On strict monotonicity of the p-torsional rigidity over annuli

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III International Conference *"Mathematical Physics, Dynamical Systems, Infinite-Dimensional Analysis,"* dedicated to the 100th anniversary of V.S. Vladimirov, the 100th anniversary of L.D. Kudryavtsev and the 85th anniversary of O.G. Smolyanov

July 05-13, 2023, Moscow Region, Dolgoprudny

Outline

1. Introduction
   - Saint-Venant inequality
   - Reverse Saint-Venant inequality
2. Monotonicity of Torsional Rigidity
   - Monotonicity via Polarization: Complete Dirichlet boundary
   - Monotonicity via Geometry and Shape calculus: Mixed boundary
   - Neumann data of $w = u_s \circ \sigma_s - u_s$ on ∂B_R
3. Geometry of the torsion function
   - Foliated Schwarz symmetry
   - Monotonicity in the affine-radial directions
   - Monotonicity in the axial direction

Introduction

For $0 < r < R < \infty$ and $0 \leq s < R - r$, let $\Omega_s = B_R(0) \setminus \overline{B}_r(se_1)$ be an annular domain in $\mathbb{R}^d$, $d \geq 2$. For $p \in (1, \infty)$, we consider the p-torsion problem in Ω_s:

$$-\Delta_p u = 1 \text{ in } \Omega_s \tag{1a}$$

$$u = 0 \text{ on } \Gamma^s; \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_s \setminus \Gamma^s, \tag{1b}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator, and

$$\text{either } \Gamma^s = \partial\Omega_s \text{ or } \Gamma^s = \partial B_R(0) \text{ or } \Gamma^s = \partial B_r(se_1). \tag{2}$$

- The problem (1a) with (1b) admits a unique solution $u_s \in W^{1,p}_{\Gamma^s}(\Omega_s)$, and it is called the *p-torsion function* of Ω_s. Further, $0 < u_s \in C^{1,\alpha}(\overline{\Omega})$.
- The p-torsional rigidity $\mathcal{T}(s)$ of Ω_s is defined as

$$\mathcal{T}(s) = \left(\int_{\Omega_s} u_s \,\mathrm{d}x\right)^{p-1} = \left(\int_{\Omega_s} |\nabla u_s|^p \,\mathrm{d}x\right)^{p-1}. \tag{3}$$

- The p-torsion function u_s and p-torsional rigidity $\mathcal{T}(s)$ are given by the variational characterizations:

$$\max_{u \in W^{1,p}_{\Gamma^s}(\Omega_s)} \int_{\Omega_s} u \,\mathrm{d}x - \frac{1}{p} \int_{\Omega_s} |\nabla u|^p \,\mathrm{d}x = \frac{p-1}{p} \int_{\Omega_s} u_s \,\mathrm{d}x; \qquad (4)$$

$$\mathcal{T}(s) = \sup \left\{ \left[\int_{\Omega_s} u \,\mathrm{d}x \right]^p : 0 \le u \in W^{1,p}_{\Gamma^s}(\Omega_s) \text{ with } \int_{\Omega_s} |\nabla u|^p \,\mathrm{d}x = 1 \right\}. \qquad (5)$$

- **Saint-Venant inequality:** for a bounded domain $\Omega \subset \mathbb{R}^d$ with $\Gamma^s = \partial\Omega_s$,

$$\mathcal{T}(\Omega) \le \mathcal{T}(\Omega^*),$$

where Ω^* is a ball in $\mathbb{R}^d$ with $|\Omega^*| = |\Omega|$.

  - Conjectured by Saint-Venant [1855] for $d = 2 = p$.
  - Pólya [1948]: For the beams with simply connected cross-sections.
  - Pólya-Weinstein [1950]: For the beams with multiply-connected cross-sections.
  - Talenti [1977]: Proved by a symmetrization argument that holds even for $p \in (1, \infty)$ and $d \ge 2$.

Reverse Saint-Venant inequality

- Reverse Saint-Venant inequality:

$$\mathcal{T}(0) < \mathcal{T}(s) \text{ for } 0 < s < R - r,$$

where $\Omega_s = B_R(0) \setminus \overline{B}_r(se_1)$ for $0 \leq s < R - r$.

  - Hersch[1] & Wienberger[2] [1960's] proved for $d = 2 = p$, using the method of interior parallels.
  - Anoop-Ashok [2020][3] proved for $p \in (1, \infty)$ and $d \geq 2$, using the method of interior parallels.

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[1] J. Hersch. *Pacific J. Math.*, 13, 1963.

[2] Wienberger. *Pacific J. Math.*, 13, 1963.

[3] T. V. Anoop and K. Ashok Kumar. *J. Math. Anal. Appl.*, 485(1), 2020.

Monotonicity of Torsional Rigidity

- We are interested in the behavior $s \longmapsto \mathcal{T}(s)$ on $[0, R-r)$.
- For the first eigenvalue problem $-\Delta_p u = \gamma_1(s)|u|^{p-2}u$ in Ω_s with (1b):
  - **For $\Gamma^s = \partial\Omega_s$:** Harrell[4] & Kesavan[5] ($p = 2$), and Anoop-Bobkov-Sasi[6] ($1 < p < \infty$) proved the monotonicity of $\gamma_1(s)$ by showing that

$$\gamma_1'(0) = 0 \text{ and } \gamma_1'(s) < 0 \text{ for } s > 0.$$

  - **For $\Gamma^s = \partial B_r(se_1)$:** Anoop-Ashok-Kesavan[7] proved for $p = 2$.
  - **For $\Gamma^s \subseteq \partial\Omega_s$ as in (2):** Anoop-Ashok[8] proved when $p > \frac{2d+2}{d+2}$ for certain translations and rotations, using the methods of polarization.

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[4] E. M. Harrell, II, P. Kröger, and K. Kurata. *SIAM J. Math. Anal.*, 33(1), 2001.

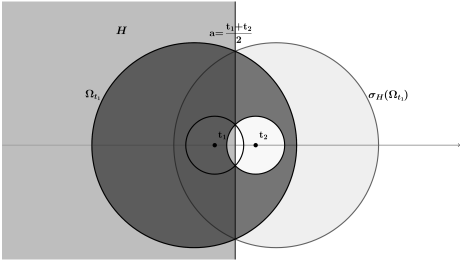
[5] S. Kesavan. *Proc. Roy. Soc. Edinburgh Sect. A*, 133(3), 2003.

[6] Anoop, T. V., V. Bobkov, and S. Sasi. Trans. Amer. Math. Soc., 370(10), 2018.

[7] Anoop, T. V., Ashok Kumar, K., and Kesavan, S. *J. Differ. Equ.* 298, 2021.

[8] Anoop, T. V. and Ashok Kumar, K. *Adv. Differential Equations* 28(7-8), 2023.

Monotoncity: when $\Gamma^s = \partial\Omega_s$ via Polarization



- Take $0 \leq s < t < R - r$ and $a := \frac{s+t}{2}$. Let $H_a = \left\{(x_1, x') \in \mathbb{R}^d : x_1 < a\right\}$ and $\sigma_a(x_1, x') = (2a - x_1, x')$ be the reflection with respect to $\partial H_a = \{x_1 = a\}$.
- For the maximizer $u_s \in W^{1,p}_{\Gamma^s}(\Omega_s)$ of $\mathcal{T}(s)$, $\widetilde{u_s}$ is the zero extension to $\mathbb{R}^d$. Define

$$P_a\widetilde{u_s}(x) = \begin{cases} \max\left\{\widetilde{u_s}(x), \widetilde{u_s}(\sigma_a(x))\right\}, & \text{for } x \in H_a, \\ \min\left\{\widetilde{u_s}(x), \widetilde{u_s}(\sigma_a(x))\right\}, & \text{for } x \in H_a^c. \end{cases} \tag{6}$$

- $\operatorname{supp}(P_a\widetilde{u_s}) \subseteq \Omega_t$, and $v := P_a\widetilde{u_s}|_{\Omega_t} \in W^{1,p}_{\Gamma^t}(\Omega_t)$ with $\int_{\Omega_t} v \,\mathrm{d}x = \int_{\Omega_s} u_s \,\mathrm{d}x$, and $\int_{\Omega_t} |\nabla v|^p \,\mathrm{d}x = \int_{\Omega_s} |\nabla u_s|^p \,\mathrm{d}x$.

Monotonicity via Polarization

- By the variational characterization (5),

$$\mathcal{T}(t) \geq \left(\int_{\Omega_t} v \, \mathrm{d}x \right)^p = \left(\int_{\Omega_s} u_s \, \mathrm{d}x \right)^p = \mathcal{T}(s).$$

- For the strict inequality, we use a version of the strong comparison principle.

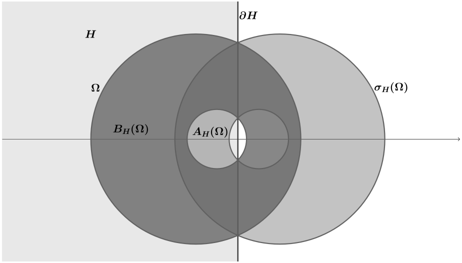
Strong Comparison Principle, Sciunzi[9]

Let $B \subset \mathbb{R}^d$ be a bounded smooth domain, and $\frac{2d+2}{d+2} < p < \infty$. Let $u, v \in \mathcal{C}^1(\overline{B})$ be positive distributional solutions of $-\Delta_p w = 1$ in B. If $v \geq u$ in B, then either $v > u$ in B or $v \equiv u$ in B.

- We find a ball $B \subset \Omega_s \cap H_a$ in which v, u_s satisfy a contradictory properties to the SCP.
- If $\mathcal{T}(s) = \mathcal{T}(t)$, then $v, u_s \in \mathcal{C}^1(\overline{B})$ are positive distributional solutions in B. This leads to a contradiction to the SCP. Thus the strict inequality $\mathcal{T}(s) < \mathcal{T}(t)$ holds.

[9] Sciunzi, B., *Commun. Contemp. Math.*, 16(6), 2014.

Existence of a ball $B \subset \Omega_s \cap H_a$



- $B_H(\Omega_s) = B_R(0) \setminus B_R(\sigma_a(0))$, and $A_H(\Omega_s) = \overline{B}_r(se_1) \setminus \overline{B}_r(\sigma_a(se_1))$ have non-empty interiors in $\Omega_t \cap H_a$.
- $\Omega_t \cap H_a = \mathcal{M}_{u_s} \sqcup \mathcal{N}_{u_s}$ with $\mathcal{M}_{u_s} = \{x \in \Omega_t \cap H_a : v(x) > u_s(x)\} \supsetneq A_H(\Omega_s)$ and $\mathcal{N}_{u_s} = \{x \in \Omega_t \cap H_a : v(x) = u_s(x)\} \supsetneq B_H(\Omega_s)$.
- $\Omega_t \cap H_a$ is connected, then $\emptyset \neq \mathcal{N}_{u_s} \cap \partial \mathcal{M}_{u_s} \ni x_0$ and $B := B_\rho(x_0) \subset \Omega_s \cap H_a$.
- $v > u_s$ in $B \cap \mathcal{M}_{u_s} \neq \emptyset$ and $v = u$ in $B \cap \mathcal{N}_{u_s} \neq \emptyset$.

Monotonicity of Torsional Rigidity

Theorem

Let $0 < r < R < \infty$ and $p \in (1, \infty)$. For $0 \leq s < R - r$, let $\mathcal{T}(s)$ be the p-torsional rigidity of $\Omega_s = B_R(0) \setminus \overline{B}_r(se_1)$ with $\Gamma_s = \partial\Omega_s$. Then $s \longmapsto \mathcal{T}(s)$ is increasing on $[0, R - r)$. Further, if $p > \frac{2d+2}{d+2}$ then $s \longmapsto \mathcal{T}(s)$ is strictly increasing on $[0, R - r)$.

Monotonicity of Torsional Rigidity

Main ingredients in proof of Harrell & Kesavan: Let $u > 0$ be the first eigenfunction corresponding to $\lambda_1(s)$ with $\|u\|_2 = 1$.

1. Shape derivative formula for $\lambda_1'(s)$,

$$\lambda_1'(s) = -\int_{\partial B_r} \left(\frac{\partial u}{\partial n}(x)\right)^2 n_1(x)\,\mathrm{d}S(x).$$

   Derived using the Hadamard perturbation formula

$$d\lambda_1(\Omega_s; V) = -\int_{\partial \Omega_s} \left(\frac{\partial u}{\partial n}(x)\right)^2 V(x)\cdot n(x)\,\mathrm{d}S(x),$$

   where V is a smooth vector field on $\mathbb{R}^d$, and $n(x) = (n_1(x), \dots, n_d(x))$ is an outward normal to Ω_s.

2. Strong maximum principle and Hopf's boundary point lemma

Shape derivative of $\mathcal{T}(s)$: when $p = 2$

Let $u_s > 0$ be the torsion function of Ω_s with Γ^s as in (2), i.e.,

$$\mathcal{T}(s) = \int_{\Omega_s} u_s \, \mathrm{d}x = \int_{\Omega_s} |\nabla u_s|^2 \, \mathrm{d}x$$

- Hadamard perturbation formula: For a smooth vector field V on $\mathbb{R}^N$,

$$d\mathcal{T}(\Omega_s; V) = \int_{\Gamma^s} \left(\frac{\partial u_s}{\partial n} \right)^2 V \cdot n \, \mathrm{d}S - \int_{\partial\Omega_s \setminus \Gamma^s} \left(|\nabla u_s|^2 - 2u_s \right) V \cdot n \, \mathrm{d}S$$

- Shape derivative formula when $\Gamma^s = \partial B_r(se_1)$: Take V that fixes the outer boundary ∂B_R and translates the inner ball $B_r(se_1)$ along e_1-axis: $V(x) = \rho(x) e_1$ with $0 < \rho \in \mathcal{C}_c^\infty(B_R(0))$ with $\rho \equiv 1$ in a neighborhood of $B_r(se_1)$.

$$\mathcal{T}'(s) = \int_{\partial B_r} \left(\frac{\partial u_s}{\partial n}(x) \right)^2 n(x) \cdot e_1 \, \mathrm{d}S(x).$$

Monotonicity of $\mathcal{T}(s)$: Approach of Harrell & Kesavan

1. Rewriting the shape derivative formula:

$$\mathcal{T}'(s) = \int_{\partial B_r \cap \{x_1 > s\}} \left[\left(\frac{\partial u_s}{\partial n}(x) \right)^2 - \left(\frac{\partial u_s}{\partial n}(\sigma_s(x)) \right)^2 \right] n_1(x) \, \mathrm{d}S(x),$$

   where $\sigma_s(x)$ is the reflection with respect to the affine-hyperplane $\{x_1 = s\}$.

2. Observe $\frac{\partial u_s}{\partial n}(\sigma_s(x)), \frac{\partial u_s}{\partial n}(x) < 0$ (by Hopf's), and $n_1(x) < 0$ for $x \in \partial B_r \cap \{x_1 > s\}$.

3. If

$$\frac{\partial u_s}{\partial n}(\sigma_s(x)) - \frac{\partial u_s}{\partial n}(x) < 0 \text{ for } x \in \partial B_r \cap \{x_1 > s\}, \tag{7}$$

   then we get $\mathcal{T}'(s) > 0$.

Monotonicity of $\mathcal{T}(s)$

- Consider $w = u_s \circ \sigma_s - u_s$ on $\widetilde{\Omega_s} = \Omega_s \cap \{x_1 > s\}$. Then w satisfies:

$$-\Delta w = 0 \text{ in } \widetilde{\Omega_s}.$$

- On $H_s \cup (\partial B_r \cap \{x_1 > s\})$: $w(x) = u_s \circ \sigma_s(x) - u_s(x) = u_s(x) - u_s(x) = 0.$
- On $\partial B_R \cap \{x_1 > s\}$: $w =?$ or $\dfrac{\partial w}{\partial n} =?$
- Since $\dfrac{\partial u_s}{\partial n} = 0$ on $\partial B_R(0)$, it is natural to expect the sign of $\dfrac{\partial w}{\partial n}$ on $\partial B_R(0)$.
- Here, the finer geometric properties of the torsion function help us to confirm

$$\frac{\partial w}{\partial n} \geq 0 \text{ on } \partial B_R \cap \{x_1 > s\}.$$

Neumann data of $w = u_s \circ \sigma_s - u_s$ on ∂B_R

- Observe that, for $x \in \partial B_R$,

$$\frac{\partial w}{\partial n}(x) = \frac{\partial u_s \circ \sigma_s}{\partial n}(x) - \frac{\partial u_s}{\partial n}(x) = \frac{\partial u_s \circ \sigma_s}{\partial n}(x).$$

- Recall that, for $x = (x_1, x') \in \mathbb{R}^d$,

$$\sigma_s(x) = (2s - x_1, x') = 2se_1 + \sigma_0(x).$$

- So $\nabla(u_s \circ \sigma_s)(x) = \nabla u_s(\sigma_s(x))A_0$, where $A_0 = \mathrm{diag}(-1, 1, \ldots, 1)$ is the reflection matrix with respect to $\{x_1 = 0\}$.

Neumann data of $w = u_s \circ \sigma_s - u_s$ on ∂B_R

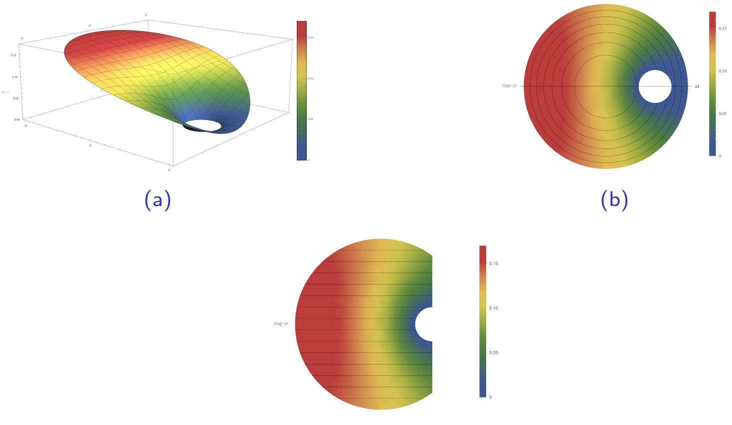
- For $x \in \partial B_R$, the unit outward normal is $n(x) = \dfrac{x}{R}$. Then

$$\begin{aligned} R\nabla(u_s \circ \sigma_s)(x) \cdot n(x) = \nabla u_s(\sigma_s(x))A_0 \cdot x &= \nabla u_s(\sigma_s(x)) \cdot \sigma_0(x) \\ &= \nabla u_s(\sigma_s(x)) \cdot (\sigma_s(x) - 2se_1). \end{aligned}$$

- Write

$$R\frac{\partial w}{\partial n}(x) = \varphi(\sigma_s(x)) - s\frac{\partial u_s}{\partial x_1}(\sigma_s(x)),$$

where $\varphi(x) = \nabla u_s(x) \cdot (x - se_1)$ for $x \in \Omega_s$.



(a) (b) (c)

Figure: (a) The torsion function of $B_5(0) \setminus \overline{B_1(3e_1)} \subset \mathbb{R}^2$. (b) The foliated Schwartz symmetry, (c) Monotonicity along the e_1-direction up to $x < 3$.

Theorem

Let $\Omega_s = B_R(0) \setminus \overline{B_r(se_1)}$ be an annular domain with $0 < s < R - r$. Let $u_s > 0$ be the torsion function and $\mathcal{T}(s)$ of Ω_s with $\Gamma^s = \partial B_r(se_1)$ be the torsional rigidity. Then

1. *u_s has the foliated Schwarz symmetry in Ω_s with respect to $-\mathbb{R}^+ e_1$;*
2. *u_s is strictly increasing along all affine-radial directions from se_1 in Ω_s, i.e.,*

$$\nabla u_s(x) \cdot (x - se_1) > 0 \text{ for } x \in \overline{\Omega_s} \setminus \{\pm Re_1\};$$

3. *u_s is strictly decreasing in the e_1-direction on the sub-region $\{x \in \Omega_s : x_1 < s\}$ of Ω_s, i.e.,*

$$\frac{\partial u_s}{\partial x_1} < 0 \text{ on } \{x \in \Omega_s : x_1 < s\}.$$

- The affine-radial monotonicity of u_s implies that $\varphi > 0$ in Ω_s,
- The axial monotonicity of u gives that

$$\frac{\partial u_s}{\partial x_1} < 0 \text{ for } x \in \Omega_s \cap \{x_1 < s\}.$$

- Therefore, for $x \in \partial B_R \cap \{x_1 > s\}$,

$$\frac{\partial w}{\partial n}(x) = \frac{1}{R}\varphi(\sigma_s(x)) - \frac{s}{R}\frac{\partial u_s}{\partial x_1}(\sigma_s(x)) \geq 0.$$

Monotonicity of $\mathcal{T}(s)$

- The map w satisfies the following boundary value problem on $\widetilde{\Omega_s} := \Omega_s \cap \{x_1 > s\}$:

$$\left.\begin{aligned} -\Delta w &= 0 \text{ in } \widetilde{\Omega_s}, \\ w &= 0 \text{ on } H_s \cup (\partial B_r \cap \{x_1 > s\}), \\ \frac{\partial w}{\partial n} &\geq 0 \text{ on } \partial B_R \cap \{x_1 > s\}, \end{aligned}\right\} \tag{8}$$

- We derived a variant of strong maximum principle that holds for the boundary value problem above.

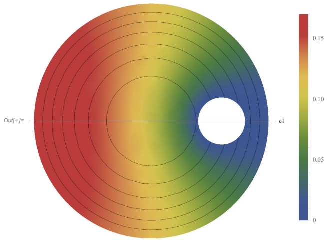
Monotonicity of $\mathcal{T}(s)$

- Since $\dfrac{\partial w}{\partial n} > 0$ on $\partial B_R \setminus \{\pm Re_1\}$, SMP implies that $w > 0$ in $\widetilde{\Omega_s}$.
- Hopf's lemma implies that $\dfrac{\partial w}{\partial n} < 0$ on $\partial B_r \cap \{x_1 > s\}$ since $w = 0$.
- For $x \in \partial B_r \cap \{x_1 > s\}$,

$$0 > \frac{\partial w}{\partial n}(x) = \frac{\partial u_s}{\partial n}(\sigma_s(x)) - \frac{\partial u_s}{\partial n}(x). \tag{9}$$

- Therefore $\mathcal{T}'(s) > 0$.

Foliated Schwarz symmetry of u



- Let $-\mathbb{R}^+ e_1$ be the half-ray from 0 in $-e_1$-direction.
- Let (r, θ) be the polar-coordinates given by $-\mathbb{R}^+ e_1$ with

$$r(x) = |x| \text{ and } \cos(\theta(x)) = -\frac{x \cdot e_1}{|x|}, \quad \theta \in [0, \pi]. \tag{10}$$

- A map $0 \leq u_s : \Omega_s \to \mathbb{R}$ is foliated Schwarz symmetric in Ω_s with respect to $-\mathbb{R}^+ e_1$, if and only if
  1. u depends only on (r, θ),
  2. for fixed $r > 0$, u decreases in θ. Equivalently, for $x, y \in \Omega_s$ with $|x| = |y| = r$ and $\theta(x) \geq \theta(y)$ we have $u(x) < u(y)$.

Foliated Schwarz symmetrization via Polarization

- Let H be an open affine half-space in $\mathbb{R}^N$ such that $-e_1 \in H$ and $se_1 \in \partial H$, and let σ_H be the reflection with respect to ∂H.
- For $0 \leq u : \mathbb{R}^d \to \mathbb{R}$, the polarization $P_H(u) : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$P_H(u)(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\}, \text{ for } x \in H, \\ \min\{u(x), u(\sigma_H(x))\}, \text{ for } x \in H^c. \end{cases} \tag{11}$$

For $0 \leq u : \Omega_s \to \mathbb{R}$ let $\widetilde{u}$ be the zero extension of u to $\mathbb{R}^d$. The polarization of u is defined as $P_H(u) := P_H(\widetilde{u})|_{\Omega_s}$.

- If $u \in H^1(\Omega_s)$ with $u|_{\partial B_r} = 0$, then $P_H(u) \in H^1(\Omega_s)$ with $P_H(u)|_{\partial B_r} = 0$ and

$$\|u\|_{2,\Omega} = \|P_H(u)\|_{2,\Omega_s} \text{ and } \|\nabla u\|_{2,\Omega} = \|\nabla P_H(u)\|_{2,\Omega_s}.$$

- The function u has foliated Schwarz symmetry with respect to $-\mathbb{R}e_1$ in Ω_s iff $P_H(u) = u$ for any affine half-space H with $-e_1 \in H$ and $se_1 \in \partial H$.
- Take $0 < u_s$ to be the torsion function. By the uniqueness,

$$P_H(u_s) = u_s \text{ in } \Omega_s, \quad \forall H.$$

- The foliated Schwarz symmetry of u_s gives that: For any $x \in \overline{\Omega_s}$ and $\eta \in S^{d-1}$ with $\eta \cdot x = 0$,

$$\nabla u_s(x) \cdot \eta > 0 \text{ iff } \eta_1 < 0.$$

- Neumann condition $0 = R\dfrac{\partial u_s}{\partial n} = \nabla u(x) \cdot x$ for $x \in \partial B_R(0)$. Since $|\nabla u_s| \neq 0$, we get $\dfrac{\partial u_s}{\partial x_1} < 0$ on $\partial B_R(0) \setminus \{\pm Re_1\}$.

Monotonicity in the affine-radial directions

- For $x \in \mathbb{R}$, the vector $x - se_1$ is called as the affine-radial direction from se_1.
- The map $\varphi(x) := \nabla u_s(x) \cdot (x - se_1)$ for $x \in \Omega_s$ satisfies:

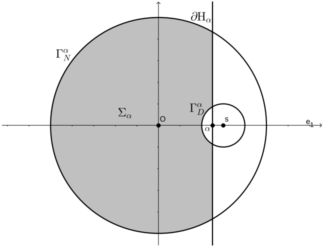
$$-\Delta\varphi = 2u_s > 0 \text{ in } \Omega_s. \tag{12}$$

- On $\partial B_R(0)$: $\varphi(x) = R\nabla u_s(x) \cdot \frac{x}{R} - s\frac{\partial u_s}{\partial x_1} \geq 0.$
- For $\partial B_r(se_1)$: The vector $x - se_1$ is an inward direction to Ω_s and $u_s(x) = 0$. Hopf's boundary lemma gives that $\varphi(x) \geq 0$. Therefore

$$\varphi(x) \geq 0 \text{ on } \partial\Omega_s.$$

- By SMP, $\varphi > 0$ in Ω_s.

Monotonicity in the axial direction

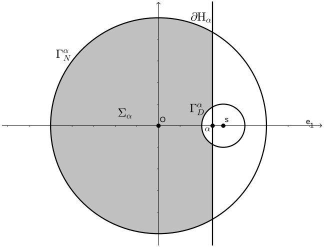


- For $\alpha \in (-R, R)$, the α-cap Σ_α is

$$\Sigma_\alpha := \Omega_s \cap \{x_1 < \alpha\}.$$

- First, by fSS of u_s, observe that: for $\alpha \leq 0$

$$\frac{\partial u_s}{\partial x_1} < 0 \text{ in } \overline{\Sigma_\alpha} \setminus \{-Re_1\}. \qquad (*)$$



- $(-R, 0] \subset \mathcal{A} := \{\alpha \in (-R, R) : (*) \text{ holds } \}$.
- $$\frac{\partial u_s}{\partial x_1} < 0 \text{ on } \partial B_r \cap \{x_1 < s\}, \text{ and } \frac{\partial u_s}{\partial x_1} \geq 0 \text{ on } \partial B_r \cap \{x_1 \geq s\}$$
- Smoothness of u_s, and compactness of $\Omega_s \cap \{x_1 = \alpha\}$: for any $\alpha \in \mathcal{A}$ with $\alpha < s$ there exists $\epsilon > 0$ such that $(*)$ holds for $\alpha + \epsilon$. Therefore

$$\sup \mathcal{A} = s.$$


Thank You!