

Subordination principle and Feynman-Kac formulae for generalized time-fractional evolution equations

Yana Kinderknecht (Butko)

U N I K A S S E L
V E R S I T Ä T

MPDSIDA 2023

Classical Setting:

$(X_t)_{t \geq 0}$ is a time-homogeneous Markov process with state space $(Q, \mathcal{B}(Q))$ ($\Rightarrow$ no memory) $\Rightarrow$ transition kernel $P(t, x, dy) = \mathbb{P}(X_t \in dy \mid X_0 = x)$. Hence:

$$\bullet \quad (T_t)_{t \geq 0}, \quad T_t u_0(x) := \int_Q u_0(y) P(t, x, dy) \equiv \mathbb{E}^x [u_0(X_t)]$$

is an operator semigroup (i.e. $T_0 = \text{Id}$, $T_t \circ T_s = T_{t+s}$).

• If $(T_t)_{t \geq 0}$ is a C_0 -SG with generator $(L, \text{Dom}(L))$ on a BS $\mathbf{X} \subset F(Q)$ then

$$u(t, x) := T_t u_0(x) \equiv \mathbb{E}^x [u_0(X_t)]$$

solves the Cauchy problem

$$\frac{\partial u(t, x)}{\partial t} = L u(t, x), \quad u(0, x) = u_0(x).$$

And

$$u(t, x) = u_0(x) + \int_0^t L u(s, x) ds.$$

$(X_t)_{t \geq 0}$ is a time-homogeneous Markov process with state space $(Q, \mathcal{B}(Q))$ ($\Rightarrow$ no memory) $\Rightarrow$ transition kernel $P(t, x, dy) = \mathbb{P}(X_t \in dy \mid X_0 = x)$. Hence:

- $(T_t)_{t \geq 0}, \quad T_t u_0(x) := \int_Q u_0(y) P(t, x, dy) \equiv \mathbb{E}^x [u_0(X_t)]$

is an operator semigroup (i.e. $T_0 = \text{Id}$, $T_t \circ T_s = T_{t+s}$).

- If $(T_t)_{t \geq 0}$ is a C_0 -SG with generator $(L, \text{Dom}(L))$ on a BS $\mathbf{X} \subset F(Q)$ then

$$u(t, x) := T_t u_0(x) \equiv \mathbb{E}^x [u_0(X_t)]$$

solves the Cauchy problem

$$\frac{\partial u(t, x)}{\partial t} = L u(t, x), \quad u(0, x) = u_0(x).$$

And

$$u(t, x) = u_0(x) + \int_0^t L u(s, x) ds.$$

Basic pair: Heat equation $\leftrightarrow$ Brownian motion.

$$u(t, x) = \mathbb{E}^x [u_0(B_t)] \quad \text{solves} \quad \frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \Delta u(t, x)$$

Classical Setting

Basic couple: Heat equation $\leftrightarrow$ Brownian motion.

$$u(t, x) = \mathbb{E}^x[u_0(B_t)] \quad \text{solves} \quad \frac{\partial u(t, x)}{\partial t} = \Delta u(t, x)$$

$\Rightarrow$ **classical diffusion**

Basic features of classical diffusion:

- $\text{Var}(B_t) = t \quad \leftrightarrow \quad \langle (\Delta x)^2 \rangle \sim \Delta t$
- Gaussian law

$$\langle (\Delta x)^2 \rangle \propto \Delta t,$$

or / and:

NO Gaussian law

$$\langle (\Delta x)^2 \rangle \sim \Delta t, \quad \text{or / and:} \quad \text{NO Gaussian law}$$

Anomalous diffusion has been observed in:

• Subdiffusion:

- many complex self-organized systems in biology (e.g., motion of macromolecules in the cell cytoplasm and membrane);
- charge carrier transport in amorphous semiconductors;
- nuclear magnetic resonance (NMR) diffusometry in percolative and porous systems;
- Rouse or reptation dynamics in polymeric systems;
- transport on fractal geometries;
- diffusion of a scalar tracer in an array of convection rolls;
- dynamics of a bead in a polymeric network;
- Sinai-type disorder;
- ageing models....

• Superdiffusion (incl. Lévy flights):

- Richardson's turbulent diffusion;
- transport in turbulent plasma;
- special domains of rotating flows;
- quantum optics;
- single molecule spectroscopy;
- diffusion with search mechanisms (e.g., in cells)....

Heat equation $\rightsquigarrow$ time- and space-fractional heat equation:

$$\partial_t^\beta u(t, x) = - \left(-\frac{1}{2} \Delta \right)^\gamma u(t, x), \quad \beta \in (0, 1], \quad \gamma \in (0, 1],$$

i.e.

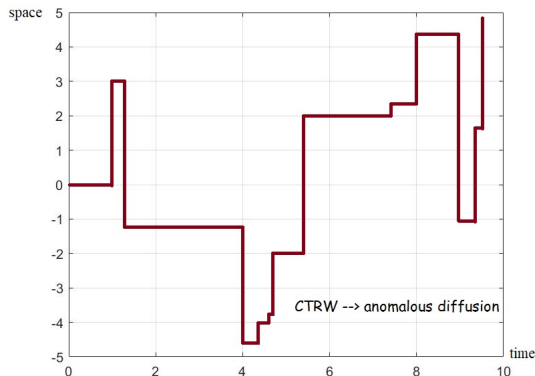
$$u(t, x) = u_0(x) - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left(-\frac{1}{2} \Delta \right)^\gamma u(s, x) ds.$$

Anomalous diffusion

Heat equation $\leadsto$ time- and space-fractional heat equation:

$$u(t, x) = u_0(x) - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left(-\frac{1}{2}\Delta\right)^\gamma u(s, x) ds, \quad \beta \in (0, 1], \gamma \in (0, 1].$$

This equation is a governing equation for scaling limits of CTRWs!

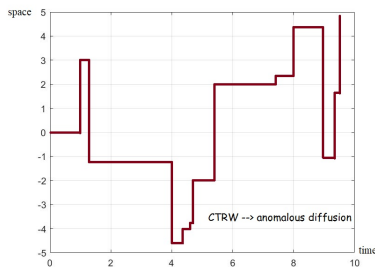


Anomalous diffusion

Heat equation $\rightsquigarrow$ time- and space-fractional heat equation:

$$u(t, x) = u_0(x) - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left(-\frac{1}{2}\Delta\right)^\gamma u(s, x) dx, \quad \beta \in (0, 1], \gamma \in (0, 1].$$

This equation is a governing equation for scaling limits of CTRWs!



The corresponding scaling limits of CTRWs are processes $(\xi_{\mathcal{E}_t})_{t \geq 0}$

Stochastic representations via inverse subordinators

Inverse subordinators are actively used to produce stochastic representations of solutions of time-fractional evolution equations of the following form

$$u(t, x) = u_0(x) + \int_0^t k(t-s) Lu(s, x) ds,$$

where $(L, \text{Dom}(L))$ is the generator of a Markov process $(\xi_t)_{t \geq 0}$, and the convolution kernel k has Laplace transform $\frac{1}{h(\cdot)}$ for some BF h . This BF h serves as the Laplace exponent of some Lévy subordinator, whose inverse $(\mathcal{E}_t^h)_{t \geq 0}$ provides stochastic solution

$$u(t, x) = \mathbb{E}^x \left[u_0 \left(\xi_{\mathcal{E}_t^h} \right) \right].$$

NOTE: $(\mathcal{E}_t^h)_{t \geq 0}$ and hence $(\xi_{\mathcal{E}_t^h})_{t \geq 0}$ are NOT Markov!

Stochastic representations via RSGP

Another approach: **randomly scaled Gaussian processes (RSGP)**.

Stochastic representations via RSGP

Another approach: **randomly scaled Gaussian processes (RSGP)**.

$$X_t := AG_t,$$

where $A \geq 0$, $A \perp G$, G is a Gaussian process.

Stochastic representations via RSGP

Another approach: **randomly scaled Gaussian processes (RSGP)**.

$$X_t := AG_t,$$

where $A \geq 0$, $A \perp G$, G is a Gaussian process.

Example: Generalized Grey Brownian Motion (GGBM)

$$X_t^{\alpha,\beta} := \sqrt{A_\beta} B_t^{\alpha/2},$$

where $(B_t^{\alpha/2})_{t \geq 0}$ is a fractional Brownian motion with $H := \alpha/2$,
 $A_\beta \perp B^{\alpha/2}$, $A_\beta \geq 0$ and $\mathbb{E}[\exp(-\lambda A_\beta)] = E_\beta(-\lambda) := \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\beta n + 1)}$.

Stochastic representations via RSGP

Another approach: **randomly scaled Gaussian processes (RSGP)**.

$$X_t := AG_t,$$

where $A \geq 0$, $A \perp G$, G is a Gaussian process.

Example: Generalized Grey Brownian Motion (GGBM)

$$X_t^{\alpha,\beta} := \sqrt{Y_\beta} B_t^{\alpha/2},$$

where $(B_t^{\alpha/2})_{t \geq 0}$ is a fractional Brownian motion with $H := \alpha/2$,
 $A_\beta \perp B^{\alpha/2}$, $A_\beta \geq 0$ and $\mathbb{E}[\exp(-\lambda A_\beta)] = E_\beta(-\lambda) := \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\beta n + 1)}$.

- $(X_t^{\alpha,\beta})_{t \geq 0}$ is $\alpha/2$ -SSSI

Stochastic representations via RSGP

Another approach: **randomly scaled Gaussian processes (RSGP)**.

$$X_t := AG_t,$$

where $A \geq 0$, $A \perp G$, G is a Gaussian process.

Example: Generalized Grey Brownian Motion (GGBM)

$$X_t^{\alpha,\beta} := \sqrt{Y_\beta} B_t^{\alpha/2},$$

where $(B_t^{\alpha/2})_{t \geq 0}$ is a fractional Brownian motion with $H := \alpha/2$, $A_\beta \perp B^{\alpha/2}$, $A_\beta \geq 0$ and $\mathbb{E}[\exp(-\lambda A_\beta)] = E_\beta(-\lambda) := \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\beta n + 1)}$.

- $(X_t^{\alpha,\beta})_{t \geq 0}$ is $\alpha/2$ -SSSI
- $u(t, x) := \mathbb{E}\left[u_0\left(x + X_t^{\alpha,\beta}\right)\right]$ solves

$$u(t, x) = u_0(x) + \frac{\alpha}{\beta \Gamma(\beta)} \int_0^t s^{\frac{\alpha}{\beta}-1} \left(t^{\frac{\alpha}{\beta}} - s^{\frac{\alpha}{\beta}}\right)^{\beta-1} \frac{1}{2} \frac{\partial^2 u(s, x)}{\partial x^2} ds,$$

Stochastic representations via RSGP

Another approach: **randomly scaled Gaussian processes (RSGP)**.

$$X_t := AG_t,$$

where $A \geq 0$, $A \perp G$, G is a Gaussian process.

Example: Generalized Grey Brownian Motion (GGBM)

$$X_t^{\alpha,\beta} := \sqrt{Y_\beta} B_t^{\alpha/2},$$

where $(B_t^{\alpha/2})_{t \geq 0}$ is a fractional Brownian motion with $H := \alpha/2$,
 $A_\beta \perp B^{\alpha/2}$, $A_\beta \geq 0$ and $\mathbb{E}[\exp(-\lambda A_\beta)] = E_\beta(-\lambda) := \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\beta n + 1)}$.

- $u(t, x) := \mathbb{E}\left[u_0\left(x + X_t^{\alpha,\beta}\right)\right]$ solves

$$u(t, x) = u_0(x) + \frac{\alpha}{\beta \Gamma(\beta)} \int_0^t s^{\frac{\alpha}{\beta}-1} \left(t^{\frac{\alpha}{\beta}} - s^{\frac{\alpha}{\beta}}\right)^{\beta-1} \frac{1}{2} \frac{\partial^2 u(s, x)}{\partial x^2} ds,$$

- Pagnini-Paradisi 2016: Yet another RSGP solves

$$\partial_t^\beta u(t, x) = -(-\Delta)^\gamma u(t, x), \quad \beta \in (0, 1], \quad \gamma \in (0, 1].$$

Natural questions:

We consider evolution equations of the form

$$u(t, x) = u_0(x) + \int_0^t k(t, s) Lu(s, x) ds, \quad (*)$$

where L generates a Markov process with a C_0 -SG on some BS $\mathbf{X}$.

Questions:

- 1 Which classes of stochastic processes $(X_t)_{t \geq 0}$ can be used to solve the evolution equation $(*)$?
- 2 What is in common between $X_t := \xi_{\mathcal{E}_t}$ and $X_t := AG_t$?
- 3 To which extent can we use the RSGPs?

 Ch. Bender, M. Bormann, Ya.A. Butko (2022)

Subordination principle and Feynman-Kac formulae for generalized time-fractional evolution equations

Fract. Calc. Appl. Anal. 25 N4, 1818-1836 (2022).

 Ch. Bender, Ya.A. Butko (2022)

Stochastic solutions of generalized time-fractional evolution equations

Frac. Calc. Appl. Anal. 25(2), 2022.

 Marie Bormann (2021)

Feynman-Kac-Formeln für Lösungen einer Klasse von Integro-Differentialgleichungen

Masterarbeit, Institut für Mathematische Stochastik, TU Braunschweig

 Merten Mlinarzik (2023)

Stochastic calculus for randomly scaled Gaussian processes related to generalized time-fractional evolution equations

Masterarbeit, Institut für Mathematische Stochastik, TU Braunschweig

Generalized time-fractional evolution equations

Generalized time-fractional evolution equations

We consider

$$u(t) = u_0 + \int_0^t k(t,s)Lu(s)ds, \quad (*)$$

where $(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$,

Generalized time-fractional evolution equations

We consider

$$u(t) = u_0 + \int_0^t k(t,s) Lu(s) ds, \quad (*)$$

where $(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$, kernel k satisfies

Assumption 1: We consider a Borel-measurable kernel $k : (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ satisfying the following condition: $\exists \alpha^* \in [0, 1)$ and $\exists \varepsilon > 0$ such that for each $T > 0$

$$K_T := \sup_{0 < t \leq T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \|k(t, \cdot)\|_{L^{1+\varepsilon}((0,t))} < \infty.$$

Generalized time-fractional evolution equations

We consider

$$u(t) = u_0 + \int_0^t k(t, s) Lu(s) ds, \quad (*)$$

where $(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$, kernel k satisfies

Assumption 1: We consider a Borel-measurable kernel

$k : (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ satisfying the following condition: $\exists \alpha^* \in [0, 1)$ and $\exists \varepsilon > 0$ such that for each $T > 0$

$$K_T := \sup_{0 < t \leq T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \|k(t, \cdot)\|_{L^{1+\varepsilon}((0, t))} < \infty.$$

Question: Which processes $(X_t)_{t \geq 0}$ solve $(*)$?

We consider

$$u(t) = u_0 + \int_0^t k(t, s) Lu(s) ds, \quad (*)$$

where $(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$, kernel k satisfies

Assumption 1: $\exists \alpha^* \in [0, 1), \exists \varepsilon > 0$ such that $\forall T > 0$

$$K_T := \sup_{0 < t \leq T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \|k(t, \cdot)\|_{L^{1+\varepsilon}((0, t))} < \infty.$$

Thm. 1: Under Assm. 1 $\forall t \geq 0$, the function $\Phi(t, \cdot) : \mathbb{C} \rightarrow \mathbb{C}$ given by

$$\Phi(t, \lambda) := \sum_{n=0}^{\infty} c_n(t) \lambda^n,$$

$$c_0(t) := 1, \quad c_n(t) := \begin{cases} \int_0^t k(t, s) c_{n-1}(s) ds, & \forall t > 0, \\ 0 & t = 0, \end{cases} \quad n \in \mathbb{N},$$

is well-defined and entire.

We consider

$$u(t) = u_0 + \int_0^t k(t,s) Lu(s) ds, \quad (*)$$

where $(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$, kernel k satisfies

Assumption 1: $\exists \alpha^* \in [0, 1), \exists \varepsilon > 0$ such that $\forall T > 0$

$$K_T := \sup_{0 < t \leq T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \|k(t, \cdot)\|_{L^{1+\varepsilon}((0,t))} < \infty.$$

Thm. 1: $k \rightsquigarrow \Phi(t, \lambda) := \sum_{n=0}^{\infty} c_n(t) \lambda^n$, $c_n(t) := \int_0^t k(t,s) c_{n-1}(s) ds$.

We consider

$$u(t) = u_0 + \int_0^t k(t, s) Lu(s) ds, \quad (*)$$

where $(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$, kernel k satisfies

Assumption 1: $\exists \alpha^* \in [0, 1), \exists \varepsilon > 0$ such that $\forall T > 0$

$$K_T := \sup_{0 < t \leq T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \|k(t, \cdot)\|_{L^{1+\varepsilon}((0, t))} < \infty.$$

Thm. 1: $k \rightsquigarrow \Phi(t, \lambda) := \sum_{n=0}^{\infty} c_n(t) \lambda^n$, $c_n(t) := \int_0^t k(t, s) c_{n-1}(s) ds$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geq 0}$:

$$\int_0^{\infty} e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \quad \forall \lambda \in \mathbb{C}, \quad \text{Re } \lambda \geq 0.$$

$$u(t) = u_0 + \int_0^t k(t,s) Lu(s) ds, \quad (*)$$

$(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$.

Assumption 1: $\exists \alpha^* \in [0, 1)$, $\exists \varepsilon > 0$ such that $\forall T > 0$

$$K_T := \sup_{0 < t \leq T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \|k(t, \cdot)\|_{L^{1+\varepsilon}((0,t))} < \infty.$$

Thm. 1: $k \rightsquigarrow \Phi(t, \lambda) := \sum_{n=0}^{\infty} c_n(t) \lambda^n$, $c_n(t) := \int_0^t k(t,s) c_{n-1}(s) ds$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geq 0}$:

$$\int_0^{\infty} e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \quad \forall \lambda \in \mathbb{C}, \quad \text{Re } \lambda \geq 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:

(i) For each $t \geq 0$, the operator $\Phi(t, L)$ given by the Bochner integral

$$\Phi(t, L)\varphi := \int_0^{\infty} T_a \varphi \mathcal{P}_{A(t)}(da), \quad \varphi \in X, \quad (1)$$

is well defined and it is a bounded linear operator on X .

$$u(t) = u_0 + \int_0^t k(t, s) L u(s) ds, \quad (*)$$

$(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$.

Assumption 1: $K_T := \sup_{0 < t \leq T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \|k(t, \cdot)\|_{L^{1+\varepsilon}((0, t))} < \infty$.

Thm. 1: $k \rightsquigarrow \Phi(t, \lambda) := \sum_{n=0}^{\infty} c_n(t) \lambda^n$, $c_n(t) := \int_0^t k(t, s) c_{n-1}(s) ds$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geq 0}$:

$$\int_0^{\infty} e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \quad \forall \lambda \in \mathbb{C}, \quad \text{Re } \lambda \geq 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:
(ii) For each $t > 0$ and each $u_0 \in \text{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^{\infty} T_a u_0 \mathcal{P}_{A(t)}(da)$$

solves equation $(*)$ and it holds $\lim_{t \searrow 0} u(t) = u_0$.

$$u(t) = u_0 + \int_0^t k(t, s) L u(s) ds, \quad (*)$$

$(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$.

Thm. 1: $k \rightsquigarrow \Phi(t, \lambda) := \sum_{n=0}^{\infty} c_n(t) \lambda^n$, $c_n(t) := \int_0^t k(t, s) c_{n-1}(s) ds$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geq 0}$:

$$\mathbb{E} \left[e^{-\lambda A(t)} \right] \equiv \int_0^{\infty} e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \quad \forall \lambda \in \mathbb{C}, \quad \text{Re } \lambda \geq 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:

(ii) For each $t > 0$ and each $u_0 \in \text{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^{\infty} T_a u_0 \mathcal{P}_{A(t)}(da) \equiv \mathbb{E} \left[T_{A(t)} u_0 \right]$$

solves equation $(*)$ and it holds $\lim_{t \searrow 0} u(t) = u_0$.

Rem.: If T_t has stoch. repr. then $\Phi(t, L)$ too!

- $L := L_0 \leftrightarrow \xi = \text{Markov process}, \xi \perp (A(t))_{t \geq 0}$:

$$T_t u_0(x) = \mathbb{E}^x [u_0(\xi_t)] \quad \Rightarrow \quad \Phi(t, L) u_0(x) = \mathbb{E}^x [u_0(\xi_{A(t)})]$$

$$u(t) = u_0 + \int_0^t k(t, s) L u(s) ds, \quad (*)$$

$(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geq 0}$:

$$\mathbb{E} \left[e^{-\lambda A(t)} \right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \quad \forall \lambda \in \mathbb{C}, \quad \text{Re } \lambda \geq 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:

(ii) For each $t > 0$ and each $u_0 \in \text{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0 \mathcal{P}_{A(t)}(da) \equiv \mathbb{E} \left[T_{A(t)} u_0 \right]$$

solves equation $(*)$ and it holds $\lim_{t \searrow 0} u(t) = u_0$.

Rem.: If T_t has stoch. repr. then $\Phi(t, L)$ too!

- $L_0 \leftrightarrow \xi =$ Markov process, $\xi \perp (A(t))_{t \geq 0}$,
- $L := L_0 + V$, then $T_t u_0(x) = \mathbb{E}^x \left[u_0(\xi_t) e^{\int_0^t V(\xi_s) ds} \right],$

$$\Phi(t, L) u_0(x) = \mathbb{E}^x \left[u_0(\xi_{A(t)}) e^{\int_0^{A(t)} V(\xi_s) ds} \right]$$

$$u(t) = u_0 + \int_0^t k(t, s) L u(s) ds, \quad (*)$$

$(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geq 0}$:

$$\mathbb{E} \left[e^{-\lambda A(t)} \right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \quad \forall \lambda \in \mathbb{C}, \quad \text{Re } \lambda \geq 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:

(ii) For each $t > 0$ and each $u_0 \in \text{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0 \mathcal{P}_{A(t)}(da) \equiv \mathbb{E} \left[T_{A(t)} u_0 \right]$$

solves equation $(*)$ and it holds $\lim_{t \searrow 0} u(t) = u_0$.

Rem.: If T_t has stoch. repr. then $\Phi(t, L)$ too!

- $L_0 \leftrightarrow \xi =$ Markov process, $\xi \perp (A(t))_{t \geq 0}$,
- $L := L_0 + V$, then $T_t u_0(x) = \mathbb{E}^x \left[u_0(\xi_t) e^{\int_0^t V(\xi_s) ds} \right]$,

$$\Phi(t, L) u_0(x) = \mathbb{E}^x \left[u_0(\xi_{A(t)}) e^{\int_0^{A(t)} V(\xi_s) ds} \right]$$

- $L :=$ generator of a subordinate semigroup; Schrödinger group.....

$$u(t) = u_0 + \int_0^t k(t, s) L u(s) ds, \quad (*)$$

$(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geq 0}$:

$$\mathbb{E} \left[e^{-\lambda A(t)} \right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \quad \forall \lambda \in \mathbb{C}, \quad \text{Re } \lambda \geq 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:

(ii) For each $t > 0$ and each $u_0 \in \text{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0 \mathcal{P}_{A(t)}(da) \equiv \mathbb{E} \left[T_{A(t)} u_0 \right]$$

solves equation $(*)$ and it holds $\lim_{t \searrow 0} u(t) = u_0$.

Thm. 3: Under Assm. 1, Assm. 2 let k be **homogeneous** of $\theta - 1$ for some $\theta > 0$, i.e.

$$k(t, ts) = t^{\theta-1} k(1, s) \quad \forall \quad t > 0, s \in (0, 1).$$

Then one can choose $A(t) := At^\theta$, where A is a nonnegative RV such that

$$\int_0^\infty e^{-\lambda a} \mathcal{P}_A(da) = \Phi(1, -\lambda) \quad \forall \lambda \in \mathbb{C}, \quad \text{Re } \lambda \geq 0.$$

$$u(t) = u_0 + \int_0^t k(t, s) L u(s) ds, \quad (*)$$

$(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geq 0}$:

$$\mathbb{E} \left[e^{-\lambda A(t)} \right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \quad \forall \lambda \in \mathbb{C}, \quad \text{Re } \lambda \geq 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:

(ii) For each $t > 0$ and each $u_0 \in \text{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0 \mathcal{P}_{A(t)}(da) \equiv \mathbb{E} \left[T_{A(t)} u_0 \right]$$

solves equation $(*)$ and it holds $\lim_{t \searrow 0} u(t) = u_0$.

Thm. 3: $k(t, ts) = t^{\theta-1} k(1, s)$. Then $A(t) := At^\theta$ is OK, $A \leftrightarrow \Phi(1, -\cdot)$.

Example:

- $k(t, s) := \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}$, $\beta \in (0, 1]$. Then $\theta = \beta$; $\Phi(t, -\lambda) = E_\beta(-t^\beta \lambda) = \text{CMF}$

$$u(t) = u_0 + \int_0^t k(t, s) L u(s) ds, \quad (*)$$

$(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geq 0}$:

$$\mathbb{E} \left[e^{-\lambda A(t)} \right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \quad \forall \lambda \in \mathbb{C}, \quad \text{Re } \lambda \geq 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:

(ii) For each $t > 0$ and each $u_0 \in \text{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0 \mathcal{P}_{A(t)}(da) \equiv \mathbb{E} \left[T_{A(t)} u_0 \right]$$

solves equation $(*)$ and it holds $\lim_{t \searrow 0} u(t) = u_0$.

Thm. 3: $k(t, ts) = t^{\theta-1} k(1, s)$. Then $A(t) := At^\theta$ is OK, $A \leftrightarrow \Phi(1, -\cdot)$.

Example: Time-fractional heat equation:

- $k(t, s) := \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}$, $\beta \in (0, 1]$. Then $\theta = \beta$; $\Phi(t, -\lambda) = E_\beta(-t^\beta \lambda) = \text{CMF}$
- $L := \frac{1}{2} \Delta \leftrightarrow (B_t)_{t \geq 0} = \text{Brownian motion}$

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds, \quad (*)$$

$(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geq 0}$:

$$\mathbb{E} \left[e^{-\lambda A(t)} \right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \quad \forall \lambda \in \mathbb{C}, \quad \text{Re } \lambda \geq 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:

(ii) For each $t > 0$ and each $u_0 \in \text{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0 \mathcal{P}_{A(t)}(da) \equiv \mathbb{E} [T_{A(t)} u_0]$$

solves equation $(*)$ and it holds $\lim_{t \searrow 0} u(t) = u_0$.

Thm. 3: $k(t, ts) = t^{\theta-1} k(1, s)$. Then $A(t) := At^\theta$ is OK, $A \leftrightarrow \Phi(1, -\cdot)$.

Example: Time-fractional heat equation. Solution: $u(t, x) = \mathbb{E}^x [u_0(X_t)]$

- $k(t, s) := \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}$, $\beta \in (0, 1]$. Then $\theta = \beta$; $\Phi(t, -\lambda) = E_\beta(-t^\beta \lambda) = \text{CMF}$
- $L := \frac{1}{2} \Delta \leftrightarrow (B_t)_{t \geq 0} = \text{Brownian motion}$

$$X_t : \quad B_{A(t)} \quad \leftrightarrow \quad B_{At^\beta} \quad \leftrightarrow \quad \sqrt{A} B_{t^\beta} \quad \leftrightarrow \quad \sqrt{A} B_t^{\beta/2}$$

$$u(t) = u_0 + \int_0^t k(t,s) Lu(s) ds, \quad (*)$$

$(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$.

Cor. 4: Under Assm. 1, Assm. 2. suppose $k(t,s) := \mathfrak{K}(t-s)$, where $\mathcal{L}\mathfrak{K} = 1/h$ for some Bernstein function h . Then

$$h \quad \leftrightarrow \quad (\eta_t^h)_{t \geq 0} \quad \leftrightarrow \quad (\mathcal{E}_t^h)_{t \geq 0},$$

where $\mathbb{E} \left[e^{-\lambda \eta_t^h} \right] = e^{-th(\lambda)}$ and $\mathcal{E}_t^h := \inf \{ s > 0 : \eta_s^h > t \}$. Hence, one may take $A(t) := \mathcal{E}_t^h$.

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds, \quad (*)$$

$(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$.

Cor. 4: Under Assm. 1, Assm. 2. suppose $k(t,s) := \mathfrak{K}(t-s)$, where $\mathcal{L}\mathfrak{K} = 1/h$ for some Bernstein function h . Then

$$h \quad \Leftrightarrow \quad (\eta_t^h)_{t \geq 0} \quad \Leftrightarrow \quad (\mathcal{E}_t^h)_{t \geq 0},$$

where $\mathbb{E} \left[e^{-\lambda \eta_t^h} \right] = e^{-th(\lambda)}$ and $\mathcal{E}_t^h := \inf \{ s > 0 : \eta_s^h > t \}$. Hence, one may take $A(t) := \mathcal{E}_t^h$.

Example: Time-fractional heat equation. Solution: $u(t,x) = \mathbb{E}^x [u_0(X_t)]$

- $k(t,s) := \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}$. Then $h(\lambda) = \lambda^\beta$.
- $L := \frac{1}{2} \Delta \Leftrightarrow (B_t)_{t \geq 0} = \text{Brownian motion}$

$$X_t : \quad B_{\mathcal{E}_t^h} \quad \Leftrightarrow \quad B_{A(t)} \quad \Leftrightarrow \quad B_{At^\beta} \quad \Leftrightarrow \quad \sqrt{A} B_{t^\beta} \quad \Leftrightarrow \quad \sqrt{A} B_t^{\beta/2}$$

$$u(t) = u_0 + \int_0^t k(t,s) Lu(s) ds, \quad (*)$$

$(L, \text{Dom}(L))$ generates a C_0 -SG $(T_t)_{t \geq 0}$ on a BS X , $u_0 \in \text{Dom}(L)$.

Cor. 4: Under Assm. 1, Assm. 2. suppose $k(t,s) := \mathfrak{K}(t-s)$, where $\mathcal{L}\mathfrak{K} = 1/h$ for some Bernstein function h . Then one may take $A(t) := \mathcal{E}_t^h$.

Rem.: In the situation of Cor. 4, eq. $(*)$ is equivalent to Cauchy problem

$$\mathcal{D}_t^h u(t, x) = Lu(t, x), \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}^d, \quad t > 0,$$

where $\mathcal{D}_t^h$ is a generalized time-fractional derivative of Caputo type:

$$(\mathcal{L}[\mathcal{D}_t^h v])(\sigma) = h(\sigma)(\mathcal{L}v)(\sigma) - \frac{h(\sigma)}{\sigma} v(+0).$$

Special cases:

- Caputo derivative of order β : $h(\lambda) = \lambda^\beta$;
- a mixture of Caputo derivatives of orders $\beta, \beta_1, \dots, \beta_m$:

$$h_1(\lambda) := \left(\lambda^{-\beta} + \sum_{j=1}^m b_j \lambda^{-\beta_j} \right)^{-1};$$
- distributed order derivative: $h(\lambda) := \int_0^1 \lambda^\beta \mu(d\beta)$.

Example 2 (Marichev-Saigo-Maeda kernels):

Example 2 (MSM kernels): Appell's third generalization F_3 of the Gauss hypergeometric function is defined by:

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) = \sum_{m, n \geq 0} \frac{(\alpha)_m (\beta)_m (\alpha')_n (\beta')_n}{(\gamma)_{m+n} n! m!} x^m y^n, \quad (1)$$

where $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$, $\gamma \notin -\mathbb{N}$, and the general Pochhammer symbol $(\lambda)_\nu$ is defined as follows:

$$(\lambda)_\nu := \begin{cases} 1, & \nu = 0, \quad \lambda \in \mathbb{C} \\ \lambda(\lambda-1) \cdot \dots \cdot (\lambda+n-1), & \nu = n \in \mathbb{N}, \quad \lambda \in \mathbb{C}. \end{cases}$$

The series in (1) converges for $|y|, |x| < 1$ and can be analytically extended to reals $x, y < 1$.

Example 2 (MSM kernels): Appell's third generalization F_3 of the Gauss hypergeometric function is defined by:

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) = \sum_{m, n \geq 0} \frac{(\alpha)_m (\beta)_m (\alpha')_n (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n, \quad (1)$$

where $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$, $\gamma \notin -\mathbb{N}$, and the general Pochhammer symbol $(\lambda)_\nu$ is defined as follows:

$$(\lambda)_\nu := \begin{cases} 1, & \nu = 0, \quad \lambda \in \mathbb{C} \\ \lambda(\lambda-1) \cdots (\lambda+n-1), & \nu = n \in \mathbb{N}, \quad \lambda \in \mathbb{C}. \end{cases}$$

The series in (1) converges for $|y|, |x| < 1$ and can be analytically extended to reals $x, y < 1$.

For $b > 0$, $a > 0$, $\mu > -1$, and $\nu > \max\{-b, -a\mu\}$ we consider the **MSM kernel**

$$k(t, s) := \frac{a}{\Gamma(b/a)} (t^a - s^a)^{\frac{b}{a}-1} t^{a-\nu} s^{\nu-1} F_3\left(\frac{\nu}{a} - 1, \frac{b}{a}, 1, \mu, \frac{b}{a}, 1 - \left(\frac{s}{t}\right)^a, 1 - \left(\frac{t}{s}\right)^a\right)$$

Thm 5: Let $b > 0$, $a > 0$, $\mu > -1$, and $\nu > \max\{-b, -a\mu\}$. Consider the kernel

$$k(t, s) := \frac{a}{\Gamma(b/a)} (t^a - s^a)^{\frac{b}{a}-1} t^{a-\nu} s^{\nu-1} F_3\left(\frac{\nu}{a} - 1, \frac{b}{a}, 1, \mu, \frac{b}{a}, 1 - \left(\frac{s}{t}\right)^a, 1 - \left(\frac{t}{s}\right)^a\right)$$

where $0 < s < t$. Then the kernel k is homogeneous of degree $b - 1$ and satisfies $k(1, \cdot) \in L^{1+\varepsilon}((0, 1))$ for some $\varepsilon > 0$. The corresponding Φ is

$$\Phi(1, z) = \Gamma(\lambda_2) E_{\lambda_1, \lambda_2}^{\lambda_3}(z),$$

where $\lambda_1 = \frac{b}{a}$, $\lambda_2 = \frac{\nu}{a} + \mu$, $\lambda_3 = 1 + \frac{\nu-a}{b}$, and $E_{\lambda_1, \lambda_2}^{\lambda_3}$ is the three parameter Mittag-Leffler (or Prabhakar) function

$$E_{\lambda_1, \lambda_2}^{\lambda_3}(z) := \sum_{n=0}^{\infty} \frac{(\lambda_3)_n}{\Gamma(\lambda_1 n + \lambda_2) n!} z^n.$$

Thm 5: Let $b \in (0, 1]$, $a > 0$, $\mu > -1$, and $\nu > \max\{-b, -a\mu\}$. Consider the kernel

$$k(t, s) := \frac{a}{\Gamma(b/a)} (t^a - s^a)^{\frac{b}{a}-1} t^{a-\nu} s^{\nu-1} F_3\left(\frac{\nu}{a} - 1, \frac{b}{a}, 1, \mu, \frac{b}{a}, 1 - \left(\frac{s}{t}\right)^a, 1 - \left(\frac{t}{s}\right)^a\right)$$

where $0 < s < t$. Then the kernel k is homogeneous of degree $b - 1$ and satisfies $k(1, \cdot) \in L^{1+\varepsilon}((0, 1))$ for some $\varepsilon > 0$. The corresponding Φ is:

$$\Phi(1, z) = \Gamma(\lambda_2) E_{\lambda_1, \lambda_2}^{\lambda_3}(z),$$

where $\lambda_1 = \frac{b}{a}$, $\lambda_2 = \frac{\nu}{a} + \mu$, $\lambda_3 = 1 + \frac{\nu-a}{b}$, and $E_{\lambda_1, \lambda_2}^{\lambda_3}$ is the three parameter Mittag-Leffler (or Prabhakar) function

$$E_{\lambda_1, \lambda_2}^{\lambda_3}(z) := \sum_{n=0}^{\infty} \frac{(\lambda_3)_n}{\Gamma(\lambda_1 n + \lambda_2) n!} z^n.$$

Sufficient conditions for $z \mapsto \Gamma(\lambda_2) E_{\lambda_1, \lambda_2}^{\lambda_3}(-z)$ to be CM are $0 < \lambda_1 \leq 1$, $0 < \lambda_3 \leq \frac{\lambda_2}{\lambda_1}$. This implies additional assumptions on parameters a, b, μ, ν :

$$b \leq a, \quad \nu > a - b, \quad \mu \geq \frac{b}{a} - 1.$$

Special cases:

(i) Let $\lambda_3 = 1$, i.e. $\nu = a$. Then $E_{\lambda_1, \lambda_2}^{\lambda_3}$ reduces to the two parameter Mittag-Leffler function E_{λ_1, λ_2} and we have

$$\Phi(t, z) = \Gamma(\mu + 1) E_{\frac{b}{a}, \mu+1}(zt^b).$$

Further, the corresponding kernel k simplifies to

$$k(t, s) = \frac{a}{\Gamma(b/a)} (t^a - s^a)^{b/a-1} t^{-a\mu} s^{a(\mu+1)-1}.$$

Special cases:

(i) Let $\lambda_3 = 1$, i.e. $\nu = a$. Then $E_{\lambda_1, \lambda_2}^{\lambda_3}$ reduces to the two parameter Mittag-Leffler function E_{λ_1, λ_2} and we have

$$\Phi(t, z) = \Gamma(\mu + 1) E_{\frac{b}{a}, \mu+1}(zt^b).$$

Further, the corresponding kernel k simplifies to

$$k(t, s) = \frac{a}{\Gamma(b/a)} (t^a - s^a)^{b/a-1} t^{-a\mu} s^{a(\mu+1)-1}.$$

(ii) Let $\lambda_2 = \lambda_3 = 1$, i.e. $\nu = a$ and $\mu = 0$. Then $E_{\lambda_1, \lambda_2}^{\lambda_3}$ reduces to the classical Mittag-Leffler function E_{λ_1} and we have

$$\Phi(t, z) = E_{\frac{b}{a}}(zt^b).$$

The corresponding kernel k simplifies to

$$k(t, s) = \frac{a}{\Gamma(b/a)} (t^a - s^a)^{b/a-1} s^{a-1}.$$

Let now $\beta \in (0, 1]$, $\alpha \in (0, 2)$. Choosing $b := \alpha$ and $a := \frac{\alpha}{\beta}$ we obtain the kernel of the governing equation of the GGBM and $\Phi(t, z) = E_{\beta}(zt^{\alpha})$. If additionally $\alpha = \beta$, we get $k(t, s) = \frac{1}{\Gamma(\beta)} (t - s)^{\beta-1}$ and $\Phi(t, z) = E_{\beta}(zt^{\beta})$.

Further Feynman–Kac formulae with RSGPs

Example 3: Let $k : k(t, ts) = t^{\theta-1} k(1, s)$, $\theta > 0$, $\gamma \in (0, 1)$, $c \leq 0$, $w \in \mathbb{R}^d$. Under Assm. 1 and Assm. 2 the function

$$u(t, x) := \mathbb{E} \left[u_0 \left(x + X_t^{\mathcal{A}, \gamma, \theta} + \mathcal{A} w t^{\theta/\gamma} \right) e^{c A t^{\theta/\gamma}} \right]$$

solves the evolution equation

$$u(t, x) = u_0(x) - \int_0^t k(t, s) \left(-\frac{1}{2} \Delta - w \nabla - c \right)^\gamma u(s, x) ds,$$

where

$$\begin{aligned} X_t^{\mathcal{A}, \gamma, \theta} &:= B_{\mathcal{A} t^{\theta/\gamma}} \quad \text{or} \quad X_t^{\mathcal{A}, \gamma, \theta} := \sqrt{\mathcal{A}} B_{t^{\theta/\gamma}} \\ \text{or} \quad X_t^{\mathcal{A}, \gamma, \theta} &:= \sqrt{\mathcal{A}} B_t^H \quad \text{if } H := \frac{\theta}{2\gamma} \in (0, 1), \end{aligned}$$

and

$$\mathcal{A} := A^{1/\gamma} \eta, \quad A \perp \eta, \quad \mathbb{E}[e^{-\lambda \eta}] = e^{-\lambda^\gamma}.$$

Variable diffusion coefficient can be also included (in some special cases)!

Next step: Stochastic calculus with respect to RSGP

Stochastic calculus with respect to RSGP



new processes and new equations!

Example: Consider a kernel $k(t, s)$ as in Assm.1 and Assm.2.

$k \rightsquigarrow X_t := \sqrt{A} B_t^H$, $H \in (0, 1)$ as above. Define $Z_t := \int_0^t \nu(s) \delta X_s$. Then:

$$u(t, x) = \mathbb{E}[u_0(x + Z_t)]$$

solves evolution equation with the time-changed kernel

$$u(t, x) = u_0 + \int_0^t k(\sigma(t), \sigma(s)) \sigma'(s) \frac{1}{2} \Delta u(s, x) ds,$$

where

$$\sigma(t) := \|M_-^H \mathbb{1}_{(0,1)} \nu\|_2^{1/H}.$$

Thanks for your attention!!!

