Classification of all constant solutions of SU(2) Yang–Mills equations

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Let us consider pseudo-Euclidean space $\mathbb{R}^{p,q}$, $n = p + q$ or Euclidean space $\mathbb{R}^n$ of arbitrary finite dimension n. We denote Cartesian coordinates by x^μ, $\mu = 1, \ldots, n$ and partial derivatives by $\partial_\mu = \partial/\partial x^\mu$. The metric tensor of $\mathbb{R}^{p,q}$ is given by the diagonal matrix $\eta = (\eta_{\mu\nu}) = (\eta^{\mu\nu}) = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ with p ones and q minus ones on the diagonal. We can raise or lower indices of components of tensor fields using metric tensor, for example, $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$.
Let us consider

$$\mathrm{G} = \mathrm{SU}(2) = \{S \in \mathrm{Mat}(2, \mathbb{C}) \,|\, S^\dagger S = I, \det S = 1\},$$
$$\mathfrak{g} = \mathfrak{su}(2) = \{S \in \mathrm{Mat}(2, \mathbb{C}) \,|\, S^\dagger = -S, \mathrm{tr} S = 0\}, \qquad \dim \mathfrak{g} = 3.$$

Let us consider the Yang–Mills equations

$$\partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = F_{\mu\nu}, \tag{1}$$
$$\partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = J^\nu, \tag{2}$$

where $A_\mu : \mathbb{R}^{p,q} \to \mathfrak{g}$ is the potential, $J^\nu : \mathbb{R}^{p,q} \to \mathfrak{g}$ is the non-Abelian current, $F_{\mu\nu} = -F_{\nu\mu} : \mathbb{R}^{p,q} \to \mathfrak{g}$ is the strength of the Yang–Mills field.

Let us substitute the components of the tensor $F^{\mu\nu}$ from (1) into (2):

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu - [A^\mu, A^\nu]) - [A_\mu, \partial^\mu A^\nu - \partial^\nu A^\mu - [A^\mu, A^\nu]] = J^\nu. \qquad (3)$$

We may verify that the current J^ν satisfies the non-Abelian conservation law

$$\partial_\nu J^\nu - [A_\nu, J^\nu] = 0. \qquad (4)$$

The Yang–Mills equations are gauge invariant w.r.t. the transformations

$$A_\mu \to S^{-1} A_\mu S - S^{-1}\partial_\mu S, \qquad F_{\mu\nu} \to S^{-1} F_{\mu\nu} S, \qquad J^\nu \to S^{-1} J^\nu S, \quad (5)$$

where $\quad S = S(x) : \mathbb{R}^{p,q} \to \mathrm{G}.$

Particular classes of solutions (monopoles, instantons, merons, etc.):
Wu T.T., Yang C.N. (1968), 't Hooft G. (1974), Polyakov A.M. (1975), Belavin A.A., Polyakov A.M., Schwartz A.S., Tyupkin Yu.S. (1975), Witten E. (1977), Atiyah M., Drinfeld V., Hitchin N., Manin Yu. (1978), de Alfaro V., Fubini S., Furlan G. (1976), ...

The well-known classes of solutions of the Yang–Mills equations are described in reviews:

Actor A., Classical solutions of SU(2) Yang–Mills theories, Rev.Mod.Phys. **51**(1979).

Suppose that A^μ and J^μ do not depend on $x \in \mathbb{R}^{p,q}$. We obtain the following algebraic system of equations

$$[A_\mu, [A^\mu, A^\nu]] = J^\nu, \qquad \nu = 1, \ldots, n. \tag{6}$$

We have the following expression for the strength of the Yang–Mills field

$$F^{\mu\nu} = -[A^\mu, A^\nu]. \tag{7}$$

We want to obtain all solutions $A^\mu \in \mathfrak{su}(2)$ of (6) for arbitrary $J^\nu \in \mathfrak{su}(2)$. Constant solutions of the Yang–Mills equations with zero current $J^\nu = 0$ were considered in the following papers:

- Schimming R.: On constant solutions of the Yang–Mills equations. Arch. Math. **24**:2, 65–73 (1988).
- Schimming R., Mundt E.: Constant potential solutions of the Yang–Mills equation. J. Math. Phys. **33**, 4250 (1992).

Let us consider the Pauli matrices σ^a, $a = 1, 2, 3$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

We can take the following basis of the Lie algebra $\mathfrak{su}(2)$

$$\tau^1 = \frac{\sigma^1}{2i}, \qquad \tau^2 = \frac{\sigma^2}{2i}, \qquad \tau^3 = \frac{\sigma^3}{2i}. \tag{9}$$

$$\text{with} \qquad (\tau^a)^\dagger = -\tau^a, \qquad \mathrm{tr}\,\tau^a = 0, \qquad [\tau^a, \tau^b] = \epsilon^{ab}{}_c \tau^c,$$

where the structural constants of the Lie algebra $\mathfrak{su}(2)$ are the antisymmetric Levi-Civita symbol, $\epsilon^{123} = 1$. For the potential and the current, we have

$$A^\mu = A^\mu{}_a \tau^a, \qquad J^\mu = J^\mu{}_a \tau^a, \qquad A^\mu{}_a, J^\mu{}_a \in \mathbb{R}. \tag{10}$$

Latin indices take values $a = 1, 2, 3$ and Greek indices take values $\mu = 1, 2, \ldots, n$. Substituting (10) into (6), we get

$$\boxed{A_{\mu c} A^\mu{}_a A^\nu{}_b \epsilon^{ab}{}_d \epsilon^{cd}{}_k = J^\nu{}_k, \qquad \nu = 1, \ldots, n, \qquad k = 1, 2, 3.} \tag{11}$$

We obtain $3n$ equations ($k = 1, 2, 3$, $\nu = 1, 2, \ldots, n$) for $3n$ unknown $A^\nu{}_k$ and $3n$ known $J^\nu{}_k$. We can consider (11) as a system of equations for elements of two matrices $A_{n\times 3} = (A^\nu{}_k)$ and $J_{n\times 3} = (J^\nu{}_k)$.

Lemma

The system of equations $\boxed{A_{\mu c}A^{\mu}{}_{a}A^{\nu}{}_{b}\epsilon^{ab}{}_{d}\epsilon^{cd}{}_{k} = J^{\nu}{}_{k}, \quad \nu = 1,\ldots,n, \quad k = 1,2,3,}$ *is invariant under the following transformations*

$$
\begin{array}{ll}
1) & A^{\mu}{}_{b} \to A^{\mu}{}_{a}p^{a}_{b}, \qquad J^{\mu}{}_{b} \to J^{\mu}{}_{a}p^{a}_{b}, \\
 & \textit{i.e.} \quad A \to AP, \qquad J \to JP, \qquad P = (p^{a}_{b}) \in \mathrm{SO}(3), \\
 & \textit{where} \quad A_{\mu} \to S^{-1}A_{\mu}S, \qquad J^{\nu} \to S^{-1}J^{\nu}S, \\
 & S^{-1}\tau^{a}S = p^{a}_{b}\tau^{b}, \qquad \pm S \in \mathrm{SU}(2) \simeq \mathrm{Spin}(3), \\
2) & A^{\nu}{}_{a} \to q^{\nu}_{\mu}A^{\mu}{}_{a}, \qquad J^{\nu}{}_{a} \to q^{\nu}_{\mu}J^{\mu}{}_{a}, \\
 & \textit{i.e.} \quad A \to QA, \qquad J \to QJ, \qquad Q = (q^{\mu}_{\nu}) \in \mathrm{O}(p,q), \\
 & \textit{where} \quad x^{\mu} \to q^{\mu}_{\nu}x^{\nu}.
\end{array}
$$

Combining gauge and orthogonal transformations, we conclude that the system is invariant under the transformation

$$
\begin{array}{l}
A^{\nu}_{b} \to q^{\nu}_{\mu}A^{\mu}{}_{a}p^{a}_{b}, \qquad J^{\nu}_{b} \to q^{\nu}_{\mu}J^{\mu}{}_{a}p^{a}_{b}, \\
\text{i.e.} \quad A \to QAP, \qquad J \to QJP, \qquad P \in \mathrm{SO}(3), \quad Q \in \mathrm{O}(p,q).
\end{array}
$$

Theorem (Singular Value Decomposition (SVD))

For an arbitrary real matrix $A_{n\times N}$ of the size $n \times N$, there exist orthogonal matrices $L_{n\times n} \in \mathrm{O}(n)$ and $R_{N\times N} \in \mathrm{O}(N)$ such that

$$L^{\mathrm{T}}_{n\times n} A_{n\times N} R_{N\times N} = D_{n\times N}, \tag{12}$$

where

$$D_{n\times N} = \mathrm{diag}(\mu_1, \ldots, \mu_s), \qquad s = \min(n, N), \qquad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_s \geq 0.$$

The numbers $\mu_1, \ldots, \mu_s$ are called the singular values, the columns l_i of the matrix L are called the left singular vectors, the columns r_i of the matrix R are called the right singular vectors.
The columns of the matrix L are eigenvectors of the matrix AA^{T}, and the columns of the matrix R are eigenvectors of the matrix $A^{\mathrm{T}}A$.
The squares of singular values are eigenvalues of the corresponding matrices.
From this fact, it follows that singular values are uniquely determined.

Theorem (for Euclidean case)

Let $A = (A^{\nu}{}_{k})$, $J = (J^{\nu}{}_{k})$ *satisfy the system of* $3n$ *cubic equations*

$$\boxed{A_{\mu c} A^{\mu}{}_{a} A^{\nu}{}_{b} \epsilon^{ab}{}_{d} \epsilon^{cd}{}_{k} = J^{\nu}{}_{k}, \qquad \nu = 1, \ldots, n, \qquad k = 1, 2, 3.} \tag{13}$$

Then there exist matrices $P \in \mathrm{SO}(3)$ *and* $Q \in \mathrm{O}(n)$ *such that* QAP *is diagonal. For all such matrices* P *and* Q, *the matrix* QJP *is diagonal too and the system (13) takes the following form under the transformation* $A \to QAP, J \to QJP$*:*

in the case $n = 2$*:*

$$\begin{aligned} -a_1(a_2)^2 &= j_1, \\ -a_2(a_1)^2 &= j_2, \end{aligned} \tag{14}$$

in the cases $n \geq 3$ *:*

$$\begin{aligned} -a_1((a_2)^2 + (a_3)^2) &= j_1, \\ -a_2((a_1)^2 + (a_3)^2) &= j_2, \\ -a_3((a_1)^2 + (a_2)^2) &= j_3. \end{aligned} \tag{15}$$

We denote diagonal elements of the matrix QAP *by* a_1, a_2, a_3 *(or* a_1, a_2*) and diagonal elements of the matrix* QJP *by* j_1, j_2, j_3 *(or* j_1, j_2*).*

Suppose we have known matrix J and want to obtain all solutions A of the system (13). We can always calculate singular values j_1, j_2, j_3 of J and solve the system (15). Finally, we obtain all solutions $A_D = \mathrm{diag}(a_1, a_2, a_3)$ of the system (13) but in some other system of coordinates depending on $Q \in \mathrm{O}(n)$ and with gauge fixing depending on $P \in \mathrm{SO}(3)$. The matrix $A = Q^{-1} A_D P^{-1}$ will be solution of the system (13) in the original system of coordinates and with the original gauge fixing.

Note that $Q^{-1} Q_1^{-1} A_D P_1^{-1} P^{-1}$, for all $Q_1 \in \mathrm{O}(n)$ and $P_1 \in \mathrm{SO}(3)$ such that $Q_1 J_D P_1 = J_D$, $J_D = \mathrm{diag}(j_1, j_2, j_3)$, will be also solutions of the system (13) in the original system of coordinates and with the original gauge fixing because of Lemma.

Example. If the matrix $J = 0$, then all singular values of this matrix equal zero and we can take $Q = P = I$ for its SVD. We solve the system (15) for $j_1 = j_2 = j_3 = 0$ and obtain all solutions $A_D = \mathrm{diag}(a_1, a_2, a_3)$ of this system. We have $Q_1 J_D P_1 = J_D$ for $J_D = 0$ and any $Q_1 \in \mathrm{O}(n)$, $P_1 \in \mathrm{SO}(3)$. Therefore, the matrices $Q_1 A_D P_1$ for all $Q_1 \in \mathrm{O}(n)$ and $P_1 \in \mathrm{SO}(3)$ will be solutions of the system (13) because of Lemma.

The systems (14), (15) can be rewritten in the following way using $b_k := -a_k$:

$$n = 2: \quad b_1 b_2^2 = j_1, \quad b_2 b_1^2 = j_2, \tag{16}$$

$$n \geq 3: \quad b_1(b_2^2 + b_3^2) = j_1, \quad b_2(b_1^2 + b_3^2) = j_2, \quad b_3(b_1^2 + b_2^2) = j_3. \tag{17}$$

The system (17) has the following symmetry (similarly for (16)): if we change the sign of some j_k, $k = 1, 2, 3$, then we must change the sign of the corresponding b_k, $k = 1, 2, 3$. Using SVD, we can always get nonnegative j_k, $k = 1, 2, 3$.

Lemma. The system of equations (16) has the following general solution:

1. in the case $j_1 = j_2 = 0$, has solutions $(b_1, 0)$, $(0, b_2)$ for all $b_1, b_2 \in \mathbb{R}$;
2. in the cases $j_1 = 0$, $j_2 \neq 0$; $j_1 \neq 0$, $j_2 = 0$, has no solutions;
3. in the case $j_1 \neq 0$, $j_2 \neq 0$, has a unique solution

$$b_1 = \sqrt[3]{\frac{j_2^2}{j_1}}, \qquad b_2 = \sqrt[3]{\frac{j_1^2}{j_2}}.$$

Lemma. If the system (17) has a solution (b_1, b_2, b_3), where $b_1 \neq 0$, $b_2 \neq 0$, $b_3 \neq 0$, then this system has also a solution $(\frac{K}{b_1}, \frac{K}{b_2}, \frac{K}{b_3})$, where $K = (b_1 b_2 b_3)^{\frac{2}{3}}$.

Example. Let us take $j_1 = 13$, $j_2 = 20$, $j_3 = 15$. Then the system (17) has solutions $(b_1, b_2, b_3) = (1, 2, 3)$ and $(6^{\frac{2}{3}}, \frac{6^{\frac{2}{3}}}{2}, \frac{6^{\frac{2}{3}}}{3})$.

Lemma. The system of equations

$$b_1(b_2^2+b_3^2)=j_1,\quad b_2(b_1^2+b_3^2)=j_2,\quad b_3(b_1^2+b_2^2)=j_3$$

has the following general solution:
1) in the case $j_1=j_2=j_3=0$, has solutions

$$(b_1,0,0),\quad (0,b_2,0),\quad \text{and}\quad (0,0,b_3),\qquad b_1,b_2,b_3\in\mathbb{R};$$

2) in the cases $j_1=j_2=0$, $j_3\neq 0$ (or similar cases with circular permutation), has no solutions;
3) in the case $j_1\neq 0$, $j_2\neq 0$, $j_3=0$ (or similar cases with circular permutation), has a unique solution

$$b_1=\sqrt[3]{\frac{j_2^2}{j_1}},\qquad b_2=\sqrt[3]{\frac{j_1^2}{j_2}},\qquad b_3=0;$$

4) in the case $j_1=j_2=j_3\neq 0$, has a unique solution

$$b_1=b_2=b_3=\sqrt[3]{\frac{j_1}{2}};$$

5) in the case of not all the same $j_1, j_2, j_3 > 0$, has two solutions

$$(b_{1+}, b_{2+}, b_{3+}), \qquad (b_{1-}, b_{2-}, b_{3-})$$

with the following expression for K from the previous lemma

$$K := b_{1+}b_{1-} = b_{2+}b_{2-} = b_{3+}b_{3-} = (b_{1+}b_{2+}b_{3+})^{\frac{2}{3}} = (b_{1-}b_{2-}b_{3-})^{\frac{2}{3}} :$$

5a) in the case $j_1 = j_2 > j_3 > 0$ (or similar cases with circular permutation)

$$b_{1\pm} = b_{2\pm} = \sqrt[3]{\frac{j_3}{2z_\pm}}, \quad b_{3\pm} = z_\pm b_{1\pm}, \quad z_\pm = \frac{j_1 \pm \sqrt{j_1^2 - j_3^2}}{j_3}.$$

$$\text{Moreover,} \quad z_+ z_- = 1, \quad K = (\frac{j_3}{2})^{\frac{2}{3}}.$$

5b) in the case $j_3 > j_1 = j_2 > 0$ (or similar cases with circular permutation):

$$b_{1\pm} = \frac{1}{w_\pm} b_3, \quad b_{2\pm} = w_\pm b_3, \quad b_{3\pm} = b_3 = \sqrt[3]{\frac{j_1}{s}},$$

$$w_\pm = \frac{s \pm \sqrt{s^2 - 4}}{2}, \quad s = \frac{j_3 + \sqrt{j_3^2 + 8j_1^2}}{2j_1}.$$

$$\text{Moreover,} \quad w_+ w_- = 1, \quad b_{1\pm} = b_{2\mp}, \quad K = (\frac{j_1}{s})^{\frac{2}{3}}.$$

5c) in the case of all different $j_1, j_2, j_3 > 0$:

$$b_{1\pm} = \sqrt[3]{\frac{j_3}{t_0 y_\pm z_\pm}}, \quad b_{2\pm} = y_\pm b_{1\pm}, \quad b_{3\pm} = z_\pm b_{1\pm},$$

$$z_\pm = \sqrt{\frac{y_\pm(j_1 - j_2 y_\pm)}{j_2 - j_1 y_\pm}}, \quad y_\pm = \frac{t_0 \pm \sqrt{t_0^2 - 4}}{2},$$

where $t_0 > 2$ is the solution (it always exists, moreover, it is bigger than $\frac{j_2}{j_1} + \frac{j_1}{j_2}$) of the cubic equation $\qquad j_1 j_2 t^3 - (j_1^2 + j_2^2 + j_3^2) t^2 + 4 j_3^2 = 0.$

$$\text{Moreover,} \quad y_+ y_- = 1, \quad z_+ z_- = 1, \quad K = (\frac{j_3}{t_0})^{\frac{2}{3}}.$$

We can use the explicit Vieta or Cardano formulas for t_0:

$$t_0 = \Omega + 2\Omega \cos(\frac{1}{3} \arccos(1 - \frac{2\beta}{\Omega^3})),$$

$$\Omega := \frac{\alpha + \beta}{3}, \qquad \alpha := A + \frac{1}{A} > 2, \qquad \beta := \frac{B^2}{A}, \qquad A := \frac{j_2}{j_1}, \qquad B := \frac{j_3}{j_1},$$

$$t_0 = \Omega + L + \frac{\Omega^2}{L}, \qquad L := \sqrt[3]{\Omega^3 - 2\beta + 2\sqrt{\beta(\beta - \Omega^3)}}.$$

Consequences for the strength of the Yang–Mills field.

In the case of the constant potential A^μ of the Yang–Mills field, we have the following expression for the strength

$$F^{\mu\nu}=-[A^\mu,A^\nu]=-[A^\mu{}_a\tau^a,A^\nu{}_b\tau^b]=-A^\mu{}_aA^\nu{}_b\epsilon^{ab}{}_c\tau^c=F^{\mu\nu}{}_c\tau^c. \qquad (18)$$

We take a system of coordinates depending on $Q\in\mathrm{O}(n)$ and a gauge fixing depending on $P\in\mathrm{SO}(3)$ such that the matrices A and J are diagonal.

In the case of dimension $n=2$:

1. In the case $J=0$, we have $A=0$ or $A=\begin{pmatrix} a & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $a\in\mathbb{R}\setminus\{0\}$, $F=0$.
2. In the case $\mathrm{rank}(J)=1$, we have no constant solutions.
3. In the case $\mathrm{rank}(J)=2$, we have a unique solution

$$A=\begin{pmatrix} a_1 & 0 & 0\\ 0 & a_2 & 0\end{pmatrix},\qquad a_1=-\sqrt[3]{\frac{j_2^2}{j_1}},\qquad a_2=\sqrt[3]{\frac{j_1^2}{j_2}}.$$

For the strength, we have the following nonzero components of the strength

$$F^{12}=-F^{21}=-\sqrt[3]{j_1j_2}\tau^3. \qquad (19)$$

We have the following expression for the invariant:

$$F^2=F_{\mu\nu}F^{\mu\nu}=-\frac{1}{2}\sqrt[3]{(j_1j_2)^2}I. \qquad (20)$$

In the cases of dimension $n \geq 3$:

1. In the case $J = 0$, we have nonzero potential A^μ but zero strength $F^{\mu\nu} = 0$.
2. In the case $\mathrm{rank}(J) = 1$, we have no constant solutions.
3. In the case $\mathrm{rank}(J) = 2$, we have a unique solution. We have again (19) and (20), where j_1, j_2, and $j_3 = 0$ are singular values of the matrix J.
4. In the case $\mathrm{rank}(J) = 3$, we have one or two solutions.
   1) In the case of all the same singular values $j := j_1 = j_2 = j_3 \neq 0$, we have a unique solution

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \end{pmatrix}, \qquad a = -\sqrt[3]{\frac{j}{2}}. \tag{21}$$

$$F^{12} = -F^{21} = -\sqrt[3]{\frac{j^2}{4}}\tau^3,\ F^{23} = -F^{32} = -\sqrt[3]{\frac{j^2}{4}}\tau^1,\ F^{31} = -F^{13} = -\sqrt[3]{\frac{j^2}{4}}\tau^2.$$

In this case, we have $F^2 = F_{\mu\nu}F^{\mu\nu} = \frac{-3}{2}\sqrt[3]{\frac{j^4}{16}} I \neq 0$.

2) In the case of not all the same singular values j_1, j_2, j_3 of the matrix J, we have two different solutions

$$A = \begin{pmatrix} -b_{1\pm} & 0 & 0 \\ 0 & -b_{2\pm} & 0 \\ 0 & 0 & -b_{3\pm} \\ 0 & 0 & 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \end{pmatrix}, \qquad (22)$$

where $b_{k\pm}$, $k = 1, 2, 3$ are from Case (v) of Lemma. We have

$$F^{12}_{\pm} = -F^{21}_{\pm} = -b_{1\pm} b_{2\pm} \tau^3, \qquad F^{23}_{\pm} = -F^{32}_{\pm} = -b_{2\pm} b_{3\pm} \tau^1,$$

$$F^{31}_{\pm} = -F^{13}_{\pm} = -b_{3\pm} b_{1\pm} \tau^2,$$

$$F^2_{\pm} = -\frac{1}{2}((b_{1\pm} b_{2\pm})^2 + (b_{2\pm} b_{3\pm})^2 + (b_{3\pm} b_{1\pm})^2) I \neq 0.$$

Lemma. In the case of not all the same j_1, j_2, j_3, the invariant $F^2_{\pm}$ takes the form:

1. in the case $j_1 = j_2 > j_3 > 0$ (or similar cases with circular permutation):

$$F^2_{\pm} = \frac{-K^2(1+2z^2_{\pm})}{2z_{\pm}^{\frac{4}{3}}} I, \qquad F^2_{+} \neq F^2_{-}, \tag{23}$$

$$\text{where} \quad z_{\pm} = \frac{j_1 \pm \sqrt{j_1^2 - j_3^2}}{j_3}, \quad K = (\frac{j_3}{2})^{\frac{2}{3}}.$$

2. in the case $j_3 > j_1 = j_2 > 0$ (or similar cases with circular permutation):

$$F^2_{\pm} = \frac{-K^2(s^2-1)}{2} I, \qquad F^2_{+} = F^2_{-}, \tag{24}$$

$$\text{where} \quad s = \frac{j_3 + \sqrt{j_3^2 + 8j_1^2}}{2j_1} > 2, \quad K = (\frac{j_1}{s})^{\frac{2}{3}}.$$

3. in the case of all different j_1, j_2, $j_3 > 0$:

$$F^2_{\pm} = \frac{-K^2(y^2_{\pm} + z^2_{\pm} + y^2_{\pm} z^2_{\pm})}{2(y_{\pm} z_{\pm})^{\frac{4}{3}}} I, \qquad F^2_{+} \neq F^2_{-}, \tag{25}$$

where $K = (\frac{j_3}{t_0})^{\frac{2}{3}}$, and $y_{\pm}$, $z_{\pm}$, t_0 are from Case 5c) of previous Lemma.

Hyperbolic singular value decomposition

- Bojanczyk A.W., Onn R., Steinhardt A.O. Existence of the hyperbolic singular value decomposition // Linear Algebra and its Applications. 1993. 185: 21–30.
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- Shirokov D.S. A note on the hyperbolic singular value decomposition without hyperexchange matrices // Journal of Computational and Applied Mathematics. 2021. 391, 113450.

Theorem (HSVD)

For arbitrary matrix $A_{n\times N}\in\mathrm{Mat}(\mathbb{R})$, there exist matrices $R\in\mathrm{O}(N)$ and $L\in\mathrm{O}(p,q)$ such that

$$L^{\mathrm{T}}AR=\Sigma^A,\qquad \Sigma^A=\left(\begin{array}{cccc} X_x & 0 & 0 & 0\\ 0 & 0 & I_d & 0\\ 0 & 0 & 0 & 0\\ \hline 0 & Y_y & 0 & 0\\ 0 & 0 & I_d & 0\\ 0 & 0 & 0 & 0\end{array}\right)\in\mathrm{Mat}_{n\times N}(\mathbb{R}), \tag{26}$$

where the first block of the matrix Σ^A has p rows and the second block has q rows, X_x and Y_y are diagonal matrices of the corresponding sizes x and y with all positive uniquely determined diagonal elements, I_d is the identity matrix of size d. Here we have

$$d=\mathrm{rank}(A)-\mathrm{rank}(A^{\mathrm{T}}\eta A),\qquad x+y=\mathrm{rank}(A^{\mathrm{T}}\eta A),$$

x is the number of positive eigenvalues of the matrix $A^{\mathrm{T}}\eta A$, y is the number of negative eigenvalues of the matrix $A^{\mathrm{T}}\eta A$.

Moreover, choosing L and R, we can change the order of all columns of the matrix Σ^A. Also we can change the order of rows in each of two blocks of the matrix Σ^A, but we can not change the order of two rows in different blocks. Thus we can always arrange diagonal elements of the matrices X_x and Y_y in decreasing order.

Let us call Σ^A (26), where all diagonal elements of the matrices X_x and Y_y are positive and in decreasing order, the *canonical form* of the matrix A. The canonical form is uniquely determined for any matrix A, the corresponding matrices L and R are not uniquely determined.

The hyperbolic singular values (elements of the diagonal matrices X_x and Y_y) of the matrix A are square roots of the modules of the eigenvalues of the matrix $A^{\mathrm{T}}\eta A$. The columns of the matrix R are eigenvectors of the matrix $A^{\mathrm{T}}\eta A$. The columns of the matrix L are eigenvectors of the matrix ηAA^{T} (in the case $d=0$) or eigenvectors and generalized eigenvectors of the matrix ηAA^{T} (in the case $d\neq 0$).

The ordinary singular value decomposition (SVD) is the particular case of the hyperbolic singular value decomposition (HSVD). In the case $n=p$ and $q=0$, the parameter d is equal to zero $d=\mathrm{rank}(A)-\mathrm{rank}(A^{\mathrm{T}}A)=0$.

Theorem. Let $A=(A^\nu{}_k)$, $J=(J^\nu{}_k)$ satisfy the system of $3n$ cubic equations

$$A_{\mu c}A^\mu{}_a A^\nu{}_b \epsilon^{ab}{}_d \epsilon^{cd}{}_k = J^\nu{}_k, \qquad \nu=1,\ldots,n, \qquad k=1,2,3. \tag{27}$$

Then: 1) There exist matrices $P\in\mathrm{SO}(3)$ and $Q\in\mathrm{O}(p,q)$ such that the matrix QAP is in the canonical form (with parameters x_A, y_A, d_A)

$$\Sigma^A = QAP = \left(\begin{array}{cccc} X_{x_A} & 0 & 0 & 0\\ 0 & 0 & I_{d_A} & 0\\ 0 & 0 & 0 & 0\\ \hline 0 & Y_{y_A} & 0 & 0\\ 0 & 0 & I_{d_A} & 0\\ 0 & 0 & 0 & 0 \end{array}\right).$$

For all such matrices P and Q, the matrix QJP has the following form

$$QJP = \left(\begin{array}{cccc} Z_{x_A} & 0 & 0 & 0\\ 0 & 0 & \alpha I_{d_A} & 0\\ 0 & 0 & 0 & 0\\ \hline 0 & W_{y_A} & 0 & 0\\ 0 & 0 & \alpha I_{d_A} & 0\\ 0 & 0 & 0 & 0 \end{array}\right),$$

where elements of the diagonal matrices Z and W are real numbers (can be zero), $\alpha\in\mathbb{R}$ (can be zero).
2) For parameters of the matrices A and J, we have:

$$x_J \le x_A, \qquad y_J \le y_A, \qquad d_J = d_A > 0 \qquad \text{or} \qquad d_J = 0,\ d_A \ge 0.$$

3) There exist matrices $P \in \mathrm{SO}(3)$ and $Q \in \mathrm{O}(p,q)$ such that the matrix QJP is in the canonical form (with parameters x_J, y_J, d_J)

$$\Sigma^J = QJP = \left(\begin{array}{cccc} X_{x_J} & 0 & 0 & 0 \\ 0 & 0 & I_{d_J} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & Y_{y_J} & 0 & 0 \\ 0 & 0 & I_{d_J} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

and QAP has the following form

$$QAP = \left(\begin{array}{ccccccc} K_{x_J} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta I_{d_J} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{x_A - x_J} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{d_A - d_J} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & M_{y_J} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta I_{d_J} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{y_A - y_J} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{d_A - d_J} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

where $\beta \in \mathbb{R} \setminus \{0\}$; elements of the diagonal matrices K, L, M, N are arbitrary nonzero real numbers.

Summary for the case $\mathbb{R}^{1,1}$:

d_J	x_J	y_J	d_A	x_A	y_A	A	F	F^2
0	1	1	0	1	1	1: see (28)	1: see (29)	1: see (30)
0	1	0				$\varnothing$	$\varnothing$	$\varnothing$
0	0	1				$\varnothing$	$\varnothing$	$\varnothing$
0	0	0	0	0	0	$A=0$	$F=0$	$F^2=0$
			0	1	0	∞: see (31)	$F=0$	$F^2=0$
			0	0	1	∞: see (32)	$F=0$	$F^2=0$
			1	0	0	1: see (33)	$F=0$	$F^2=0$
1	0	0				$\varnothing$	$\varnothing$	$\varnothing$

$$A=\left(\begin{array}{ccc} a_1 & 0 & 0 \\ \hline 0 & a_2 & 0 \end{array}\right),\qquad a_1=\sqrt[3]{\frac{j_2^2}{j_1}},\qquad a_2=-\sqrt[3]{\frac{j_1^2}{j_2}}. \tag{28}$$

$$F^{12}=-F^{21}=\sqrt[3]{j_1 j_2}\,\tau^3, \tag{29}$$

$$F^2=F_{\mu\nu}F^{\mu\nu}=\frac{1}{2}\sqrt[3]{(j_1 j_2)^2}\,I_2\neq 0. \tag{30}$$

$$A=\left(\begin{array}{ccc} a_1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right),\qquad a_1\in\mathbb{R}\setminus\{0\}. \tag{31}$$

$$A=\left(\begin{array}{ccc} 0 & 0 & 0 \\ \hline a_1 & 0 & 0 \end{array}\right),\qquad a_1\in\mathbb{R}\setminus\{0\}. \tag{32}$$

$$A=\left(\begin{array}{ccc} 1 & 0 & 0 \\ \hline 1 & 0 & 0 \end{array}\right). \tag{33}$$

Summary for the case $\mathbb{R}^{p,q}$, $p+q\geq 2$:

Let we have the Yang-Mills equations (11) in pseudo-Euclidean space $\mathbb{R}^{p,q}$, $p\geq 1$, $q\geq 1$, with the known constant current $J^\mu = J^\mu_a \tau^a$, the unknown constant potential $A^\mu = A^\mu_a \tau^a$, and the corresponding unknown strength $F^{\mu\nu}$.

1. For the matrix $J=(J^\mu_a)$, we calculate three parameters d_J, x_J, y_J, which are uniquely determined, $d_J+x_J+y_J=\mathrm{rank}(J)$, x_A is the number of positive eigenvalues of the matrix $J^{\mathrm{T}}\eta J$, y_A is the number of negative eigenvalues of the matrix $J^{\mathrm{T}}\eta J$, $d_J=\mathrm{rank}(J)-\mathrm{rank}(J^{\mathrm{T}}\eta J)$.
2. We calculate the hyperbolic singular values j_1, j_2, j_3 of the matrix J, which are uniquely determined. The corresponding matrices $Q\in\mathrm{O}(p,q)$, $P\in\mathrm{SO}(3)$ related to the HSVD are not uniquely determined.
3. For the corresponding p, q, d_J, x_J, y_J, j_1, j_2, j_3, we get all solutions of (11).
   1. For each current J, the explicit form of these solutions in terms of the potential A and the strength F is given in specific coordinate system (which is determined by the matrix $Q\in\mathrm{O}(p,q)$ related to the HSVD) and with specific gauge fixing (which is determined by the matrix $P\in\mathrm{SO}(3)$ related to the HSVD). The connection $S\in\mathrm{SU}(2)$, which we use in the gauge fixing, is the two-sheeted covering of the matrix $P\in\mathrm{SO}(3)$.
   2. We calculate the invariant F^2 for all constant solutions, which is important from a physical point of view. The expression F^2 is gauge invariant and is invariant under pseudo-orthogonal transformations of coordinates. It is present in the Lagrangian of the Yang-Mills field.

We summarize the results for the case of arbitrary pseudo-Euclidean space $\mathbb{R}^{p,q}$ in the tables. We remind that we consider only the cases with positive numbers (hyperbolic singular values of J) j_1, j_2, j_3. In the columns "A", "F", and "F^2", we indicate the number of solutions in terms of A, F, and F^2. In the column "add.cond.", the additional conditions on the hyperbolic singular values j_1, j_2, and j_3 are indicated.
We use the notation $B^* := \frac{(s^*)^2+2}{s^*} \approx 7.66486 > 2\sqrt{2}$, where

$$s^* := \sqrt{13 + \sqrt{193 - 6^{\frac{4}{3}}} + \sqrt{386 + 6^{\frac{4}{3}} + \frac{5362}{193 - 6^{\frac{4}{3}}}}} \approx 7.39438.$$

p, q	d_J	x_J	y_J	add.cond.	d_A	x_A	y_A	A	F	F^2
$p \geq 3, q \geq 1$	0	3	0	$j_1 = j_2 = j_3$	0	3	0	1	1	1
$p \geq 3, q \geq 1$	0	3	0	$j_1 = j_2 > j_3$	0	3	0	2	2	2
$p \geq 3, q \geq 1$	0	3	0	$j_3 > j_1 = j_2$	0	3	0	2	2	1
$p \geq 3, q \geq 1$	0	3	0	all different j_1, j_2, j_3	0	3	0	2	2	2
$p \geq 1, q \geq 3$	0	0	3	$j_1 = j_2 = j_3$	0	0	3	1	1	1
$p \geq 1, q \geq 3$	0	0	3	$j_1 = j_2 > j_3$	0	0	3	2	2	2
$p \geq 1, q \geq 3$	0	0	3	$j_3 > j_1 = j_2$	0	0	3	2	2	1
$p \geq 1, q \geq 3$	0	0	3	all different j_1, j_2, j_3	0	0	3	2	2	2
$p \geq 2, q \geq 1$	0	2	1	$j_1 = j_2 < \frac{j_3}{2\sqrt{2}}, \frac{j_3}{j_1} = B^*$	0	2	1	6	6	3
$p \geq 2, q \geq 1$	0	2	1	$j_1 = j_2 < \frac{j_3}{2\sqrt{2}}, \frac{j_3}{j_1} \neq B^*$	0	2	1	6	6	4
$p \geq 2, q \geq 1$	0	2	1	$j_1 = j_2 = \frac{j_3}{2\sqrt{2}}$	0	2	1	4	4	$F^2 = 0$ and 2
$p \geq 2, q \geq 1$	0	2	1	$j_1 = j_2 > \frac{j_3}{2\sqrt{2}}$	0	2	1	2	2	2
$p \geq 2, q \geq 1$	0	2	1	$j_1 \neq j_2, j_3^{\frac{2}{3}} > j_2^{\frac{2}{3}} + j_1^{\frac{2}{3}}$	0	2	1	6	6	2-6
$p \geq 2, q \geq 1$	0	2	1	$j_1 \neq j_2, j_3^{\frac{2}{3}} = j_2^{\frac{2}{3}} + j_1^{\frac{2}{3}}$	0	2	1	4	4	$F^2 = 0$ and 3
$p \geq 2, q \geq 1$	0	2	1	$j_1 \neq j_2, j_3^{\frac{2}{3}} < j_2^{\frac{2}{3}} + j_1^{\frac{2}{3}}$	0	2	1	2	2	2
$p \geq 1, q \geq 2$	0	1	2	$j_3 = j_1 < \frac{j_2}{2\sqrt{2}}, \frac{j_2}{j_3} = B^*$	0	1	2	6	6	3
$p \geq 1, q \geq 2$	0	1	2	$j_3 = j_1 < \frac{j_2}{2\sqrt{2}}, \frac{j_2}{j_3} \neq B^*$	0	1	2	6	6	4
$p \geq 1, q \geq 2$	0	1	2	$j_3 = j_1 = \frac{j_2}{2\sqrt{2}}$	0	1	2	4	4	$F^2 = 0$ and 2
$p \geq 1, q \geq 2$	0	1	2	$j_3 = j_1 > \frac{j_2}{2\sqrt{2}}$	0	1	2	2	2	2
$p \geq 1, q \geq 2$	0	1	2	$j_3 \neq j_1, j_2^{\frac{2}{3}} > j_3^{\frac{2}{3}} + j_1^{\frac{2}{3}}$	0	1	2	6	6	2-6
$p \geq 1, q \geq 2$	0	1	2	$j_3 \neq j_1, j_2^{\frac{2}{3}} = j_3^{\frac{2}{3}} + j_1^{\frac{2}{3}}$	0	1	2	4	4	$F^2 = 0$ and 3
$p \geq 1, q \geq 2$	0	1	2	$j_3 \neq j_1, j_2^{\frac{2}{3}} < j_3^{\frac{2}{3}} + j_1^{\frac{2}{3}}$	0	1	2	2	2	2

p, q	d_J	x_J	y_J	add.cond.	d_A	x_A	y_A	A	F	F^2
$p \geq 2, q \geq 1$	0	2	0		0	2	0	1	1	1
$p \geq 1, q \geq 2$	0	0	2		0	0	2	1	1	1
$p \geq 1, q \geq 1$	0	1	1		0	1	1	1	1	1
$p \geq 2, q \geq 2$	0	1	1	$j_1 = j_2$	0	2	2	1	1	1
$p \geq 2, q \geq 1$	0	1	1	$j_2 > j_1$	0	2	1	4	4	2
$p \geq 1, q \geq 2$	0	1	1	$j_1 > j_2$	0	1	2	4	4	2
$p \geq 1, q = 1$	0	1	0					$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 1, q \geq 2$	0	1	0		0	1	2	4	4	1
$p = 1, q \geq 1$	0	0	1					$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 2, q \geq 1$	0	0	1		0	2	1	4	4	1
$p \geq 1, q \geq 1$	0	0	0		0	0	0	$A = 0$	$F = 0$	$F^2 = 0$
$p \geq 1, q \geq 1$	0	0	0		0	1	0	∞	$F = 0$	$F^2 = 0$
$p \geq 1, q \geq 1$	0	0	0		0	0	1	∞	$F = 0$	$F^2 = 0$
$p \geq 1, q \geq 1$	0	0	0		1	0	0	1	$F = 0$	$F^2 = 0$
$p \geq 2, q \geq 2$	0	0	0		2	0	0	1	1	$F^2 = 0$
$p \geq 3, q \geq 3$	0	0	0		3	0	0	1	1	$F^2 = 0$
$p \geq 3, q \geq 1$	1	2	0		1	2	0	1	1	1
$p \geq 1, q \geq 3$	1	0	2		1	0	2	1	1	1
$p \geq 2, q \geq 2$	1	1	1	$j_1 = j_2$				$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 2, q \geq 2$	1	1	1	$j_1 \neq j_2$	1	1	1	1	1	1
$p \geq 2, q \geq 1$	1	1	0					$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 1, q \geq 2$	1	0	1					$\varnothing$	$\varnothing$	$\varnothing$
$p = 1, q = 1$	1	0	0					$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 2, q \geq 1$	1	0	0		1	1	0	∞	∞	$F^2 = 0$
$p \geq 1, q \geq 2$	1	0	0		1	0	1	∞	∞	$F^2 = 0$
$p \geq 3, q \geq 2$	2	1	0					$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 2, q \geq 3$	2	0	1					$\varnothing$	$\varnothing$	$\varnothing$
$p = 2, q = 2$	2	0	0					$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 3, q \geq 2$	2	0	0		2	1	0	∞	∞	$F^2 = 0$
$p \geq 2, q \geq 3$	2	0	0		2	0	1	∞	∞	$F^2 = 0$
$p \geq 3, q \geq 3$	3	0	0					$\varnothing$	$\varnothing$	$\varnothing$

Nonconstant solutions in form of perturbation theory series

Let us consider nonconstant solutions of the Yang–Mills equations

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu - [A^\mu, A^\nu]) - [A_\mu, \partial^\mu A^\nu - \partial^\nu A^\mu - [A^\mu, A^\nu]] = J^\nu. \tag{34}$$

in the form of series of perturbation theory near the constant solutions. Denote the known constant solutions by $\overset{0}{A}{}^\mu$. We can take the small parameter $\varepsilon << 1$ ans substitute the expression $A^\mu = \sum_{k=0}^\infty \varepsilon^k \overset{k}{A}{}^\mu$ into (34) with constant J^ν. We get $\sum_{k=0}^\infty \varepsilon^k \overset{k}{Q}{}^\nu = J^\nu$, where $\overset{k}{Q}{}^\nu$ are some differential expressions that depend on $\overset{0}{A}{}^\mu, \ldots, \overset{k}{A}{}^\mu$ for each $k = 0, 1, \ldots$. For the first approximation, we get the system of linear partial differential equations with constant coefficients $\overset{1}{Q}{}^\nu(\overset{0}{A}{}^\mu, \overset{1}{A}{}^\mu) = 0$. We can take some solution of this system $\overset{1}{A}{}^\mu$ and substitute it and the constant solution $\overset{0}{A}{}^\mu$ into the system $\overset{2}{Q}{}^\nu(\overset{0}{A}{}^\mu, \overset{1}{A}{}^\mu, \overset{2}{A}{}^\mu) = 0$. We get a system of linear partial differential equations with variable coefficients (dependent on $x \in \mathbb{R}^{1,3}$) for variables $\overset{2}{A}{}^\mu$. In the same way, we can get $\overset{k}{A}{}^\mu$ for any $k = 0, 1, 2, \ldots$ This procedure allow us to obtain approximate solutions of the $\mathrm{SU}(2)$ Yang–Mills equations up to terms of order k.

- We obtain all constant solutions of $\mathrm{SU}(2)$ Yang–Mills equations in $\mathbb{R}^n$ for arbitrary current.
- We prove that the number (0, 1, or 2) of solutions in terms of the strength F depends on the singular values of the matrix J. The explicit form of these solutions and the invariant F^2 can always be written using these singular values.
- We solve the same problem in the case of pseudo-Euclidean space $\mathbb{R}^{p,q}$ of arbitrary finite dimension $n = p + q$, in particular, for the case of Minkowski space $\mathbb{R}^{1,3}$. We use HSVD.
- We obtain all constant solutions of Yang–Mills–Dirac equations and Yang–Mills–Proca equations in the case of the Lie group $\mathrm{SU}(2)$.
- We can consider nonconstant solutions of the Yang–Mills equations in the form of series of perturbation theory using all constant solutions as a zeroth approximation. The problem reduces to solving systems of linear partial differential equations.
- We obtain all plane-wave solutions of the Yang–Mills equations with $\mathrm{SU}(2)$ gauge symmetry and zero current in (pseudo)Euclidean space of arbitrary dimension.
- We hope that the results can be useful for solving some problems in Particle physics, in particular, in describing physical vacuum.

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Thank you for your attention!