NONLOCAL PROBLEMS WITH INTEGRAL CONDITION FOR MIXED-TYPE EQUATIONS

ZAITSEVA NATALYA VLADIMIROVNA

Lomonosov Moscow State University

III International Conference
«MATHEMATICAL PHYSICS, DYNAMICAL SYSTEMS, INFINITE-DIMENSIONAL ANALYSIS»
July 10, 2023

Bessel Operator

$$B_x u := x^{-\gamma}\frac{\partial}{\partial x}\left(x^{\gamma}\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\gamma}{x}\frac{\partial u}{\partial x},$$

where $\gamma \neq 0$ is given real number.

I. A. Kipriyanov, *Singular Elliptic Boundary-Value Problems* (Moscow, Nauka, 1997) [in Russian].

B-elliptic, B-parabolic, and B-hyperbolic equations.

B-hyperbolic equation

R. W. Carroll and R. E. Showalter, *Singular and Degenerate Cauchy Problems* (New York, Academic Press, 1976).

B-hyperbolic equations

E. L. Shishkina, General Euler–Poisson–Darboux equation and hyperbolic B-potentials // Contemporary Mathematics. Fundamental Directions. **65**, 2019 [in Russian].

B-elliptic equations

V. V. Katrakhov and S. M. Sitnik, The transmutation method and boundary-value problems for singular elliptic equations // Contemporary Mathematics. Fundamental Directions. **64**, 2018 [in Russian].

B-parabolic equations

A. B. Muravnik, Functional differential parabolic equations: integral transformations and qualitative properties of solutions of the Cauchy problem // Journal of Math. Sciences **216** (3), 2016.

E. L. Shishkina and S. M. Sitnik

Method of Transformation Operators for Differential Equations with Bessel Operators (Moscow, Fizmatlit, 2019) [in Russian].

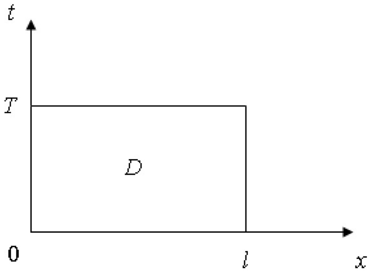
E. L. Shishkina and S. M. Sitnik

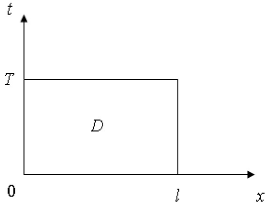
Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics, Mathematics in Science and Engineering (Elsevier. Academic Press, 2020).

N. V. Zaitseva

Mixed Problems with Integral Conditions for Hyperbolic Equations with the Bessel Operator (Moscow, Publishing house of Moscow University, 2021) [in Russian].

$$u_{tt} = u_{xx} + \frac{p}{x} u_x, \quad (x,t) \in D$$

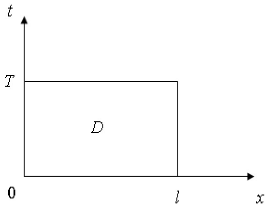




$p \geq 1$; $|p| < 1$, and $p \neq 0$

$$\int\limits_0^l x^p u(x,t)\,dx = A, \quad 0 \leq t \leq T;$$

$$u_x(l,t) + \int\limits_0^l u(x,t)\,x^p\,dx = 0, \quad 0 \leq t \leq T$$



$p \leq -1$

$$\int\limits_0^l x\,u(x,t)\,dx = A, \quad 0 \leq t \leq T;$$

$$\left(x^{p-1}u(x,t)\right)'_x\Big|_{x=l} + \int\limits_0^l x\,u(x,t)\,dx = 0, \quad 0 \leq t \leq T$$

$D = \{(x,y) :\ 0 < x < l, -\alpha < y < \beta\}$;
$D_+ = D \cap \{y > 0\}$, $D_- = D \cap \{y < 0\}$.

$p \geq 1$; $|p| < 1$, and $p \neq 0$

$$u_{xx} + (\operatorname{sgn} y) u_{yy} + \frac{p}{x} u_x = 0, \quad (x,y) \in D_+ \cup D_-$$

$$\int\limits_0^l x^p u(x,y)\, dx = A, \quad -\alpha \leq y \leq \beta$$

N. V. Zaitseva

Transmutation Operators and Applications, Trends in Mathematics (Birkhäuser, Cham, 2020);
Differential Equations **57** (2), 2021.

Statement of the Problem 1; $p \geq 1$, $|q| < 1$, and $q \neq 0$

$$D = \{(x, y) : \ 0 < x < l, -\alpha < y < \beta\}$$

$$u(x,y) \in C(\overline{D}) \cap C^2(D_+ \cup D_-),$$

$$u_{xx} + (\operatorname{sgn} y) u_{yy} + \frac{p}{x} u_x + \frac{q}{|y|} u_y = 0, \quad (x,y) \in D_+ \cup D_-,$$

$$u(x,\beta) = \varphi(x), \quad u(x,-\alpha) = \psi(x), \quad 0 \leq x \leq l,$$

$$\lim_{y \to 0+} y^q u_y(x,y) = \lim_{y \to 0-} (-y)^q u_y(x,y), \quad 0 < x < l,$$

$$\int\limits_0^l x^p u(x,y)\, dx = A, \quad -\alpha \leq y \leq \beta$$

M. V. Keldysh [Dokl. Akad. Nauk SSSR **77**(2), 1951].

$$u_x(0,y) = 0, \quad -\alpha \leq y \leq \beta$$

$$u(x,y)\in C(\overline{D})\cap C^2(D_+\cup D_-),$$

$$u_{xx}+(\operatorname{sgn} y)u_{yy}+\frac{p}{x}u_x+\frac{q}{|y|}u_y=0,\quad (x,y)\in D_+\cup D_-,$$

$$u(x,\beta)=\varphi(x),\quad u(x,-\alpha)=\psi(x),\quad 0\leq x\leq l,$$

$$\lim_{x\to 0+} x^p u_x(x,y)=0,\quad -\alpha\leq y\leq\beta,$$

$$\lim_{y\to 0+} y^q u_y(x,y)=\lim_{y\to 0-}(-y)^q u_y(x,y),\quad 0<x<l,$$

$$\int\limits_0^l x^p u(x,y)\,dx=A,\quad -\alpha\leq y\leq\beta$$

$$\int\limits_0^l x^p\varphi(x)\,dx=\int\limits_0^l x^p\psi(x)\,dx=A$$

Formal Fourier–Bessel series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t) X_n(x)$$

$$X_n(x) = \frac{\widetilde{X}_n(x)}{||\widetilde{X}_n(x)||}, \quad n = 0, 1, 2, \ldots,$$

$$||\widetilde{X}_n(x)||^2 = \int_0^l x^p \widetilde{X}_n^2(x)\,dx,$$

$$\widetilde{X}_0(x) = 1, \quad \widetilde{X}_n(x) = x^{\frac{1-p}{2}} J_{\frac{p-1}{2}}(\lambda_n x), \quad n \in \mathbb{N},$$

$$J_{\frac{p+1}{2}}(\lambda_n l) = 0$$

$$u_n(y) = \begin{cases} \frac{\varphi_n \sqrt{(\alpha y)^{1-q}} \triangle_n(\alpha,y) + \psi_n \sqrt{(\beta y)^{1-q}} A_n(y,\beta)}{\triangle_n(\alpha,\beta)\sqrt{(\alpha\beta)^{1-q}}}, & y > 0, \\ \frac{\varphi_n \sqrt{(-\alpha y)^{1-q}} B_n(\alpha,-y) + \psi_n \sqrt{(-\beta y)^{1-q}} \triangle_n(-y,\beta)}{\triangle_n(\alpha,\beta)\sqrt{(\alpha\beta)^{1-q}}}, & y < 0, \end{cases}$$

where

$$A_n(y,\beta) = \left(-J_{\frac{q-1}{2}}(i\lambda_n\beta) J_{\frac{1-q}{2}}(i\lambda_n y) + J_{\frac{1-q}{2}}(i\lambda_n\beta) J_{\frac{q-1}{2}}(i\lambda_n y)\right) i,$$

$$B_n(\alpha,-y) = -J_{\frac{q-1}{2}}(\lambda_n\alpha) J_{\frac{1-q}{2}}(-\lambda_n y) + J_{\frac{1-q}{2}}(\lambda_n\alpha) J_{\frac{q-1}{2}}(-\lambda_n y),$$

$$\triangle_n(\alpha,\beta) = iJ_{\frac{q-1}{2}}(\lambda_n\alpha) J_{\frac{1-q}{2}}(i\lambda_n\beta) + J_{\frac{1-q}{2}}(\lambda_n\alpha) J_{\frac{q-1}{2}}(i\lambda_n\beta)$$

Theorem.

If there exists a solution of the Problem 1, then it is unique if and only if condition

$$\triangle_n(\alpha,\beta) = iJ_{\frac{q-1}{2}}(\lambda_n\alpha)J_{\frac{1-q}{2}}(i\lambda_n\beta) + J_{\frac{1-q}{2}}(\lambda_n\alpha)J_{\frac{q-1}{2}}(i\lambda_n\beta) \neq 0$$

holds for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

THANK FOR YOUR ATTENTION!