Reconstruction by modules of measurements of a vector-signal in finite-dimensional and infinite-dimensional spaces

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- There is a situation in many applied researches: we have a system of so-called measuring vectors $\Phi = \{\varphi_i\}_{i=1}^N$ in Euclidean or unitary space $\mathbb{H}^D$. The researcher has access to the measurement results of the unknown vector-signal $\mathbf{x}$ in the form of modules of scalar products $|\langle \mathbf{x}, \varphi_i \rangle|$, the phases (or signs) of these products are unknown.
- Is it possible to restore the vector $\mathbf{x}$?

Factorization

- Since the modules of the scalar products do not change by passing from the vector $\mathbf{x}$ to the vector $h\,\mathbf{x}$ with $|h| = 1$, a factorization is performed beforehand for a neat formulation of the problem . Let $\mathbb{T} = \{h \in \mathbb{H} : |h| = 1\}$.
- The factor space $\mathbb{H}^D/\mathbb{T}$ is introduced as a set of equivalence classes: $\mathbf{x} \sim \mathbf{y}$, if there exists $h \in \mathbb{T} : \mathbf{x} = h\,\mathbf{y}$.
- Thus, the problem arises of vector reconstruction from modules of measurements.

(RMM)-property

- Firstly we formulate the reconstruction problem as a problem about the property of a system of measurement vectors.
- **Definition.** A family of vectors $\Phi = \{\varphi_i\}_{i=1}^N$ in $\mathbf{R}^D$ (or $\mathbf{C}^D$) does the reconstruction by modules of measurements (RMM), if for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^D$ (or $\mathbf{C}^D$), satisfying $|\langle \mathbf{x}, \varphi_i \rangle| = |\langle \mathbf{y}, \varphi_i \rangle|$ for all $i = 1, \dots, N$, we have $\mathbf{x} = c\mathbf{y}$, where $c = \pm 1$ for $\mathbf{R}^D$ (and $c \in \mathbf{T}$ for $\mathbf{C}^D$.)

(RMM) and the injectivity

- This definition can be formulated as a property of the injectivity of the nonlinear operator:

$$\mathcal{A} : \mathbb{H}^D/\mathbb{T} \to \mathbb{R}^N,\ (\mathcal{A}(\mathbf{x}))(n) := |\langle \mathbf{x}, \varphi_n \rangle|^2,$$

$$n = 1, \ldots, N.$$

Complement Property

- **Definition.** A family of vectors $\Phi = \{\varphi_n\}_{n=1}^{N}$ in $\mathbb{H}^D$ has the complement property (CP) if for each subset $T \subseteq \{1, \ldots, N\}$, at least one of the sets $\{\varphi_n\}_{n \in T}$ or $\{\varphi_n\}_{n \in T^C}$ is complete in $\mathbb{H}^D$.
- Properties (RMM) and (CP) are equivalent for $\mathbb{R}^D$.

  Firstly, in [BCE06] the part $(\Rightarrow)$ of the next theorem was "proved"also for the complex case, but then it was detected a mistake in the complex case. Nowadays this part also proved in the complex case also. The part $(\Leftarrow)$ in the complex case is wrong, sufficient conditions for (RMM) in this case are more complicated.

(RMM) and Complement Property

- **Theorem.** If the family of vectors $\Phi = \{\varphi_i\}_{i=1}^N$ in $\mathbb{H}^D$ does (RMM), then it has complement property.
  If the family of vectors $\Phi = \{\varphi_i\}_{i=1}^N$ in $\mathbb{R}^D$ has (CP), then it has (RMM).
- Proof. We present the proof of $(\Rightarrow)$, which is suited for the both cases.
  Firstly we note that if $\Phi = \{\varphi_n\}_{n=1}^N \in$ (RMM), then $\text{span}(\{\varphi_n\}_{n=1}^N) = \mathbb{H}^D$. Otherwise, there is $\mathbf{x} \neq \mathbf{0}$, $\langle \mathbf{x}, \varphi_n \rangle = 0, \ n = 1, \ldots, N$,

$$|\langle \mathbf{x}, \varphi_n \rangle| = |\langle \mathbf{0}, \varphi_n \rangle| = 0, \ n = 1, \ldots, N,$$

Proof (cont)

- which is incompatible with the injectivity of the operator $\mathcal{A}$.
- Assume that $\Phi \notin$ (CP). Once again we choose the index set $T \subset \{1, \dots, N\}$ so that $\{\varphi_n\}_{n \in T}$ and $\{\varphi_n\}_{n \in T^C}$ are not complete in $\mathbb{H}^D$.

Proof (cont)

- Then we choose two vectors $\|\mathbf{x}\| = 1 = \|\mathbf{y}\|$ so that $\mathbf{x} \perp \varphi_n$ for $n \in T$, and $\mathbf{y} \perp \varphi_n$ for $n \in T^C$. So we have

$$|\langle \mathbf{x} + \mathbf{y}, \varphi_n \rangle| = |\langle \mathbf{x} - \mathbf{y}, \varphi_n \rangle|, \ n = 1, \dots, N.$$

- Since $\Phi \in$ (RMM), there exists $|\theta| = 1$ such that $\mathbf{x} + \mathbf{y} = \theta(\mathbf{x} - \mathbf{y})$, or $(1 - \theta)\mathbf{x} = -(1 + \theta)\mathbf{y}$.

Proof (cont)

- If $\theta = 1$, then $\mathbf{y} = \mathbf{0}$; if $\theta = -1$, then $\mathbf{x} = \mathbf{0}$, which contradicts the choice of vectors.
  If $\theta \neq \pm 1$, then

$$\mathbf{x} = -\frac{1+\theta}{1-\theta}\mathbf{y} := \alpha \mathbf{y} \tag{1}$$

  where $\alpha \neq 0$.
- Now we get the following. If $n \in T$, then $\langle \mathbf{x}, \varphi_n \rangle = 0$, and by virtue of (1), $\langle \mathbf{y}, \varphi_n \rangle = 0$. On the other hand, $\langle \mathbf{y}, \varphi_n \rangle = 0$ for every $n \in T^C$.

Proof (end)

- Therefore, $\langle \mathbf{y}, \varphi_n \rangle = 0$ for every n and $\|\mathbf{y}\| = 1$, i. e. the set $\{\varphi_n\}_{n=1}^N$ is not complete in $\mathbb{H}^D$, which contradicts the first remark of this proof. A similar situation takes place for $n \in T^C$.

Minimal number of measurements

- What is the minimal number $N = N(D)$ of measurement vectors $\{\varphi_j\}_{j=1}^N$, which have property (RMM) in $\mathbb{H}^D$?
- It immediately follows from the theorem:
  $N(D) \geq 2D - 1$ in $\mathbb{R}^D$;
  any system with (RMM)-property is a complete system in $\mathbb{H}^D$, that is equivalent (in finite dimensional space) for a system to be a frame.

Definition of Frame

- Another equivalent definition of a frame: a family of vectors $\{\varphi_j\}_{j=1}^N$ is a *frame* for a real or complex $\mathbb{H}^D$ if there are constants $0 < A \leq B < \infty$ such that for all $\mathbf{x} \in \mathbb{H}^D$,

$$A\|\mathbf{x}\|^2 \leq \sum_{j=1}^{N} |\langle \mathbf{x}, \varphi_j \rangle|^2 \leq B\|\mathbf{x}\|^2.$$

What is Spark?

- We recall that the *spark* of the system Φ is the smallest number of linearly dependent vectors of the system. We can define this number formally by arranging the vectors $\{\varphi_i\}_{i=1}^N$ as columns of $D \times N$-matrix Φ, and use the notation $\|\mathbf{x}\|_0$ for the number of nonzero coordinates of vector $\mathbf{x}$.
- With this approach we have

$$\text{spark}(\mathbf{\Phi}) =$$

$$= \min \left\{ \|\mathbf{x}\|_0 : \mathbf{x} \in \mathbb{R}^N \setminus \{0\}, \text{ such that } \mathbf{\Phi x} = \mathbf{0} \right\}.$$

[ACM12]

Full Spark systems

- For the system with full spark $\text{spark}\,(\mathbf{\Phi}) = D + 1$, i. e. each subsystem of D vectors of the system Φ consists of linearly independent vectors in $\mathbb{H}^D$.

Exact number for $\mathbb{R}^D$

- Vandermonde matrices allow to construct systems of vectors $\Phi = \{\varphi_i\}_{i=1}^N$ in $\mathbb{R}^D$ with full spark for any $N \geq D$.
- Thus for $\mathbb{R}^D$ there is the equality $N(D) = 2D - 1$, and this equality is realized on systems with full spark and only on them.
- The question about $N(D)$ in $\mathbb{C}^D$ is still open.

Frame for infinite dimensional Hilbert space

- **Definition.** Frame for the separable Hilbert space $\mathcal{H}$ is a family of vectors $\{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{H}$ such that there are constants $0 < A \leq B < \infty$ and for all $\mathbf{x} \in \mathcal{H}$,

$$A\|\mathbf{x}\|^2 \leq \sum_{j=1}^{\infty} |\langle \mathbf{x}, \varphi_j \rangle|^2 \leq B\|\mathbf{x}\|^2.$$

- The notions of the Frame and the complete system differs in infinite dimensional case. The completeness of the system follows from the definition of the frame, but there are complete systems, which are not frames.

Riesz basis and Example

- **Definition.** The sequence $\{\psi_j\}_{j=1}^{\infty}$ in the Hilbert space $\mathcal{H}$ is called a *Riesz sequence*, if there exist constants $0 < A \leq B < \infty$ such that for any sequence of scalars $\{a_j\}_{j=1}^{\infty}$ the following inequalities hold

$$A \sum_{j=1}^{\infty} |a_i|^2 \leq \left\| \sum_{j=1}^{\infty} a_j \psi_j \right\|^2 \leq B \sum_{j=1}^{\infty} |a_i|^2.$$

  If $\overline{\text{span}\left(\{\psi_j\}\right)} = \mathcal{H}$, $\{\psi_j\}_{j=1}^{\infty}$ is called *Riesz basis*.
- If $\{\psi_j\}_{j=1}^{\infty}$ is a Riesz basis, then $\{\psi_j + \psi_{j+1}\}_{j=1}^{\infty}$ is a complete system, but not a frame. [Christ02]

Feightinger conjectute

- The following theorem plays an important role in infinite dimensional case.
  Theorem. Any Frame $\{\varphi_j\}_{j=1}^{\infty}$ for Hilbert space $\mathcal{H}$ with $\|\varphi_j\| \geq \delta > 0, \ j = 1, 2, \ldots$ is a finite union of Riesz sequences.
  This theorem was called "Feightinger conjectute"for a long time.
- From this theorem we deduce that a frame with (CP) has norms tending to zero, that is uncomfortable for calculations.

(RMM) in Hilbert space

- (RMM)-property in Hilbert space can be formulated as follows:
  the equalities $|\langle \mathbf{x}, \varphi_j \rangle| = |\langle \mathbf{y}, \varphi_j \rangle|$ for every j imply that there is a unimodular scalar α such that $\mathbf{x} = \alpha \mathbf{y}$.
- **Theorem.** If the system $\Phi = \{\varphi_j\}_{j=1}^{\infty}$ has (RMM)-property, then Φ is complete and has (CP).

Proof

- If Φ is not complete, there exists $\mathbf{x} \neq \mathbf{0}$ such that $\langle \mathbf{x}, \varphi_j \rangle = 0$, $n = 1, 2, \ldots$. As $|\langle \mathbf{x}, \varphi_n \rangle| = |\langle \mathbf{0}, \varphi_n \rangle| = 0$, it contradicts to (RMM).
- The rest part of the proof repeats the finite dimensional case.

Inverse Theorem

- **Theorem.** If $\Phi \in (CP)$ in *real* Hilbert space $\mathcal{H}$, then $\Phi \in (RMM)$.
- Proof. Suppose $\Phi \notin (RMM)$. It means that there exist $\mathbf{x}, \mathbf{y} \in \mathcal{H}$ such that $|\langle \mathbf{x}, \varphi_n \rangle| = |\langle \mathbf{y}, \varphi_n \rangle| \, \forall n$ and $\mathbf{x} \neq \pm \mathbf{y}$.
- Let $T := \{n \in \mathbb{N} : \langle \mathbf{x}, \varphi_n \rangle = \langle \mathbf{y}, \varphi_n \rangle\}$, then $\langle \mathbf{x} - \mathbf{y}, \varphi_n \rangle = 0$, $n \in T$ and $\mathbf{x} - \mathbf{y} \neq \mathbf{0}$.

Proof (end)

- It means that $\overline{\text{span}}\,\{\varphi_n\}_{n\in T} \neq \mathcal{H}$.
- Likewise for $\mathbf{x}+\mathbf{y}$ and T^C.
- It contradicts to (CP).

(CP)-system

- There are only few examples of systems with (CP) or Full Spark in real ℓ_2. May be the simplest one was built in [ACCHT17].
- **Example.** Let $\{e_i\}_{i \in \mathbb{N}}$ be the canonical orthonormal basis for ℓ_2. Then the the family of vectors $\{e_i + e_j\}_{i<j}$ has (CP), and hence (RMM)-property.

Full Spark systems in ℓ_2

- **Definition** A family of vectors $\{\varphi_n\}_{n\in\mathbb{N}}$ is full spark in ℓ_2 if every infinite subset spans ℓ_2.
- A full spark set clearly has complement property and hence does phase retrieval in the infinite dimensional case.
- The existence of full spark systems for ℓ_2 was proved by R. Vershinin in 1998.
- More simple proof of such existence was given by P. Casazza [ACCHT17] by using the isomorphism between ℓ_2 and $L_2[0,1]$.

Full Spark systems in $L_2[0,1]$.

- Let

$$f_n(t) = e^{a_n t},\ 0 \le t \le 1,\ n = 1, 2, \ldots$$

be a sequence of functions in $L_2[0,1]$, where $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers (real or complex) such that $a_n \to a,\ n \to \infty$. The sequence $\{f_n\}_{n=1}^{\infty}$ is complete in $L_2[0,1]$, and any its subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ is also complete, as $a_{n_k} \to a,\ k \to \infty$.
- There are no exact constructions of such families in ℓ_2.

Unions of Riesz sequences

- After the proof of Feichtinger conjecture the interest increased to the unions of Riesz sequences. In particular, the following question was stated: what is the minimal number of such sequences, which union ensures (RMM)?
- The answer is known in $\mathbb{R}^D$, this number is 2. [CCPW16]
  For the real ℓ_2 the exact number is unknown, but it will be 2 or 3. [ACCHT17]

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