω_{in}, or might possess no “holes” at all, in which case $\Omega_{\text{in}} = \emptyset$.



Theorem (Anoop, B., Drábek)

Let $0 \leq \alpha < \beta$ *be such that* $B_\alpha \subset \Omega_{\text{in}}$ *and* $|\Omega| = |B_\beta \setminus \overline{B}_\alpha|$. *Then the following assertions hold:*

(i) *If* Ω *is either 2-symmetric or centrally symmetric, then*

$$\mu_2(\Omega) \leq \mu_2(B_\beta \setminus \overline{B}_\alpha) \; (< \mu_2(B)). \tag{5}$$

(ii) *If* Ω *is 4-symmetric, then*

$$\mu_i(\Omega) \leq \mu_i(B_\beta \setminus \overline{B}_\alpha) \quad \text{for } i = 3, \dots, N+2. \tag{6}$$

(iii) *If* $N = 2$ *and* Ω *is 8-symmetric, then*

$$\mu_5(\Omega) \leq \mu_5(B_\beta \setminus \overline{B}_\alpha). \tag{7}$$

Equality holds in (5), (6), (7) *if and only if* $\Omega = B_\beta \setminus \overline{B}_\alpha$.

A direct corollary of Theorem is the domain monotonicity of several higher Neumann eigenvalues on the class of equimeasurable spherical shells.

Corollary

Let $0 < \alpha_1 < \alpha$, $0 < \beta_1 < \beta$, *and a ball* B *be such that* $|B_{\beta_1} \setminus \overline{B}_{\alpha_1}| = |B_\beta \setminus \overline{B}_\alpha| = |B|$. *Then*

$$\mu_i(B_\beta \setminus \overline{B}_\alpha) < \mu_i(B_{\beta_1} \setminus \overline{B}_{\alpha_1}) < \mu_i(B) \quad \textit{for } i = 2, 3, \ldots, N+2,$$

and, in the case $N = 2$, *also*

$$\mu_5(B_\beta \setminus \overline{B}_\alpha) < \mu_5(B_{\beta_1} \setminus \overline{B}_{\alpha_1}) < \mu_5(B).$$

Remark

This Corollary shows that the inequalities given in Theorem provide the best upper bounds with respect to α if B_α is chosen to be the maximal ball (centred at the origin) contained in Ω_{in}.

The global idea of the proof...

...is to construct an appropriate test finite-dimensional subspaces X of $H^1(\Omega)$ for the Courant-Fischer minimax characterization of the eigenvalues:

$$\mu_k(\Omega) = \min_{X \in \mathcal{X}_k} \max_{u \in X \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx},$$

where $\mathcal{X}_k$ is the collection of all k-dimensional subspaces of $H^1(\Omega)$.

To this end, we significantly use the structure and properties of eigenfunctions of the problem $(\mathcal{EP})$ on $B_\beta \setminus \overline{B}_\alpha$. Namely, one can find a complete orthogonal system of eigenfunctions of $(\mathcal{EP})$ on $B_\beta \setminus \overline{B}_\alpha$ in the form

$$\varphi(x) = h\left(\frac{x}{|x|}\right) v(|x|),$$

Here, h is a spherical harmonic corresponding to the eigenvalue $-l(l+N-2)$ of $\Delta_{S^{N-1}}$, and v is an eigenfunction of the Sturm-Liouville problem (with zero Neumann boundary conditions)

$$-v'' - \frac{N-1}{r} v' + \frac{l(l+N-2)}{r^2} v = \mu v, \quad r \in (\alpha, \beta). \tag{8}$$

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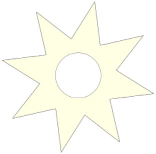
$N = 2$, symmetry of order 8

For example, let us discuss the inequality



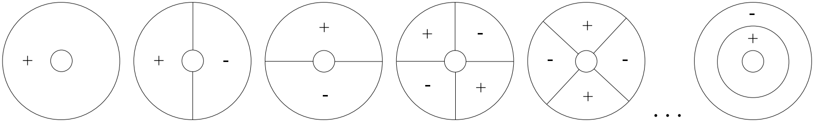
$$\mu_5(\Omega) \leq \mu_5(B_\beta \setminus \overline{B}_\alpha),$$

assuming that Ω is 8-symmetric.

First, we prove that

$$\mu_5(B_\beta \setminus \overline{B}_\alpha) = \mu_{2,1},$$

where $\mu_{2,1}$ is the first eigenvalue of the SL-problem (8) with $l = 2$.



μ_1	μ_2	μ_3	μ_4	μ_5	μ_k
$\shortparallel$	$\shortparallel$	$\shortparallel$	$\shortparallel$	$\shortparallel$	$\shortparallel$
$\mu_{0,1}$	$\mu_{1,1}$	$\mu_{1,1}$	$\mu_{2,1}$	$\mu_{2,1}$	$\mu_{0,2}$

Recall that

$$\mu_5(\Omega) = \min_{X \in \mathcal{X}_5} \max_{u \in X \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx},$$

where $\mathcal{X}_5$ is the collection of all 5-dimensional subspaces of $H^1(\Omega)$.

Let v be a first (e.g., positive) eigenfunction corresponding to $\mu_{2,1}$.

Define

$$G(r) = \begin{cases} v(r) & \text{if } r \in (\alpha, \beta), \\ v(\beta) & \text{if } r \geq \beta. \end{cases}$$

Setting $r = |x|$, we consider the set

$$X_5 = \operatorname{span}\left\{1, \frac{G(r)}{r}x_1, \frac{G(r)}{r}x_2, \frac{G(r)}{r^2}x_1x_2, \frac{G(r)}{r^2}(x_1^2 - x_2^2)\right\}.$$

We have $X_5 \in \mathcal{X}_5$, i.e., X_5 is admissible for $\mu_5(\Omega)$.

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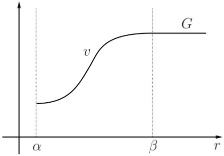

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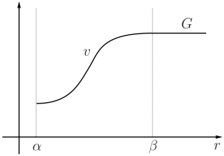

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We have $X_5 \in \mathcal{X}_5$, i.e., X_5 is admissible for $\mu_5(\Omega)$.

$N = 2$, symmetry of order 8

With this choice of X_5 and the 8-symmetry of Ω, one can show that

$$\mu_5(\Omega_{\text{out}} \setminus \overline{\Omega}_{\text{in}}) \leq \max_{u \in X_5 \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} \leq \frac{\int_\Omega \left((G'(r))^2 + \frac{2NG^2(r)}{r^2} \right) dx}{\int_\Omega G^2(r) \, dx}.$$

The following proposition is the final ingredient.

Proposition

Let $0 \leq \alpha < \beta$ be such that $B_\alpha \subset \Omega_{\text{in}}$ and $|\Omega| = |B_\beta \setminus \overline{B}_\alpha|$. Then

$$\frac{\int_\Omega \left((G'(r))^2 + \frac{2NG^2(r)}{r^2} \right) dx}{\int_\Omega G^2(r) \, dx} \leq \frac{\int_{B_\beta \setminus \overline{B}_\alpha} \left((G'(r))^2 + \frac{2NG^2(r)}{r^2} \right) dx}{\int_{B_\beta \setminus \overline{B}_\alpha} G^2(r) \, dx} = \mu_{2,1},$$

and equality holds if and only if $\Omega = B_\beta \setminus \overline{B}_\alpha$.

Recalling now that $\mu_{2,1} = \mu_5(B_\beta \setminus \overline{B}_\alpha)$, we finish the proof. □

Counterexamples for planar domains with less symmetries:

$$\mu_2(\Omega) > \mu_2(B_\beta \setminus \overline{B}_\alpha)$$

when Ω is an eccentric ring $B_\beta \setminus \overline{B_\alpha(s)}$ with certain values of parameters, see KUTTLER [1984].

$$\mu_i(\Omega) > \mu_i(B_\beta \setminus \overline{B}_\alpha) \quad i = 3, 4$$

when $\Omega_{\text{out}} = \left(-\frac{a}{2}, \frac{a}{2}\right) \times \left(-\frac{1}{2a}, \frac{1}{2a}\right)$ with $a = \sqrt{3}$ and $\Omega_{\text{in}} = B_\alpha$ with sufficiently small α.

$$\mu_5(\Omega) > \mu_5(B_\beta \setminus \overline{B}_\alpha)$$

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Thank you for your attention!

$$\sqrt{2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{G(r)}{r^2}x_i x_j+\sum_{i=1}^{N-1}\frac{G(r)}{\sqrt{i(i+1)}r^2}\left(\sum_{j=1}^{i}x_j^2-ix_{i+1}^2\right).$$