# Integrable billiard systems on multidimensional CW-complexes

V. Kibkalo

(joint with V. Vedyushkina)

11.07.2023

# Brief review: **IHS** and **IB**

- Integrable Hamiltonian systems (**IHS**) and integrable billiards (**IB**):
  energy $H$ and integral $F$ are independent, constant on trajectories.
- Results on integrable billiards (IB):
  from Poncelet, Jacobi, Birkhoff to V. Dragovic and M. Radnovic, V. Kozlov, D. Treschev, S. Tabachnikov, A. Glutsyuk, V. Kaloshin and A. Sorrentino, A. Mironov and M. Bialy.

1. Birkhoff conjecture is true (A. Glutsyuk, 18):
   Billiard with $H = |v|^2$ on a flat compact table with $^2$-smooth boundary (and not piece-wise linear) is polynomially integrable (**IB**) $\Leftrightarrow$ its domain is bounded by confocal quadrics.
2. Billiard book (V. Vedyushkina, 18):
   Billiard on CW-complex glued from domains of flat confocal **IB** is integrable.

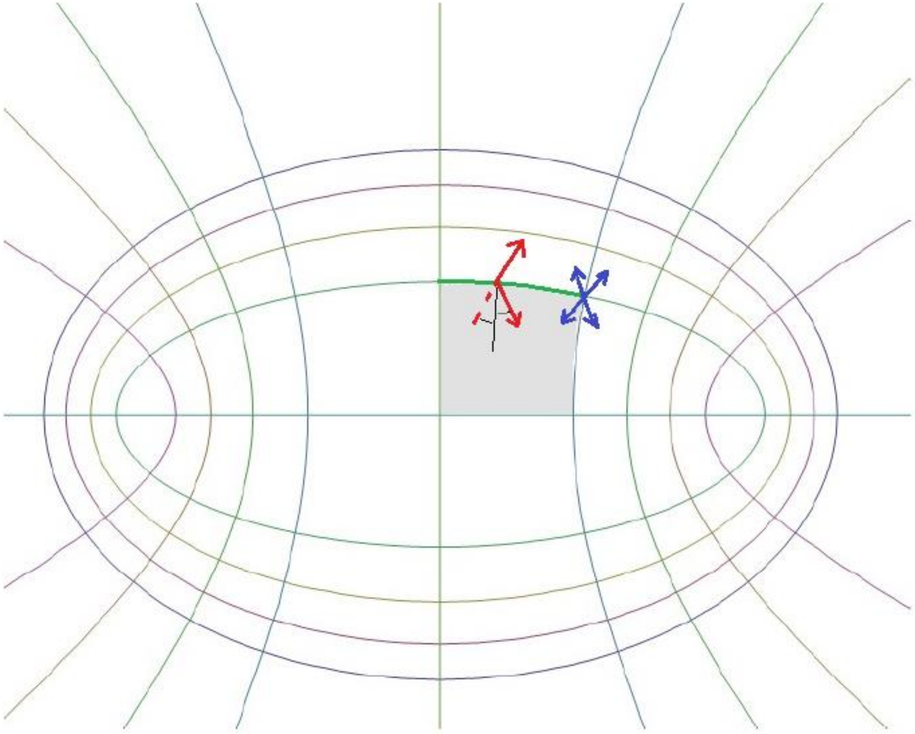
Consider integrable billiards (**IB**) on this wide class of CW-complexes equipped with permutations on 1-edges.

# Family of confocal quadrics

$$(b-\lambda)x^2 + (a-\lambda)y^2 = (a-\lambda)(b-\lambda), \lambda \le a.$$

Billiard domain: $\Omega \subset \mathbb{R}^2(x,y)$.

Phase space: $M^4 := \{(P,v) |\ P \in \Omega, v \in T_P\Omega, |v| > 0\} /$ reflections.

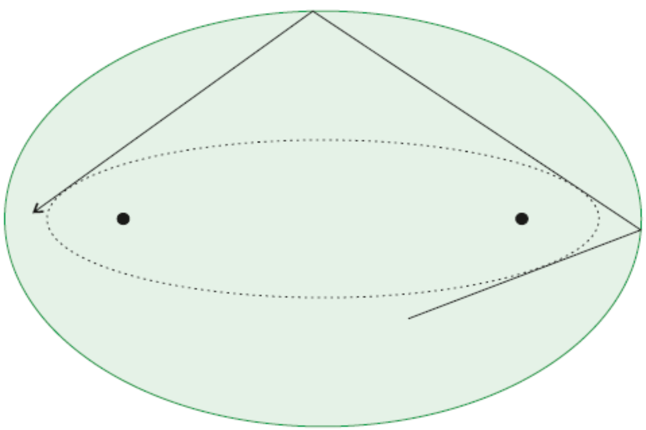


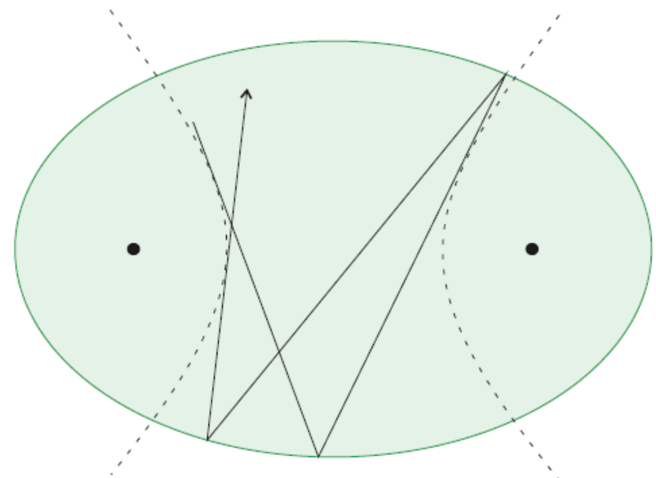
$H = |v|^2 = v_x^2 + v_y^2$, $\qquad Q_h^3 \subset M^4 : \ H = h.$

# Integrability of confocal billiards

$H = |v|^2 = v_x^2 + v_y^2, \qquad Q_h^3 \subset M^4 : \ H = h.$

Caustic is a confocal quadric from the same family for some $\lambda$:

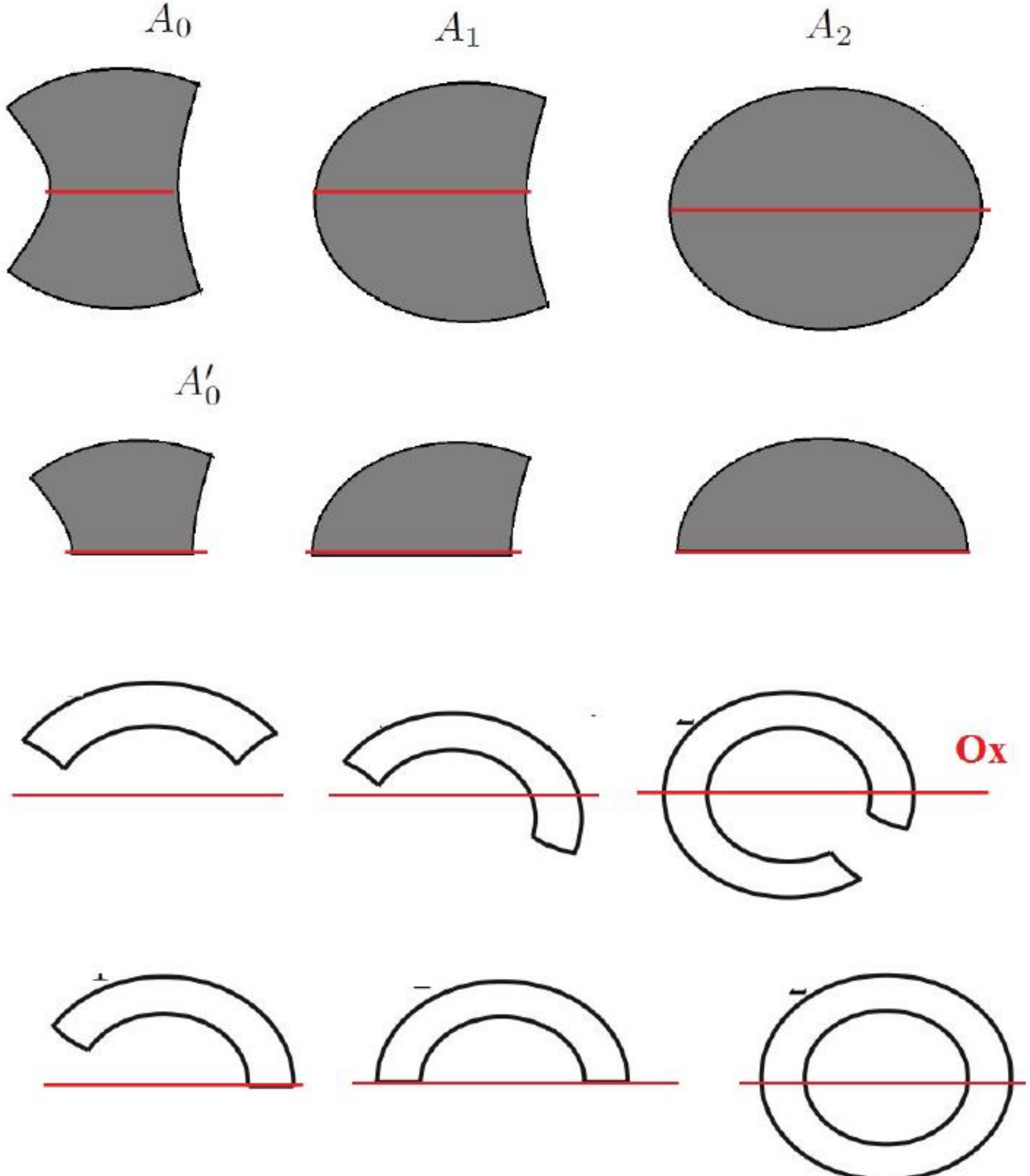




$$\lambda = \Lambda(x, y, v_x, v_y) = \frac{b v_x^2 + a v_y^2 - (x v_y - y v_x)^2}{v_x^2 + v_y^2}.$$

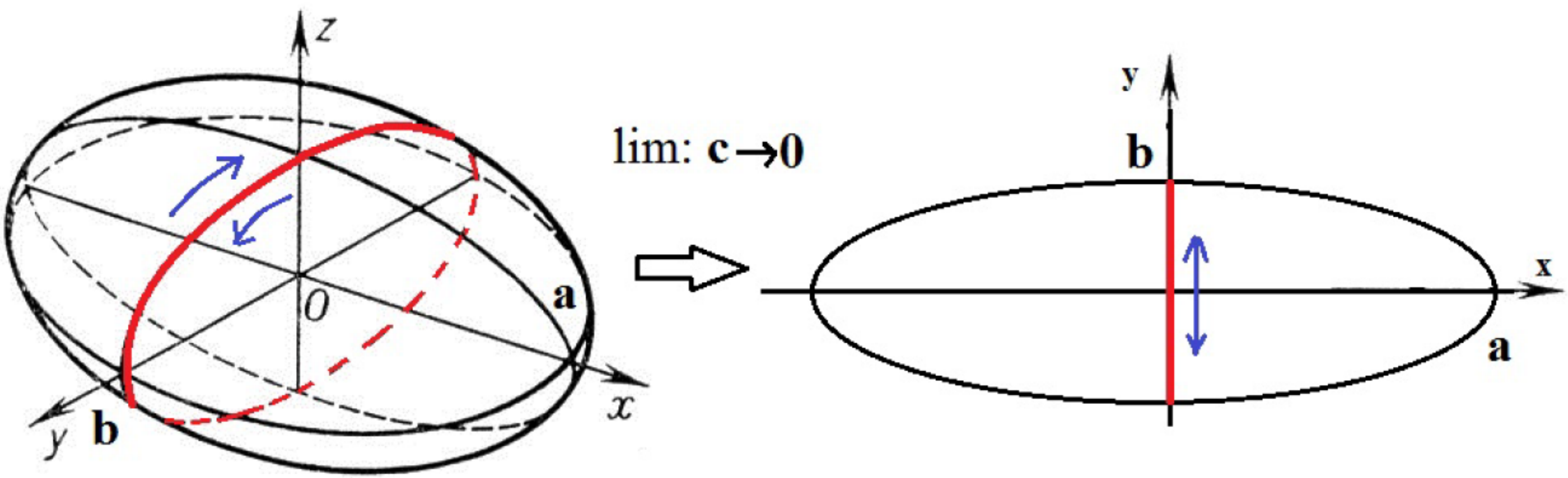
$\Lambda \cdot H$ is a polynomial integral of confocal billiards (quadratic on $v_x$, $v_y$).

# Complete list of flat confocal billiards



# Geodesic flow on an ellipsoid and billiard in an ellipse

Birkhoff: integrability of billiard follows from integrability of geodesic flow on an ellipsoid.

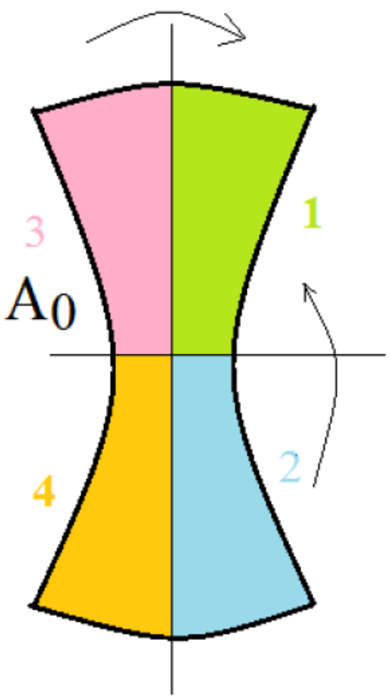


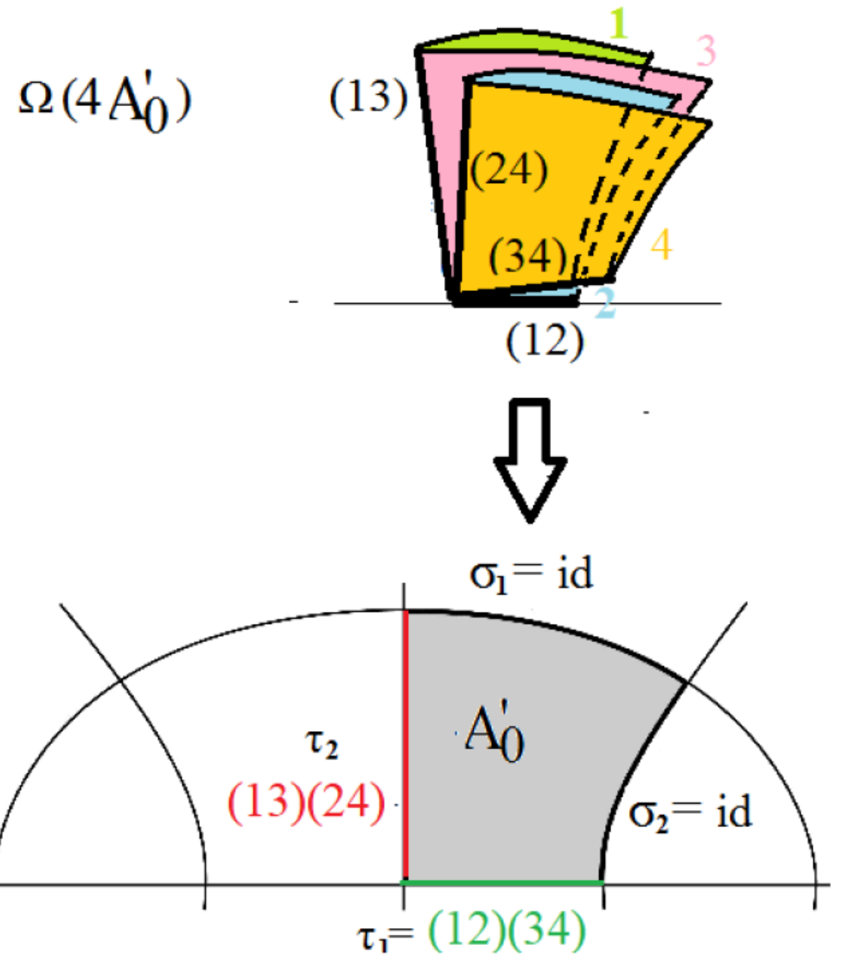

Different numbers of periodic trajectories in phase $Q_h^3$ for these systems.

Different properties of the phase space

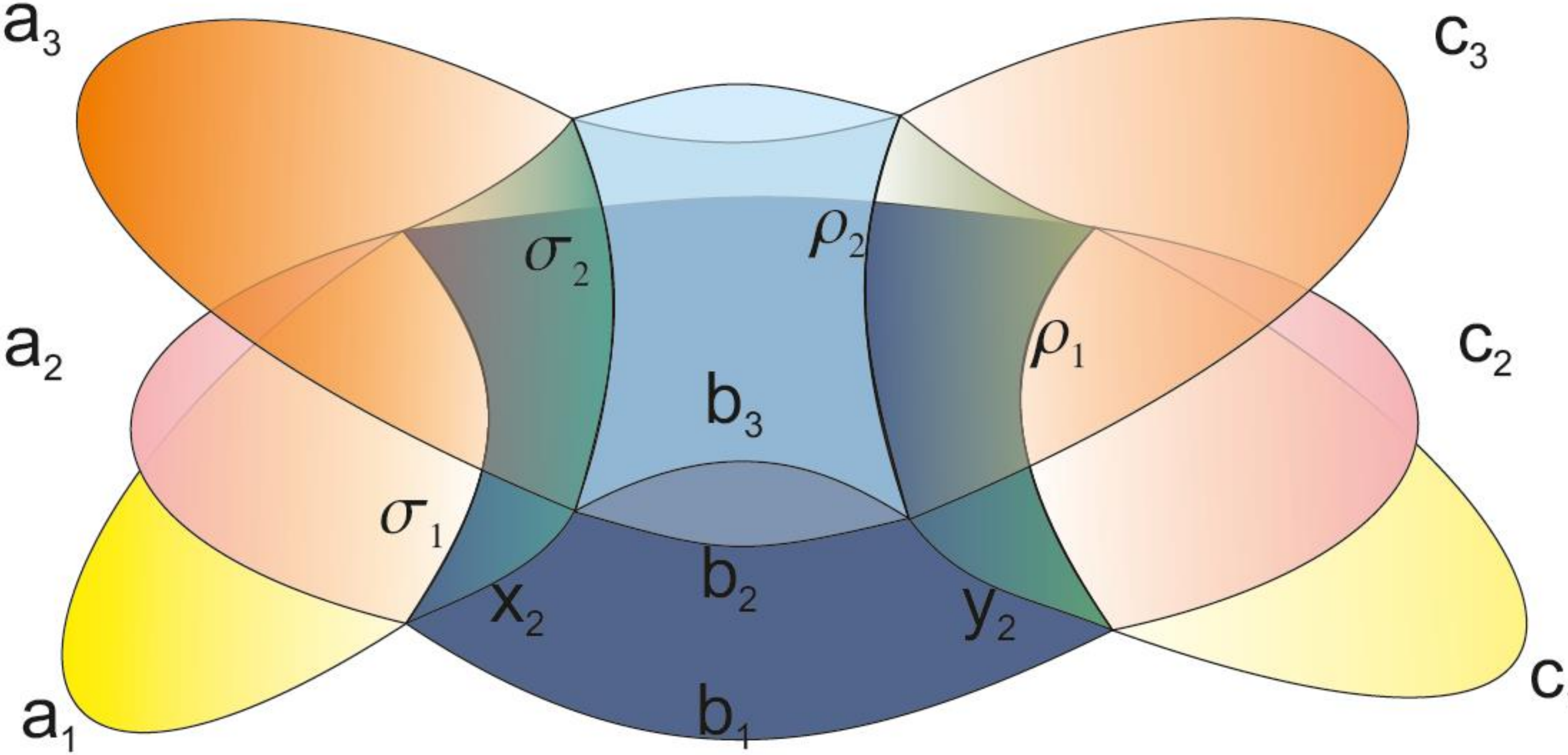
# Billiard book: example 1

Billiard books were introduced by V.Vedyushkina, [3], 2018.
It is a CW-complex w. permutations on 1-edges and projection to $\mathbb{R}^2$:



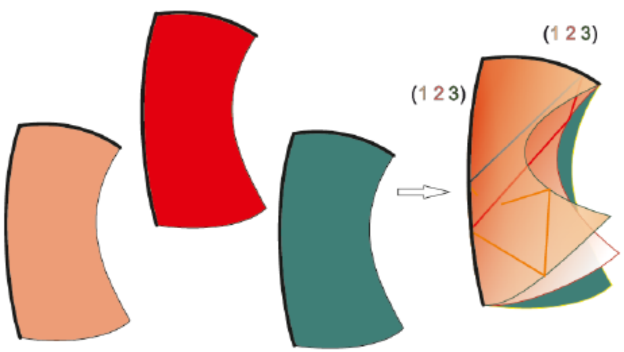
# Billiard book: example 2

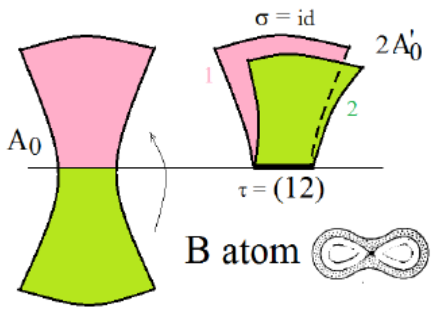


# Billiard book: CW-complex $X^2$ with permutations

Billiard book (V.Vedyushkina, 18)

- 2-cells of $\Omega$ correspond to flat confocal domains $\Omega_i \subset \mathbb{R}^2(x, y)$ and projection $\pi : \Omega \to \mathbb{R}^2(x, y)$ is an isometry and bijection.
- $\Omega_i$ are bounded by confocal quadrics of one family.
- 2-cells are glued along 1-cells, i.e. their smooth boundary arcs. $\pi$-images of such arcs coincide and belong to a quadric.
- 1-cell $\gamma_i$ has a cyclic permutation $\sigma_i$ on set of 2-cells glued along $\gamma_i$.
- Permutations for $\pi$-images (in $\mathbb{R}^2$) of intersecting $\gamma_i, \gamma_j$ commute.
- Particle moves from 2-cell $\alpha$ to $\sigma_i(\alpha)$ after reaching $\gamma_i$.

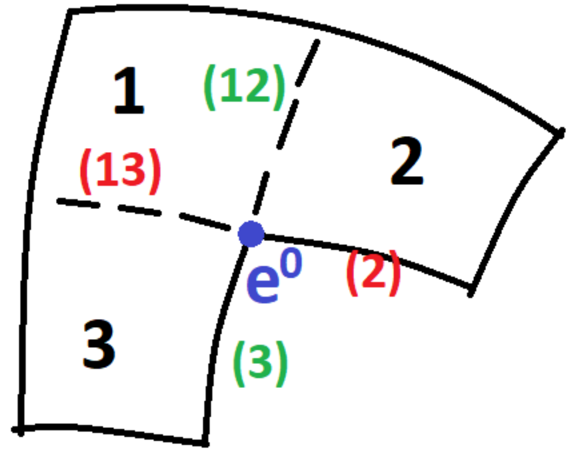


# Incorrect billiard books

$(12)(3) \circ (13)(2) \neq (13)(2) \circ (12)(3)$

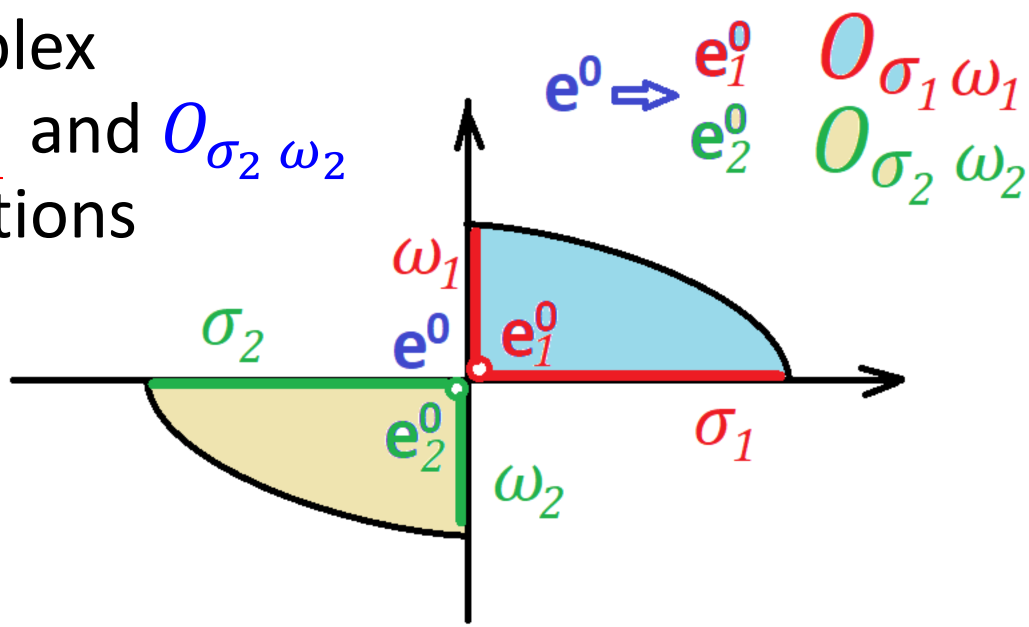
- **1.** Non-commuting permutations on two quadrics for a vertex $e^0$.




**Example:**

non-convex" $3\pi/2$ angles

- **2.** Vertices of the CW-complex correspond to orbits $O_{\sigma_1\,\omega_1}$ and $O_{\sigma_2\,\omega_2}$ of the pair of two permutations on two quadrics $Ox$, $Oy$:



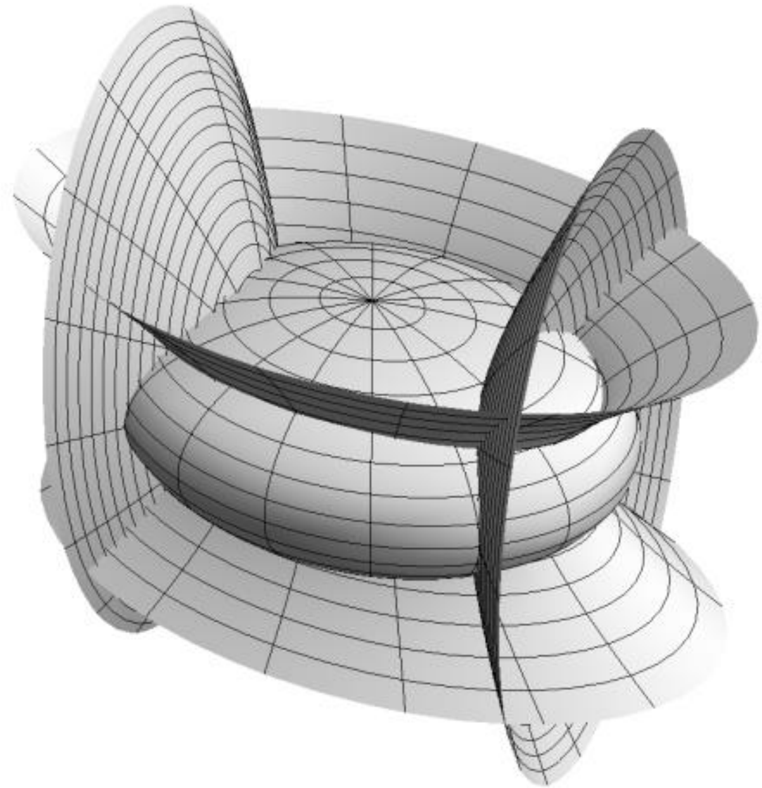

- Ox: $\sigma_1, \sigma_2$ ;    Oy: $\omega_1, \omega_2$ ;

# Multi-dimensional case: quadrics

- Family of confocal quadrics $Q_\lambda$ in $R^n$

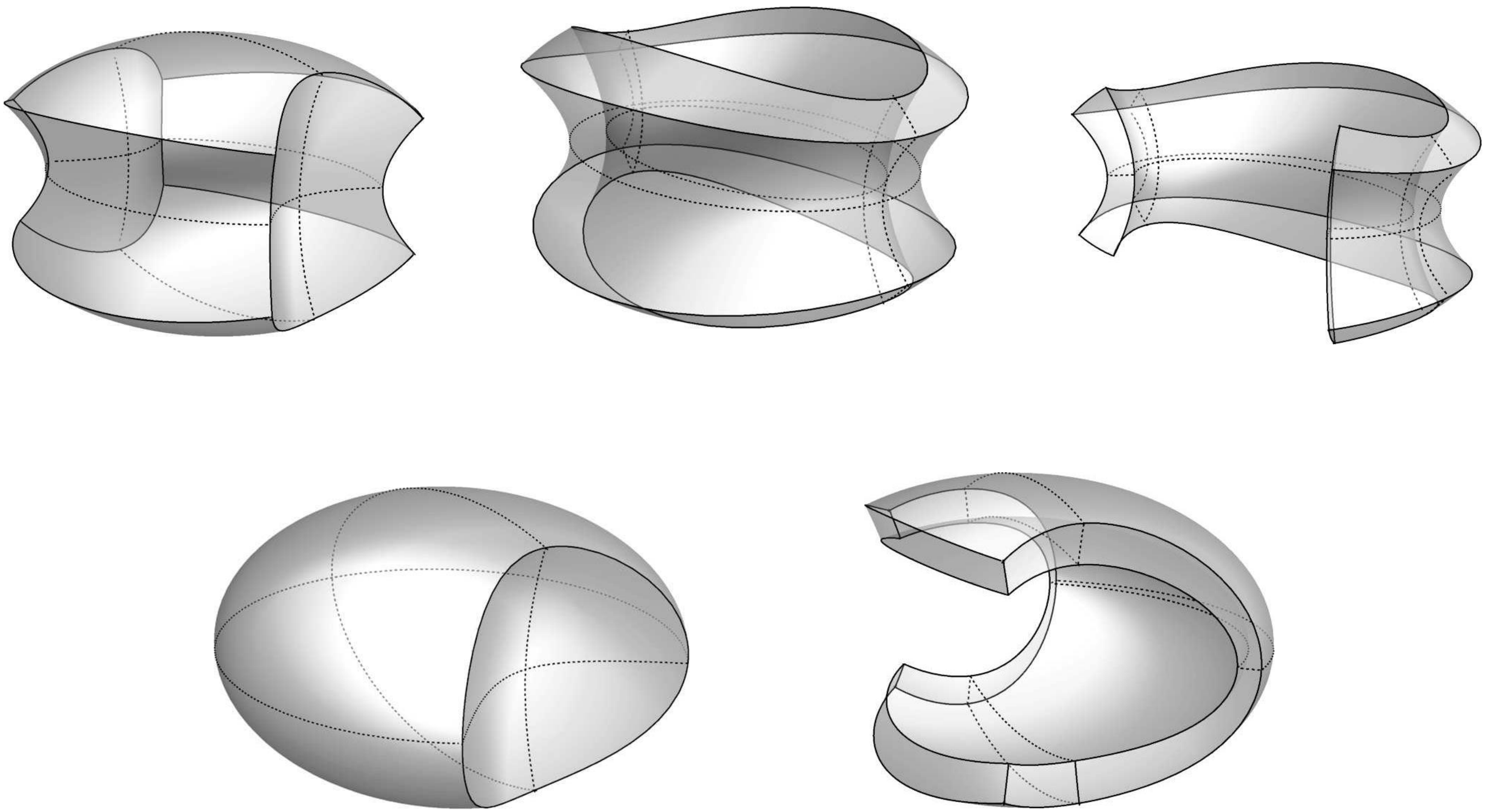
$$\frac{x_1^2}{a_1-\lambda}+\cdots+\frac{x_n^2}{a_n-\lambda}=1, \qquad 0<a_n<\cdots<a_1$$

- n =3:
  - an ellipsoid for $\lambda < a_3$,
  - a one-sheet hyperboloid for $a_3 < \lambda < a_2$, axes O
  - a two-sheet hyperboloid for $a_2 < \lambda < a_1$.



- $\vec{x}=(x_1, \dots x_n)$: $\forall x_i > 0 \ \rightarrow (E_{\lambda 1}, \dots, E_{\lambda n})$.

# Examples of confocal domains



# CW-complexes. Polyhedral complexes

**Definition.** Hausdorff topological space $X^n$ with a collection of maps $\phi_\alpha^k : D^k \to X$ is called a **CW-complex** if

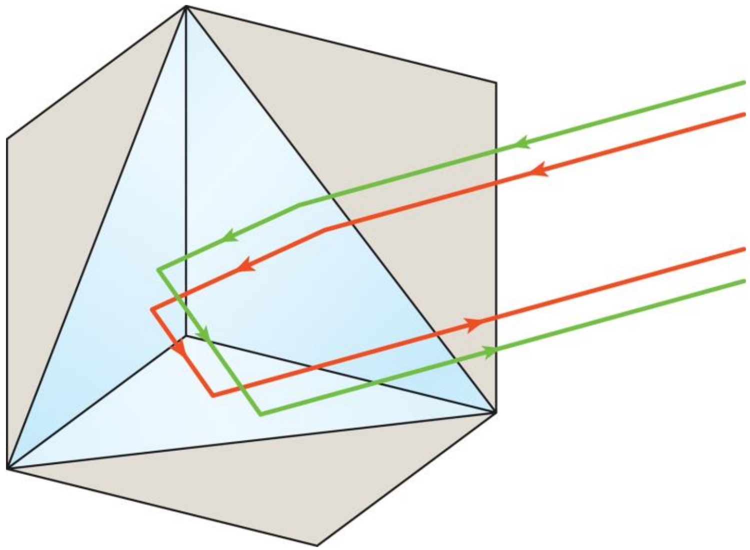
- the restriction $\phi_i^k|_{Int\ D^n}$ of each $\phi_i^k$ to the interior of $D^k$ is an embedding,
- $X$ is a disjoint union of cells $e_i^k = \phi_i^k(Int\ D^k)$: $\mathrm{X} = \coprod_{i,k} e_i^k$,
- $\phi_i^k\left(S^{k-1} = Int\ D^k\right) \subset \bigcup_{j=1}^{s} e_{i_j}^{k_j}$ for a finite $j$ and $k_j < k$.
- $W \subset X$ is closed $\Leftrightarrow (\phi_i^k)^{-1}(W) \subset D^k$ is closed $\forall k, i$.

- In a neighb. $U_x$ of $x \in \partial e^n$ the closure $\overline{e^n}$ is equivalent to $V^k \times R^{n-k}$ for a neighbourhood $0 \in V^k \subset R^k$.

- Such finite CW-complexes are **polyhedral complexes**.

# Billiard books as CW-complexes (1)

- n =2 ⇒ n-1 = 1, n-2 = 0. For $k \geq 3$: $e^{n-k}$ are empty.
  Gluing $e_i^2$ along $e^1$ with commutation conditions $\forall e^0$.
- Comm. conditions: to provide continuity of reflection of a ball



- n =3: gluing $e_i^3$ along $e^2$, comm. cond. $\forall e^1$. $\boldsymbol{e^0} = e^{n-3}$- ?
  Each $e_j^2$: $\pi\left(e_j^2\right) \subset Q_\alpha$ , is equipped with $\sigma_j$.
- Math. Induction: having tran codim = 2 \sim dim = 1

# Billiard books as CW-complexes (2)

Billiard book is a CW-complex w. a projection $\pi: X^n \to R^n$ s.th.

1) projection $\pi$ is continuous,

2) restriction of $\pi$ on closure $e_i^n$ is a bijection and isometry,

3) Projection of each $\overline{e_i^n}$ is a confocal n-dim flat domain

4) Each $e^{n-1}$ is equipped with a cyclic permutation

5) Each $e^{n-2} \subset \cap \overline{e_i^{n-1}}$ which are projected on two quadrics $E_1, E_2$ is equipped commutativity condition of two permutations correspond to these two quadrics

6) For each cell $e^{n-k}$ of codim k the set of cyclic permutations on n-1-facets $\overline{e_i^{n-1}}$ incidental to it acts transversely on the set of n-cells incidental to it.

# Billiard motion on a book

- Phase space of $X^n$ (glued from $e_i^n$) is glued from $T^* e_i^n$.
- Inside an n-cell motion is lifted from $R^n$
- Reflection law at $e^{n-1}$: generalization of standard reflection

$$(x, v_1) \sim (x, v_2), \qquad x \in e^{n-1} \subset \partial e^n, \;\; v_1 - v_2 \perp e^{n-1}$$

- $x \in X^n, \;\; e_i^n \cap e_{\sigma(i)}^n = e^{n-1}$ : on quadric $E_\alpha$.

$(i, x, v_1) \sim (\sigma(i), x, v_2)$:

$$x \in e^{n-1} \subset \bar{e}_1^n \cap \bar{e}_2^n, \quad |v_1| = |v_2|, \quad (v_1 - v_2) \perp E_\alpha$$

$$sgn_\alpha v_1 \cdot sgn_\alpha e_1^n = -sgn_\alpha v_2 \cdot sgn_\alpha e_2^n.$$

- Phase space of a billiard book is foliated into common level surfaces of the Hamiltonian $|v|^2$ and n-1 first integrals.

# Correctness of the definition

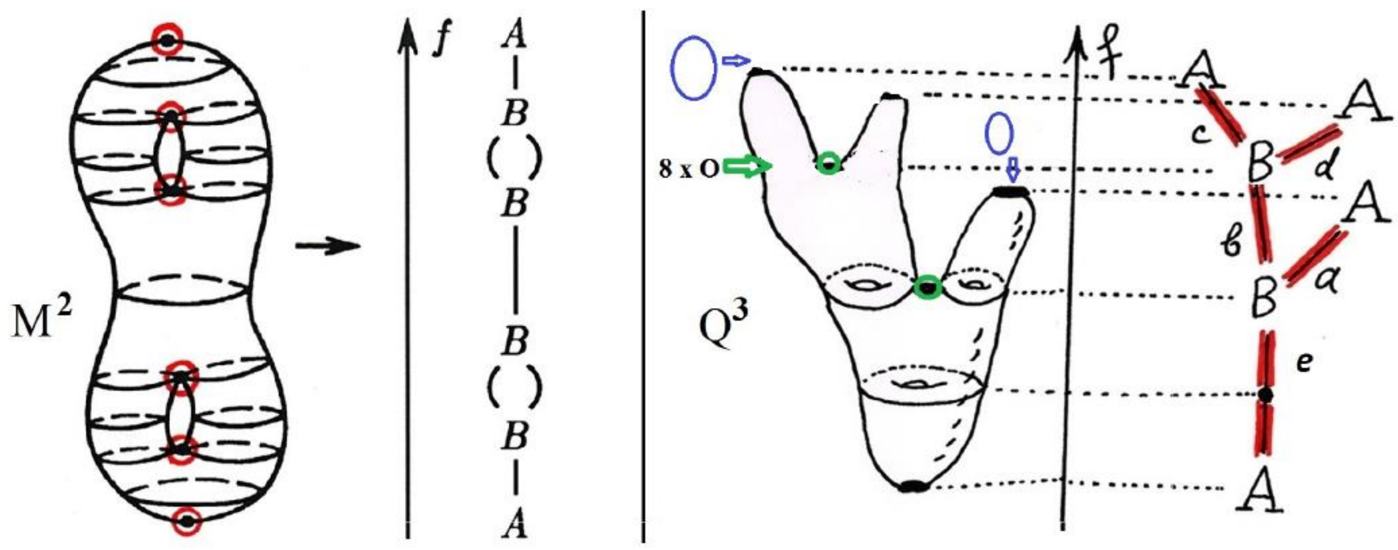
- **Theorem.** Let transitivity condition for each $e^{n-k}$ of dim $0 \leq n-k \leq n-1$ is true (action of group generated by of cyclic permutations on n-1-facets). Then
- Each cell $e^{n}$ containing $e^{n-k}$ in its boundary appears exactly k times in this set,
- Set of cyclic permutations can be uniquely partitioned into $k$ permutations $\sigma_1, \dots, \sigma_k$ corresponding to quadrics $E_1, \dots, E_k$ intersecting on $\pi(e^{n-k})$,
- Let a path from the cell $e^{n}$ to the cell $e^{n\prime}$ exists through a chain of cells $e^{n} = e_0^n, e_1^n, \dots e_k^n = e^{n\prime}$, where two neighbours are incidental by n-1-facets $e_1^{n-1}, \dots, e_k^{n-1}$ containing $e^{n-k}$ and are projected on pairwise diff. quadrics $E_1, \dots, E_k$: $\pi(e_s^{n-1}) \subset E_s$. Then for each order $E_{\phi(1)}, \dots, E_{\phi(k)}$ of quadrics there exists a path $e_0^n, \hat{e}_1^n, \dots \hat{e}_k^n = e^{n\prime}$ from the cell $e^{n}$ to the cell $e^{n\prime}$ of the same umber of steps s.th. $\pi(\hat{e}_s^{n-1} \subset E_{\phi(s)})$.

# Invariant of a foliation on isoenergy surface $Q^3$

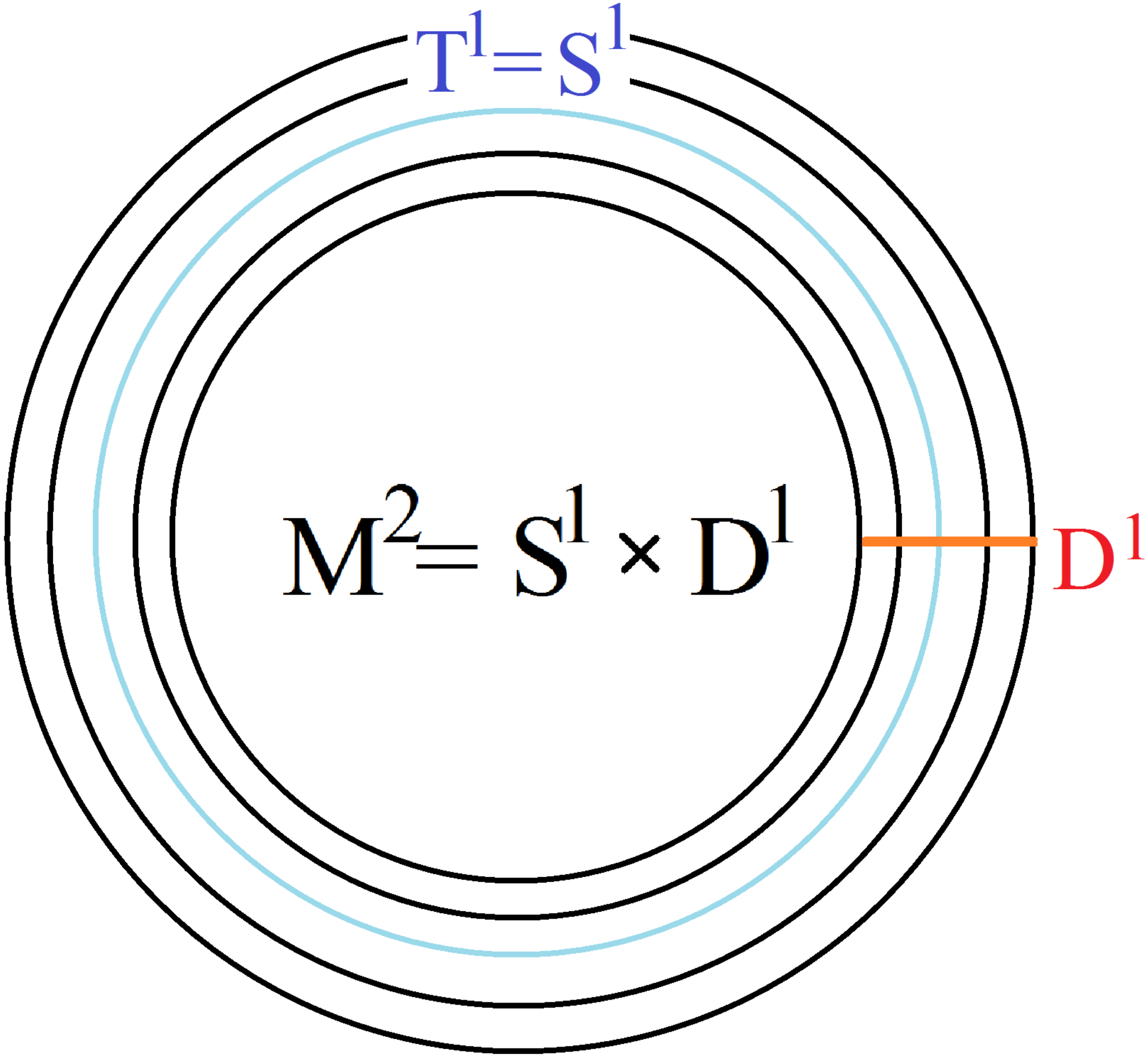
- IHS: $(M^4, \omega, H, F)$. Energy level $Q^3_h : H = h$.
- Quotient space of foliated $Q^3_h$ is a graph.

Every edge is a families of tori $T^2$, vertex — singular fibers.

- Singularity: neighbourhood of a vertex (i.e. neighb. of a sing. fiber).
- Two such singularities are glued by their boundary torus. Automorphism of $\pi_1(T^2)$. Matrixes $Mat(\mathbb{Z}, 2)$ with det $= -1$.

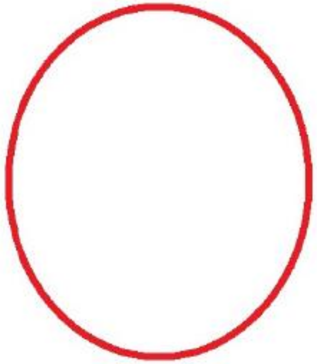


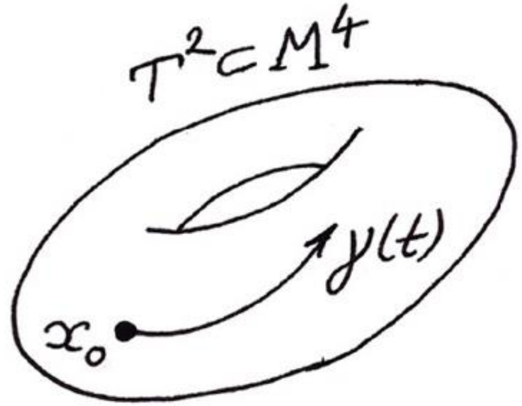
# Liouville theorem: example

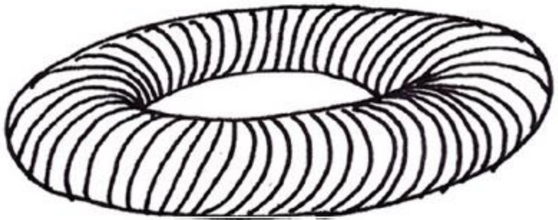


# Liouville tori and their bifurcations: examples

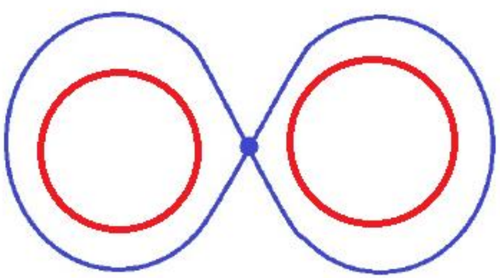
torus $T^1 = S^1$



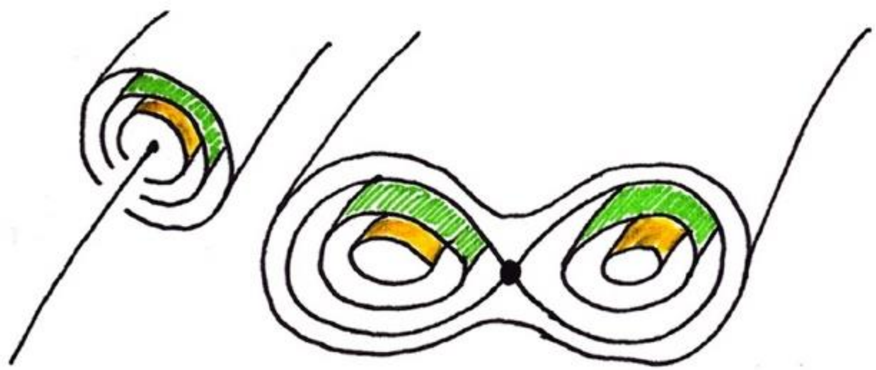


bifurcation of two $S^1$



bifurcation of tori



# Nondegenerate singularities of IHS (1)

- **IHS** on a symplectic $(M^4, \omega)$ is a pair $\mathfrak{F} = (H, F)$ of functions s.th.

$$\omega(\mathrm{sgrad}\, H, \mathrm{sgrad}\, F) = 0.$$

- **Liouville foliation**: $M^4 = \coprod_{h,f} \text{connected}\, \xi^2_{h,f} = \{H = h, F = f\}$.
- It usually has **singularities**: points $x$ where $rk\, d\mathfrak{F}|_x < 2$.

Classes of fiber-wise homeomorphic singularities:

- **local** problem: in neighbourhood of point $x$,
- **semi-local** problem: (in neighbourhood of fiber $\xi_{h,f} \ni x$.

- Point $rk = 0$ (e.g. an **equilibrium** of IHS) is **nondegenerate** if $d^2(H - \lambda F)$ has 4 nonzero eigenvalues for some $\lambda, \alpha, \beta \in \mathbb{R}$:

- center-center (elliptic-elliptic): $i\alpha, -i\alpha, i\beta, -i\beta$,
- center-saddle (elliptic-hyperbolic): $\alpha, -\alpha, i\beta, -i\beta$,
- saddle-saddle (hyperbolic-hyperbolic): $\alpha, -\alpha, \beta, -\beta$,
- focus-focus: $\alpha \pm i\beta, -\alpha \pm i\beta$.

- **Eliasson theorem**: near a point of $rk = 0$ Liouville foliation of IHS is **locally** equivalent to **direct product** of canonical singularities of center ($\dim = 2$), saddle ($\dim = 2$) and focus ($\dim = 4$) types.

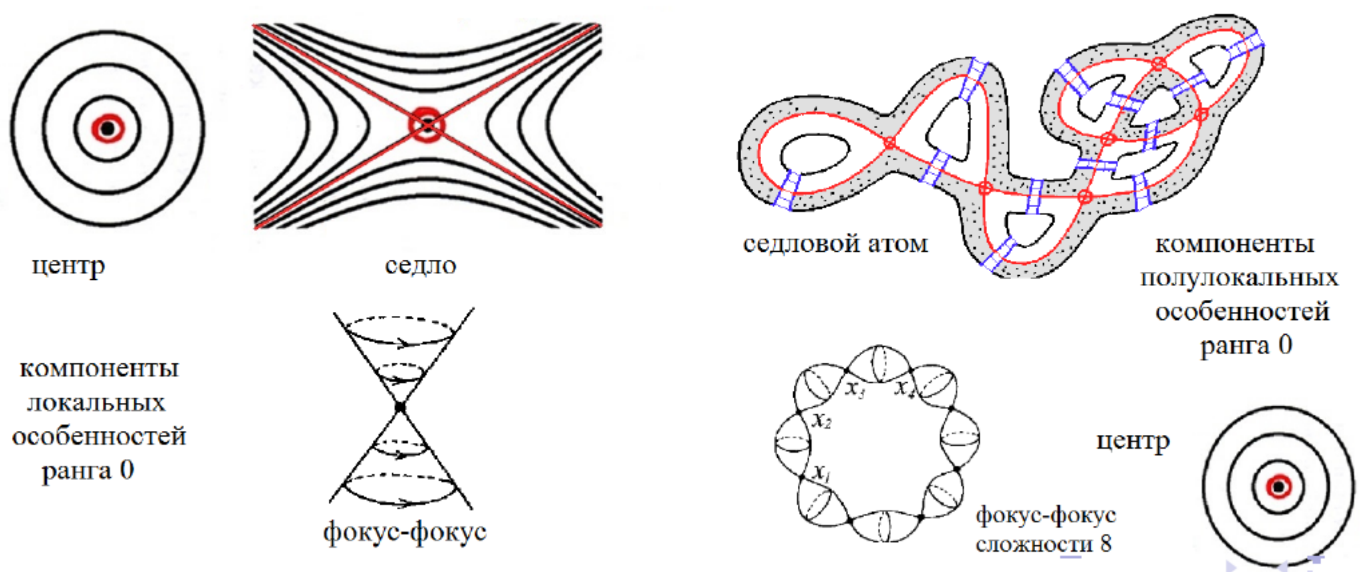
# Nondegenerate singularities of IHS (2)

Let $x : H(x) = h^*$ be nondegenerate point of rk 0 (for pair $H, F$).
Then $sgrad\, F = sgrad\, H = \vec{0}$ in $x$ and $\exists t \in R$ s.th. $det(d^2H + t\, d^2F) \neq 0$

Local (Eliasson) and semi-local (N.T. Zung) singularity of **every** d.o.f.:

1) singularity is fiber-wise homeomorphic to product of "standard" components with identification by finite group action.

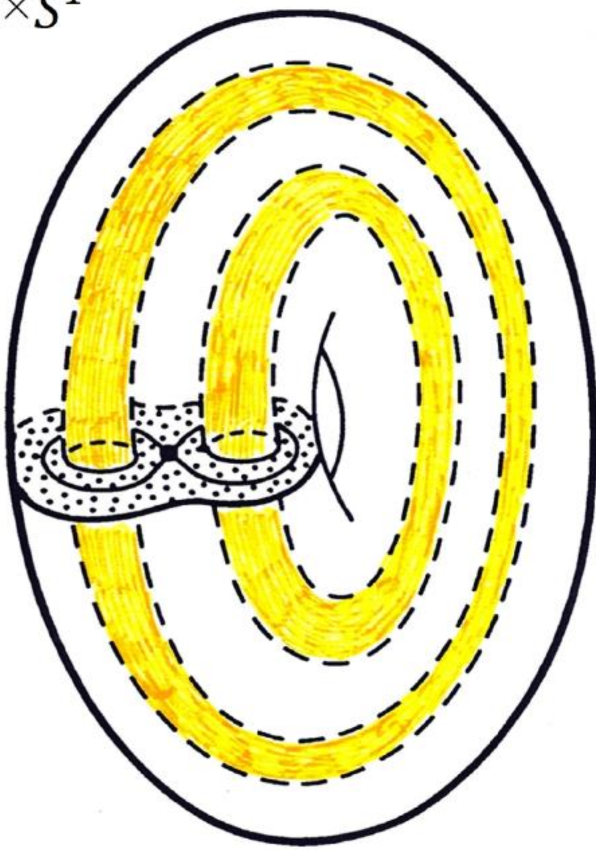
2) every component is a semi-local sing. of a Hamiltonian system with 1 d.o.f. on $M^2$ (some saddle 2-atom $V$ or the atom $A$) or a semi-local focal sing. (neigh. of a torus with n pinches).



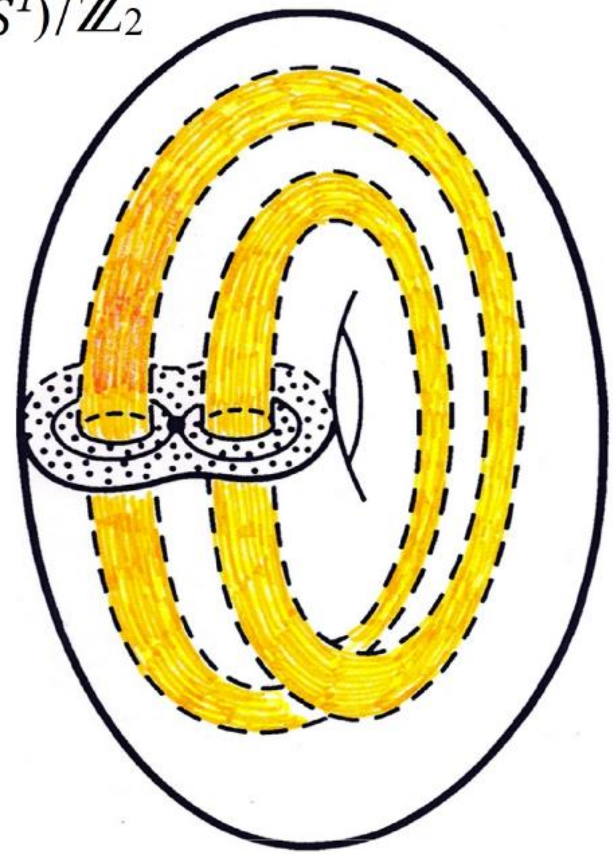
# Example of nondegenerate singularities in $Q^3$

- Consider oriented $M^2$ and neighbourhood of a directed graph with vetrices of degree 4. They correspond to Morse saddles.
- Every nondegenerate singularity in $Q^3$ is a product of such object and $S^1$, may be with $\mathbb{Z}_2$ action (right).

$\boldsymbol{B}\times S^1$



$\boldsymbol{A}^*=(\boldsymbol{B}\times S^1)/\mathbb{Z}_2$



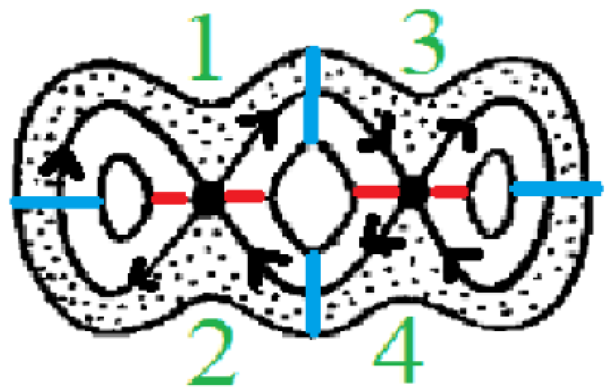
# Nondegenerate singularities

- 2 d.o.f.: $(M^4, H, F)$: nondegenerate corank 1 singularity is a product. "Atom" is a 1-parameter family $H = h, F \in [f_0 - \epsilon, f_0 + \epsilon]$ of Liouville tori and one singular fiber
$$V^2 \times S^1 \times D^1$$

$k$ equilibria, perm. $\sigma \in S_{2k}$, $\tau = (12)(34) \ldots (2k-1, 2k)$

$\mathrm{V} = \mathrm{D}_1$

k=2



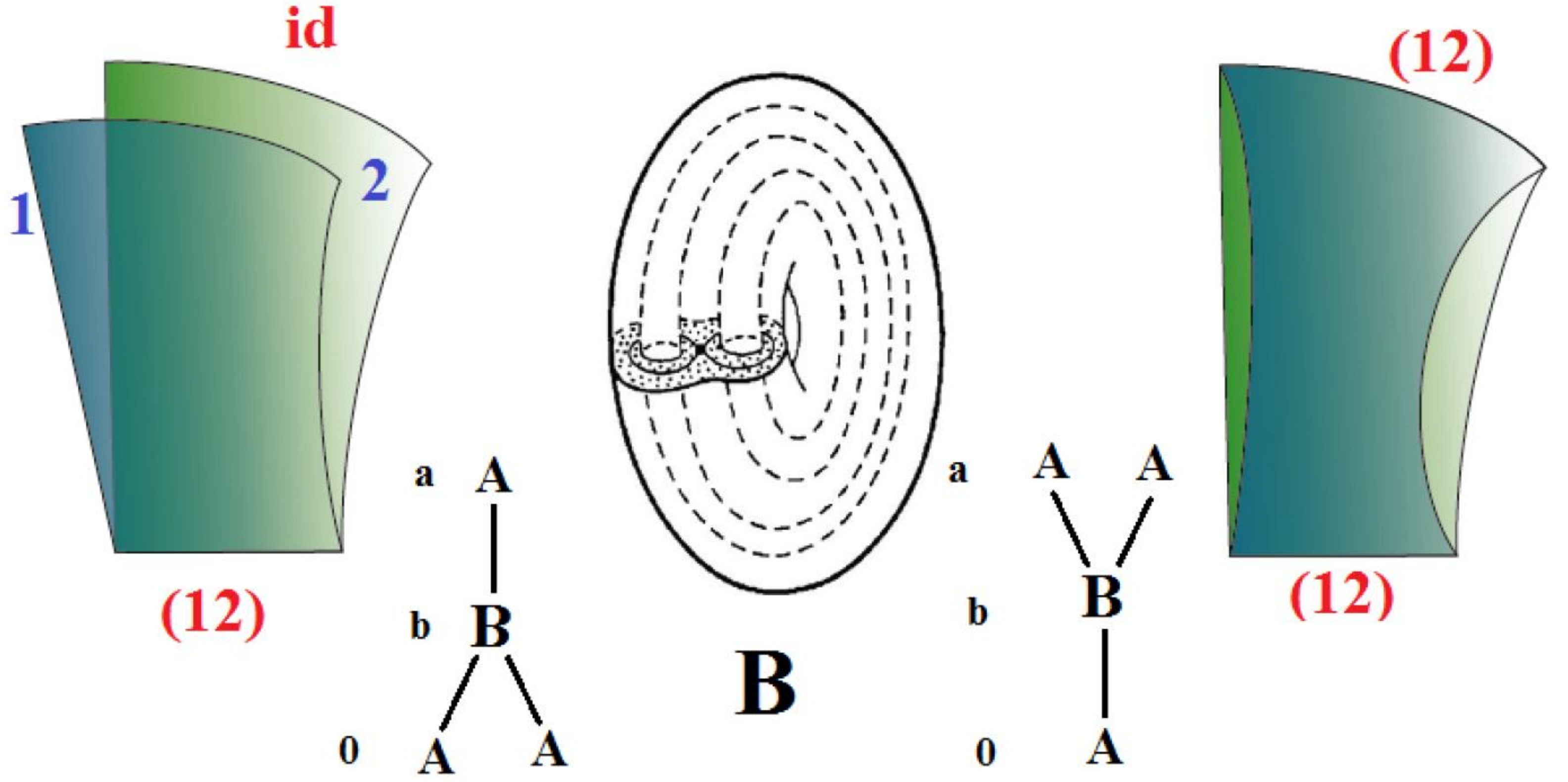

$\sigma = (1342)$

$\tau = (12)(34)$

- n d.o.f.: $M^{2n}$, $H = F_1, F_2 \ldots, F_n$.
Nondeg. corank 1 singularity is a foliated level surface in $M^{2n}$
$$F_1 = f_1, \ldots, F_{n-1} = f_{n-1}, F_n = F \in [f_0 - \epsilon, f_0 + \epsilon]$$
- Singularities of direct product type:
$$k\, T^n \to \xi^n \to s\, T^n = T^{n-1} \times V^2$$

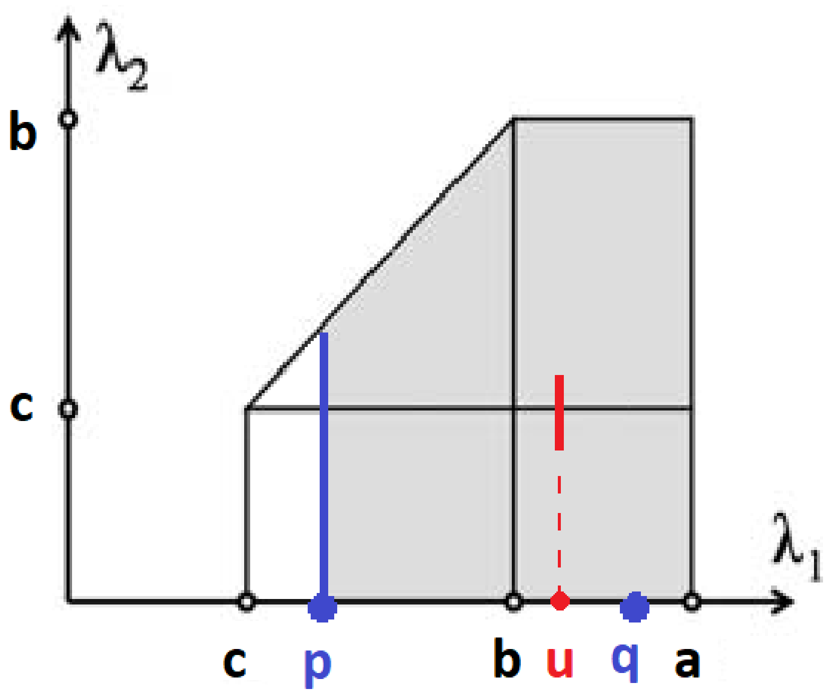
# Billiard realization of nondeg. sing: dim = 2

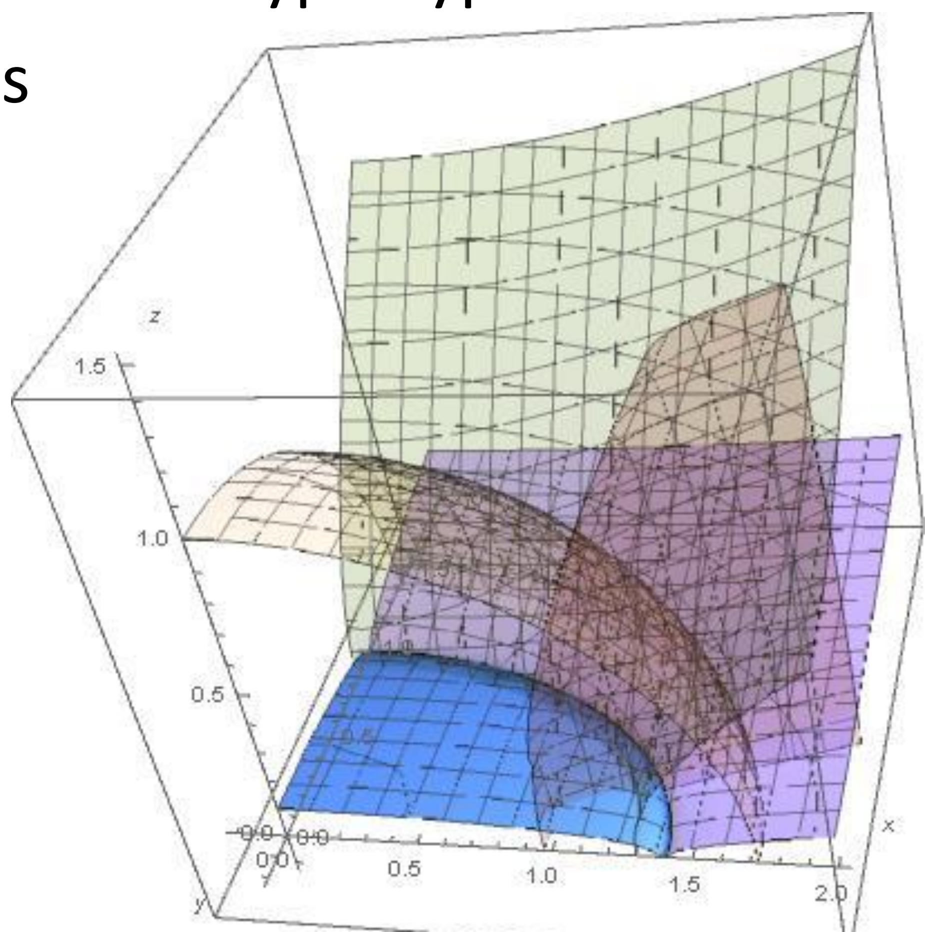
**Theorem** (Vedyushkina, Kharcheva, 18): for an atom w. k equilibria and permutations $\sigma, \tau$, the book consists of 2k cells glued by their focal (x = 0) and elliptic boundary edges.

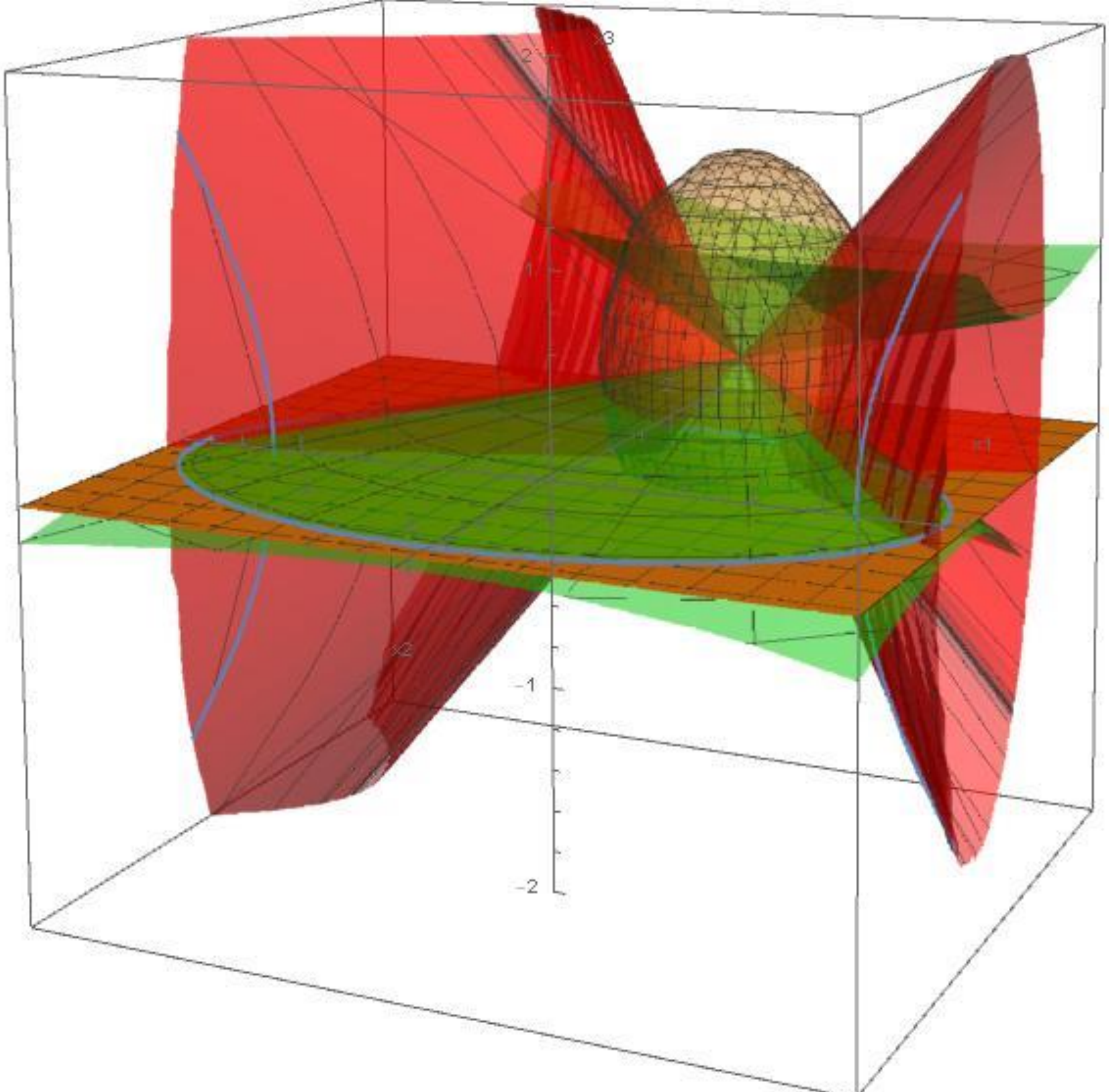


# Billiard realization of direct product atoms

- Theorem. Each direct product type atom of system with 3 d.o.f. Can be topologically modeled by a billiard book.
- It is is glued by 2k items of a standard domain, bounded by coordinate planes, ellipsoid and 2 diff. type hyperboloids
- Integrals: two confocal quadrics (Jacobi-Chasles theorem)




x3
2
1
x1
x2
−1
−2

# Thank you
# for your attention!

The work is supported by
Russian Science Foundation,
grant 22-71-10106