# Uniformization of Degenerate Equations and Semiclassical Asymptotics

Vladimir Nazaikinskii

Ishlinsky Institute for Problems in Mechanics RAS, Moscow, Russia

Joint work with S. Dobrokhotov, A. Anikin, D. Minenkov, and A. Tolchennikov

Russian Science Foundation projects nos. 21-11-00341, 21-71-30011

and Russian Foundation for Basic Research project no. 21-51-12006

III International Conference “Mathematical Physics, Dynamical Systems, Infinite-Dimensional Analysis”,
dedicated to the 100th anniversary of V.S. Vladimirov, the 100th anniversary of L.D. Kudryavtsev,
and the 85th anniversary of O.G. Smolyanov

July 5–13, 2023, Dolgoprudny , Moscow Region

# Statement of the problem

- $\hat{P} = P\left( x, -ih\frac{\partial}{\partial x} \right)$ is a pseudodifferential operator on a manifold $X$
- $h$ is a small parameter
- Consider the equation $\hat{P}u = \lambda u$ or $-ih\frac{\partial u}{\partial t} + \hat{P}u = 0$
- Asymptotic solutions as $h \to 0$ are given by Maslov's canonical operator
- What to do if the equation has degeneration or singularities?

# Uniformization

- $M$ is a $C^\infty$ manifold; $G$ is a compact Lie group
- $(g,m) \mapsto g \cdot m$ is a smooth action of $G$ on $M$

$X = M/G$ is the orbit space (a stratified manifold)

Space of smooth functions: $C^\infty(X) := C^\infty(M)^G$

Sobolev spaces: $H^s(X) := H^s(M)^G, \quad L^2(X) := H^0(X)$

$\mathcal{P} : C^\infty(M) \to C^\infty(M)$ is a $G$-invariant $(\psi)$DO on $M$

$\hat{P} = \mathcal{P}|_{C^\infty(x)}$ is a $(\psi)$DO on $M$

# Example

Let $M = \mathbb{R}^2 \ni y = (y_1, y_2)$

$G = \mathbb{S}^1$ acts by rotations on $M$

$\varphi \mapsto$ rotation by the angle $\varphi$ counterclockwise

$$X = M / G = \mathbb{R}_+ = [0, \infty) \ni x$$

$M \to X$

$$y \mapsto \frac{y_1^2 + y_2^2}{4}$$

$$\mathcal{P} = -h^2 \Delta = -h^2 \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right)$$

$$\hat{P} = -h^2 \frac{\partial}{\partial x} x \frac{\partial}{\partial x}$$

Shallow water equations

$\eta$ is the free surface elevation

depth

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left( x \frac{\partial \eta}{\partial x} \right) = 0$$

# General Problem

$$\begin{array}{ccc} & M & \\ G & \downarrow & \\ & X & \end{array}$$

We need to construct semiclassical asymptotics on $M$ invariant with respect to $G$

How to ensure the $G$-invariance?

Semiclassical asymptotics are provided by the canonical operator

$\Rightarrow$ we need to consider its structure

# Reminder: Canonical Operator as a Black Box

**Main geometric objects:**

- Lagrangian manifold $\Lambda \subset T^{*}M$
- Measure (volume form) $d\mu$ on $M$

**Quantization condition:**

$$\frac{2}{\pi h}\left[\omega^{1}\right] + \text{ ind } \equiv 0 \mod 4 \qquad (\omega^{1} = p\,dq)$$

Then one can define the canonical operator

$$K = K_{(\Lambda, dM)} : C_0^{\infty}(\Lambda) \to C_h^{\infty}(M)$$

$h \in (0,1], \quad h \to 0,$ is a small parameter

# Main Properties: Commutation Formulas

Consider a $\psi$DO $\hat{H} = H\left(x, -ih\frac{\partial}{\partial x}, h\right), \quad H(x,p) = H_0 + hH_1 + \ldots$

Then

- $\hat{H}KA = K\left(H_0|_\Lambda A\right) + O(h)$ (first commutation formula)
- If $H_0|_\Lambda = 0$ and $\mathcal{L}_{V(H_0)}(d\mu) = 0,$ then

$$\hat{H}KA = -ihK(\Pi A) + O\left(h^2\right)$$

(second commutation formula), where $\Pi$ is the transport operator,

$$\Pi = V(H) + iH_{\text{sub}}|_\Lambda, \qquad H_{\text{sub}} = H_1 + \frac{i}{2}\sum_{j=1}^{n}\frac{\partial^2 H}{\partial x_j \partial p_j}(x,p)$$

Key question: When is $KA$ $G$-invariant?

# $G$-invariant canonical operator

- It suffices to require $\mathfrak{g}$-invariance, where $\mathfrak{g}$ is the Lie algebra of $G$:

$$\xi \in \mathfrak{g} \Rightarrow v_\xi \in \mathrm{Vect}\, M \qquad\qquad v_\xi(u) = 0 \quad \forall \xi \in \mathfrak{g}$$

- We can write $\hat{H}_\xi = -ihv_\xi, \quad H_\xi(x,p) = \omega^1(v_\xi).$
- Let us use the commutation formulas:

$$1) \qquad \hat{H}_\xi KA = 0 \quad \mathrm{mod}\, O(h)$$

$$H_\xi(x,p)|_\Lambda = 0 \Rightarrow V(H_\xi)$$ is tangent to $\Lambda \Rightarrow \Lambda$ is $G$-invariant

$$2) \qquad \hat{H}_\xi KA = 0 \quad \mathrm{mod}\, O(h^2) \Rightarrow \quad A$$ is constant on $G$-orbits

This leads to a well-known construction

# Momentum Map

- General: $G:W, \quad W$ is a symplectic space, $V:\mathfrak{g}\to \mathrm{Vect}\,W;$

  $\pi : W \to \mathfrak{g}^*$ is a momentum map if

  $H_\xi(z) := \langle \pi(z), \xi \rangle$ is the Hamiltonian of $V(\xi) \quad \forall \xi \in \mathfrak{g}$

If 0 is a weakly regular value, then $\pi^{-1}(0)/G = W//G$

(symplectic reduction, Marsden-Weinstein 1974)

- In our specific case:

$$\pi = T^*M \to \mathfrak{g}^*, \quad \pi(x,p)(\xi) = H_\xi(x,p) := \omega^1(v_\xi),$$

$\tilde{\Phi} \subset \pi^{-1}(0) \subset T^*M$ is the set of weakly regular points

$\Phi := \tilde{\Phi}/G$ is the reduced phase space

# Lagrangian manifolds

- We have a Lagrangian manifold $\Lambda \subset \tilde{\Phi} \subset \pi^{-1}(0)$.
- This manifold is projected into a Lagrangian manifold $L \subset \Phi$.
- We can proceed from $K_{(\Lambda, d\mu)}$ to $K_{(L, d\tilde{\mu})}$, where $d\tilde{\mu} \otimes dh = d\mu$

  ( $dh$ is the Haar measure; $d\mu$ is assumed to be $G$ -invariant).
- $\Lambda$ satisfies quantization conditions iff so does $L$ .

$$K_{(L, d\tilde{\mu})} B = K_{(\Lambda, d\mu)} A,$$

where $A$ is the lift of $B$ as a function constant along the orbits.

# Example: 1D (half-line)

$$M = \mathbb{R}^2 \ni y = \left(y_1, y_2\right) \qquad G = \mathbb{S}^1 \qquad X = M / G = \mathbb{R}_+ = [0,\infty) \ni x$$

$$M \to X \qquad y \mapsto \frac{y_1^2 + y_2^2}{4}$$

$$T^*M \ni (y,\xi) \qquad\qquad T^*X \ni (x,p)$$

- Momentum map $\quad p_\varphi = y_1\xi_2 - y_2\xi_1$
- Singular points $\quad \{dp_\varphi = 0\} = \{(0,0)\}$
- $\tilde{\Phi} = p_\varphi^{-1}(0) \setminus \{0,0\} = \{y = \eta \cdot n(\varphi), \quad \xi = \rho \cdot n(\varphi), \quad \eta^2 + \rho^2 > 0\}.$

$$n(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

# Example (half-line): continued

- Thus, we have $\tilde{\Phi} = p_\varphi^{-1}(0) \setminus \{0,0\} = \{y = \eta \cdot n(\varphi), \quad \xi = \rho \cdot n(\varphi), \quad \eta^2 + \rho^2 > 0\}.$

$$n(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

- The reduced phase space is: $\Phi : \{(\eta, \rho)\}$

$$\omega^1 = \xi\, dy = \rho\, d\eta \qquad \omega^2 = d\xi \wedge dy = d\rho \wedge d\eta$$

$$\varphi \mapsto \varphi + \pi, \quad \eta \mapsto -\eta, \quad \xi \mapsto -\xi$$

(the point remains the same). We are interested in: $\boxed{\rho > 0, \quad \eta \in (-1,1)}$

# Example (half-line): continued

- Now consider a Lagrangian manifold $L \subset \Phi$
- $\omega^2 = d\xi \wedge dy = d\rho \wedge d\eta \Rightarrow (\rho,\eta)$ are Darboux coordinates on $\Phi$
- Consequently, $L$ can be given by generating function $\Psi(\eta,\theta)$ as follows:

$$C_\Psi = \left\{(\eta,\theta) | \frac{\partial \psi}{\partial \theta} = 0\right\}; \qquad L = \left\{(\eta,\rho) \,\middle|\, \rho = \frac{\partial \Psi}{\partial \eta}, \quad (\eta,\rho) \in C_\Psi \right\}$$

- Now $S(y,\theta,\varphi) = \Psi(\langle y, n(\varphi)\rangle, \theta)$ is a generating function of $\Lambda \subset \tilde{\Phi} \subset T^*M$
- Canonical operator on $\Lambda$:

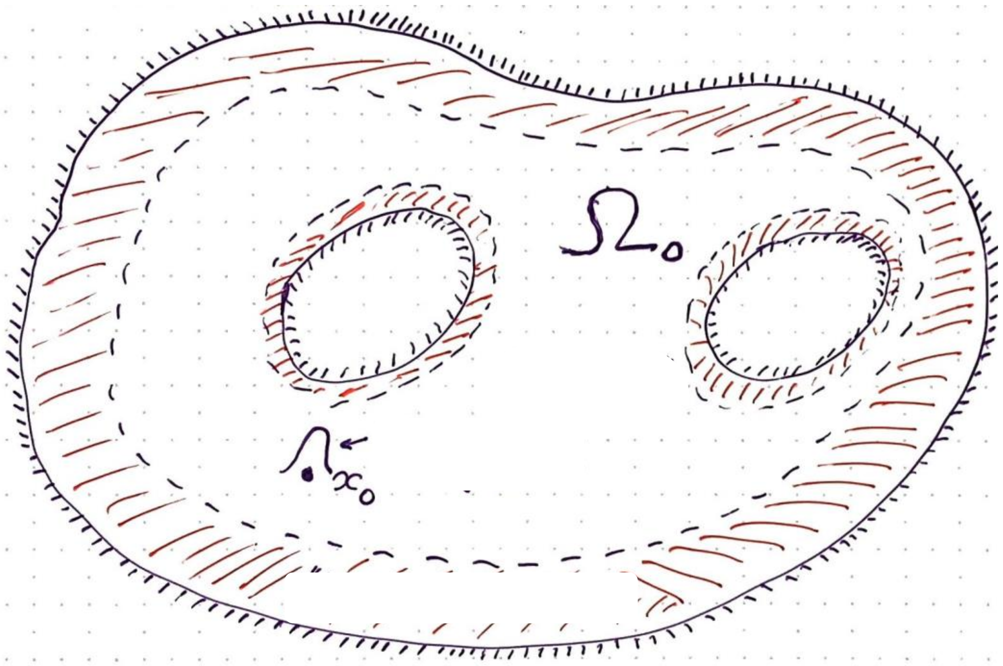
$$K_{(\Lambda,d\mu)} A(y,h) = \left(\frac{i}{2\pi h}\right)^{(m+1)/2} \iint e^{\frac{i}{h} S(y,\theta,\varphi)} (A\sqrt{F})(y,\theta,\varphi)\, d\varphi\, d\theta$$

$$x = \frac{y^2}{4}$$

- The integral $\int e^{\frac{i}{h}\Psi(\langle y, n(\varphi)\rangle,\theta)} a(\langle y, n(\varphi)\rangle, \theta)\, d\varphi$ can be expressed via $J_{0,1}\left(\frac{|y|}{h}\right)$

# Application:
# Shallow water equations

# Statement of the problem




$D(x)$ is defined in a neighborhood of $\bar{\Omega}_0$

$D(x) > 0$ in $\Omega_0$

$D(x) < 0$ outside $\bar{\Omega}_0$

$\nabla D(x) \neq 0$ on $\partial\Omega_0$

$D \in C^\infty$

(Basin with a smooth shallow beach)

$$\eta_t + \langle \nabla, (D(x) + \eta)\mathbf{u} \rangle = 0, \quad \mathbf{u}_t + \langle \mathbf{u}, \nabla \rangle \mathbf{u} + g\nabla\eta = 0, \quad t \in [0, T],$$

$$\eta \big|_{t=0} = \eta^{(0)}(x), \quad \mathbf{u} \big|_{t=0} = \mathbf{u}^{(0)}(x)$$

$$x \in \bar{\Omega}_t = \{x |\ D(x) + \eta(x,t) \geq 0\}$$

Cauchy problem for nonlinear shallow water equations

$u^{(0)} = 0, \eta^{(0)}$ localized: piston model of tsunami generation

# Linearized problem

- The linearized problem has the form

$$\eta_t + \langle \nabla, D(x)\mathbf{u} \rangle = 0, \quad \mathbf{u}_t + g\nabla\eta = 0, \quad t \in [0,T], \quad x \in \Omega_0$$

$$\eta \big|_{t=0} = \eta^{(0)}(x), \quad \mathbf{u} \big|_{t=0} = \mathbf{u}^{(0)}(x)$$

and can be written as a system with the matrix

$$\mathscr{L} = \begin{pmatrix} \partial_t & \nabla \circ D(x) \\ g\nabla & \partial_t \end{pmatrix}$$

# Main Theorem about the Linearized Problem

**Theorem**

The problem

$$\mathcal{L} u = v, \qquad u\big|_{t=0} = u_0$$

where $u_0 \in C^\infty(\bar{\Omega}_0)$ and $v \in C^\infty(\bar{\Omega}_0 \times [0,T])$, has a unique solution $u \in C^\infty(\bar{\Omega}_0 \times [0,T])$.

The operator

$$\mathscr{L} = \begin{pmatrix} \partial_t & \nabla \circ D(x) \\ g\nabla & \partial_t \end{pmatrix}$$

degenerates on the boundary, and to study it, we need ***uniformization.***

# Reduction to a single equation

First, we proceed from the original linear problem to a Cauchy problem for a scalar second-order operator.

$$\eta_t + \langle \nabla, D(x)\mathbf{u} \rangle = f_1(x,t), \quad \mathbf{u}_t + g\nabla\eta = f_2(x,t),$$
$$\eta \big|_{t=0} = \eta^{(0)}(x), \quad \mathbf{u} \big|_{t=0} = \mathbf{u}^{(0)}(x)$$

$$\Downarrow$$

$$\eta_{tt} - \langle \nabla, gD(x)\nabla\eta \rangle = f_{1t} - \langle \nabla, D(x)f_2 \rangle, \quad \eta \big|_{t=0} = \eta^{(0)}, \quad \eta_t \big|_{t=0} = f_1 - \langle \nabla, D\mathbf{u}^{(0)} \rangle$$

$$u = u^{(0)} + \int_0^t (f_2 - g\nabla\eta)\, dt$$ (reconstructed once $\eta$ is known)

The velocity $c(x) = \sqrt{gD(x)}$ in this wave equation vanishes on the boundary, so we need to study well-posedness of the problem and smoothness of solutions

# Uniformization of the linearized equations and their analysis

- Idea of uniformization: lift a degenerate equation to a higher-dimensional manifold, where the degeneration disappears or becomes weaker…
- Let us describe the general uniformization procedure for operators with Bessel type degeneration on the boundary

# Class of operators

$X$ a smooth compact $n$-dimensional manifold with boundary $\partial X$ and interior $X^\circ = X \setminus \partial X$

$d\,\mathrm{vol}_X$ a volume form on $X$ ; we consider only local coordinate systems consistent with it

$$U \simeq \left[0, \frac{1}{4}\right) \times \partial X \ni (x_1, x')$$ a collar neighborhood of the boundary

We consider differential operators of the divergence form $L = -\langle \nabla, B(x)\nabla \rangle$, where

$B(x) = B^*(x) > 0, \quad x \in X^\circ$, and $B(x) = x_1 A(x)$, $A(x) = A^*(x) > 0, \quad x \in X$.

In $L^2(X, d\,\mathrm{vol}_X)$, will always be viewed as a closure from $C^\infty(X)$

(=Friedrichs extension from $C_0^\infty(X^\circ)$).

- Cauchy problem $u_{tt} + Lu = 0, \; u|_{t=0} = V(h^{-1}(x - x_0)), \quad u_t|_{t=0} = 0$
- Eigenvalue problem $Lv = \lambda v$
- More convenient: multiply by $h^2 (= \lambda^{-1})$

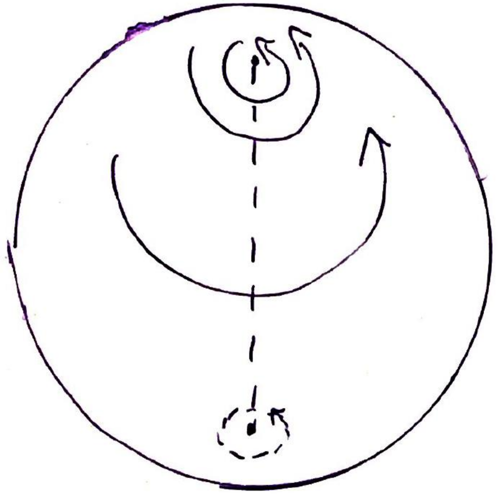
$$-h^2 u_{tt} - \widehat{H} u = 0$$
$$\widehat{H} v = v$$
$$\widehat{H} = h^2 L = \langle -ih\nabla, B(x)(-ih\nabla) \rangle$$

# Manifold with Boundary as the Space of $\mathbb{S}^1$-Orbits

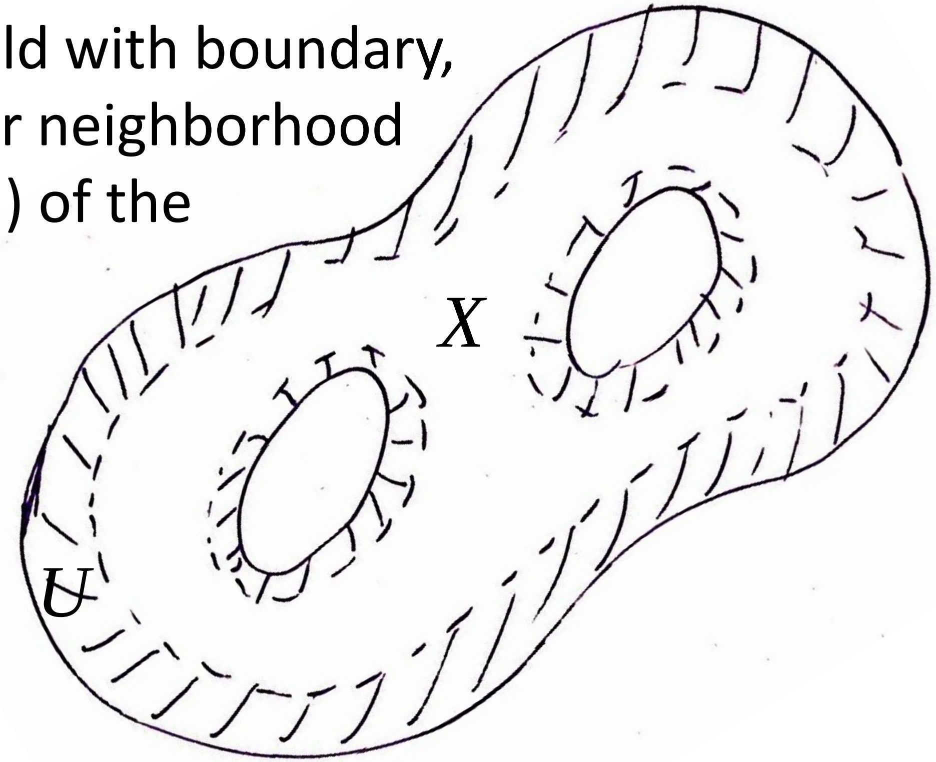
The simplest example

$$[-1,1] = \mathbb{S}^2/\mathbb{S}^1$$

($\mathbb{S}^1$ acts on $\mathbb{S}^2$ by rotations about the polar axis)



General case:
In a manifold with boundary, take a collar neighborhood $U$ (dashed) of the boundary




$$U \simeq [0, \tfrac{1}{4}) \times \partial X \ni (x_1, x')$$

Consider the open disk

$$\mathbb{D}^2 = \{y \in \mathbb{R}^2 \colon y^2 < 1\}, \qquad y^2 = y_1^2 + y_2^2.$$

Define a closed manifold $M$ and a smooth projection $\pi\colon M \to X$:

$$\pi^{-1}(X^\circ) \simeq \mathbb{S}^1 \times X^\circ, \qquad \pi(\phi, x) = x;$$
$$\pi^{-1}(U) \simeq \mathbb{D}^2 \times \partial X, \qquad \pi(y, x') = (y^2/4, x').$$

Transition map over $U^\circ = X^\circ \cap U$:

$$\chi\colon \mathbb{S}^1 \times U^\circ \to (\mathbb{D}^2 \setminus \{0\}) \times \partial X,$$
$$\chi(\phi, x) = (2\sqrt{x_1}\,\mathbf{n}(\phi), x'), \text{ where } \mathbf{n}(\phi) = (\cos\phi, \sin\phi)^\top.$$

There is a natural semifree action of $\mathbb{S}^1$ on $M$ with orbits the fibers of $\pi$, and

$$X = M/\mathbb{S}^1$$

Well known in topology; we need the specific smooth structure of the projection $\pi$.

# Lift to $M$ of boundary degenerate operators on $X$

$C^\infty(X) \simeq C^\infty(M)^{\mathbb{S}^1}$ (the subspace of $\mathbb{S}^1$-invariant functions in $C^\infty(M)$).

$P : C^\infty(M) \to C^\infty(M)$, $[P, \mathbb{S}^1] = 0 \implies P$ is called a *lift* of $L = P|_{C^\infty(X)}$.

$$\text{Let} \quad L = -\langle \nabla, B(x)\nabla \rangle, \quad B(x) = x_1 A(x), \quad A(x) = A^\top(x) > 0.$$

$$\text{Define} \quad P = L - f(x)\frac{\partial^2}{\partial \phi^2} \quad \text{on} \quad \pi^{-1}(X^\circ) \simeq \mathbb{S}^1 \times X^\circ, \quad f(x) = b_{11}(x)/(4x_1^2).$$

THEOREM 1. *$P$ is a lift of $L$. In local coordinates $(y_1, y_2, x')$ on $\pi^{-1}(U)$,*

$$P = -\langle Y, A(y^2/4, x')Y \rangle - \langle Z, A(y^2/4, x')Z \rangle,$$

*where* $Y = (Y_1, \dots, Y_n)^\top$, $Z = (Z_1, \dots, Z_n)^\top$,

$$Y_1 = \frac{\partial}{\partial y_1}, \quad Z_1 = \frac{\partial}{\partial y_2}, \quad Y' = \frac{y_1}{2}\frac{\partial}{\partial x'}, \quad Z' = \frac{y_2}{2}\frac{\partial}{\partial x'}.$$

*The operator $P$ is hypoelliptic.*

*Simplest one-dimensional examples*

- Let $\eta$ be a solution of the degenerate wave equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left(x \frac{\partial \eta}{\partial x}\right) = 0, \qquad x \in [0, \infty)$$

Then $N(y,t) = \eta\left(\frac{y_1^2 + y_2^2}{4}, t\right)$ is a radially symmetric solution of the wave equation with constant velocity

$$\frac{\partial^2 N}{\partial t^2} - \frac{\partial^2 N}{\partial y_1^2} - \frac{\partial^2 N}{\partial y_2^2} = 0, \qquad y \in \mathbb{R}^2$$

Thus, run-up waves are reduced to waves away from shores

- Similar picture holds for shallow wave equations themselves:

$$\eta_t + (xu)_x = 0, \qquad u_t + \eta_x = 0, \qquad x \geq 0$$

Consider the functions

$$N(y,t) = \eta\left(\frac{y_1^2 + y_2^2}{4}, t\right), \; W(y,t) = \frac{y_1 + iy_2}{2} u\left(\frac{y_1^2 + y_2^2}{4}, t\right), \quad y = (y_1, y_2) \in \mathbb{R}^2$$

They satisfy the system of equations

$$\frac{\partial}{\partial t}\begin{pmatrix} N \\ W \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2} \\ \frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_2} & 0 \end{pmatrix}\begin{pmatrix} N \\ W \end{pmatrix}$$

(Dirac equation for electrons in graphene)

Set $W_1 = \operatorname{Re} W, \, W_2 = \operatorname{Im} W$ ; then we obtain the two-dimensional linear shallow water equation with a constant bottom

$$\frac{\partial N}{\partial t} + \frac{\partial W_1}{\partial y_1} + \frac{\partial W_2}{\partial y_2} = 0, \quad \frac{\partial W_1}{\partial t} + \frac{\partial N}{\partial y_1} = 0, \quad \frac{\partial W_2}{\partial t} + \frac{\partial N}{\partial y_2} = 0$$

and with the potentiality condition $\dfrac{\partial W_1}{\partial y_2} - \dfrac{\partial W_2}{\partial y_1} = 0.$

# *2D shallow water equations in the half-plane*

- Now let $\eta(x,t), u_1(x,t), u_2(x,t)$ be a solution of the linear shallow water equations in the half-plane:

$$\eta_t + (x_1 u_1)_{x_1} + (x_1 u_2)_{x_2} = 0, \quad u_{1t} + \eta_{x_1} = 0, \quad u_{2t} + \eta_{x_2} = 0, \qquad x_1 \geq 0$$

Then the functions

$$N(y,t) = \eta\left(\frac{y_1^2 + y_2^2}{4}, y_3, t\right), \qquad W_3(y,t) = \frac{y_1^2 + y_2^2}{4} u_2\left(\frac{y_1^2 + y_2^2}{4}, y_3, t\right)$$

$$W(y,t) = W_1(y,t) + iW_2(y,t) = \frac{y_1 + iy_2}{2} u_1\left(\frac{y_1^2 + y_2^2}{4}, y_3, t\right)$$

satisfy the system of equations

$$\frac{\partial N}{\partial t} + \frac{\partial W_1}{\partial y_1} + \frac{\partial W_2}{\partial y_2} + \frac{\partial W_3}{\partial y_3} = 0, \qquad y = (y_1, y_2, y_3) \in \mathbb{R}^3$$

$$\frac{\partial W_1}{\partial t} + \frac{\partial N}{\partial y_1} = 0, \quad \frac{\partial W_2}{\partial t} + \frac{\partial N}{\partial y_2} = 0, \quad \frac{\partial W_3}{\partial t} + \frac{y_1^2 + y_2^2}{4} \frac{\partial N}{\partial y_3} = 0$$

and the potentiality conditions

$$\frac{\partial W_2}{\partial y_1} - \frac{\partial W_1}{\partial y_2} = 0 \qquad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} \implies \frac{(y_1^2 + y_2^2)^2}{4} \frac{\partial W_j}{\partial y_3} = y_j\left(y_1 \frac{\partial W_3}{\partial y_1} + y_2 \frac{\partial W_3}{\partial y_2} - \frac{W_3}{2}\right),$$

$$j = 1, 2$$

- Thus, we obtain «almost» shallow water equations in 3D. For the wave equation, we obtain

$$\frac{\partial^2 N}{\partial t^2} - \frac{\partial^2 N}{\partial y_1^2} - \frac{\partial^2 N}{\partial y_2^2} - \frac{y_1^2 + y_2^2}{4}\frac{\partial^2 N}{\partial y_3^2} = 0, \qquad y = (y_1, y_2, y_3) \in \mathbb{R}^3$$

- This equation is still degenerate for $y_1 = y_2 = 0$. But this degeneration is of weaker type than that of the original equation. The vector fields

$$\frac{\partial}{\partial y_1}, \ \frac{\partial}{\partial y_2}, \ y_1\frac{\partial}{\partial y_3}, \ y_2\frac{\partial}{\partial y_3}$$

and their commutators generate the entire tangent space at each point (the spatial part is hypoelliptic in the sense of Hörmander–Oleinik–Radkevich). We will see that for such equations one can construct asymptotic solutions by the usual canonical operator: the trajectories of the Hamiltonian system no longer hit the degeneration set of the symbol.

# Asymptotic Solutions of the Degenerate Equations. Symplectic Reduction

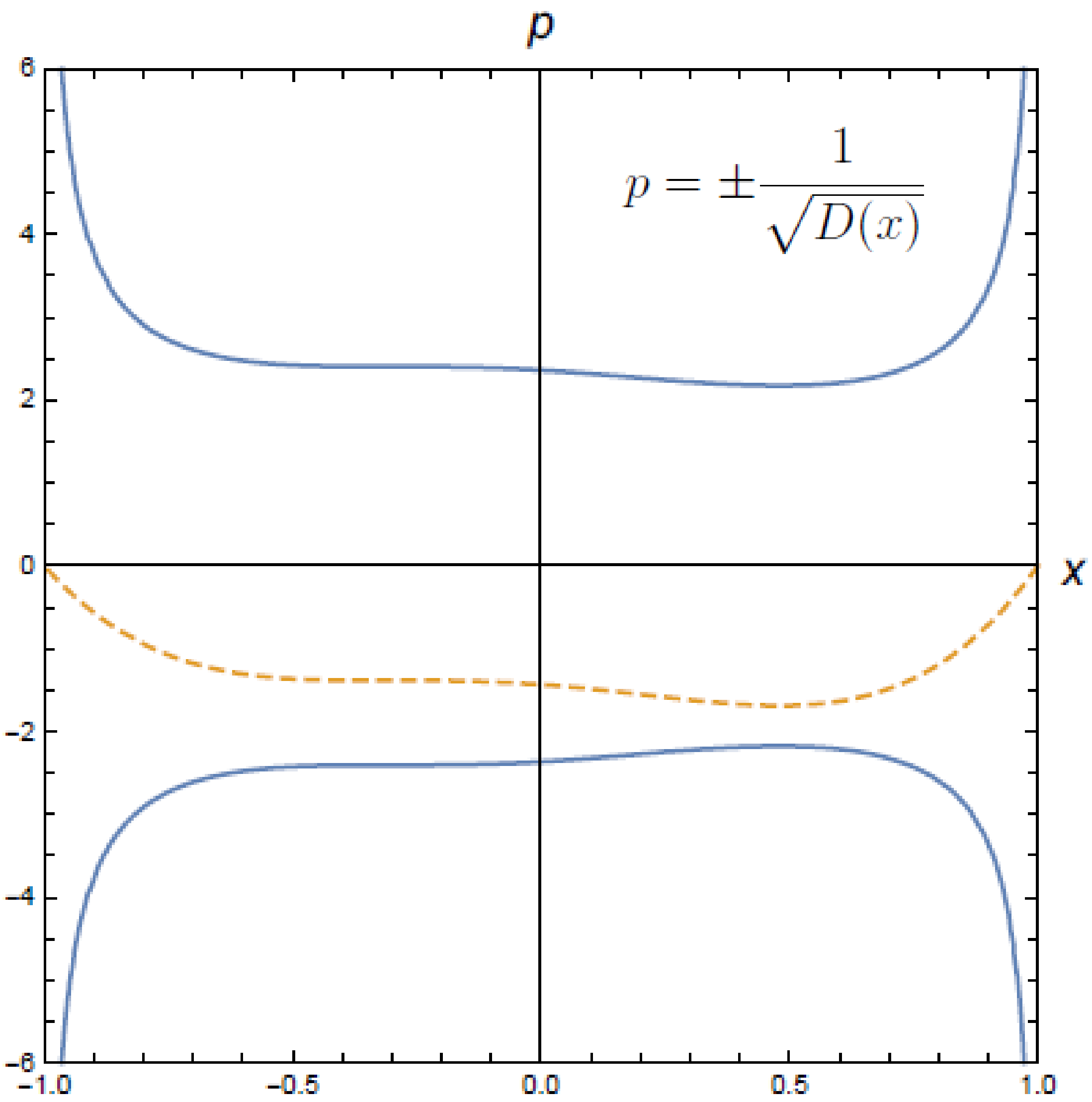
$$L\eta = \omega^2 \eta$$

$$L = -\frac{d}{dx} D(x) \frac{d}{dx}, \qquad x \in (-1,1)$$

$D(x)$ degenerates on the boundary

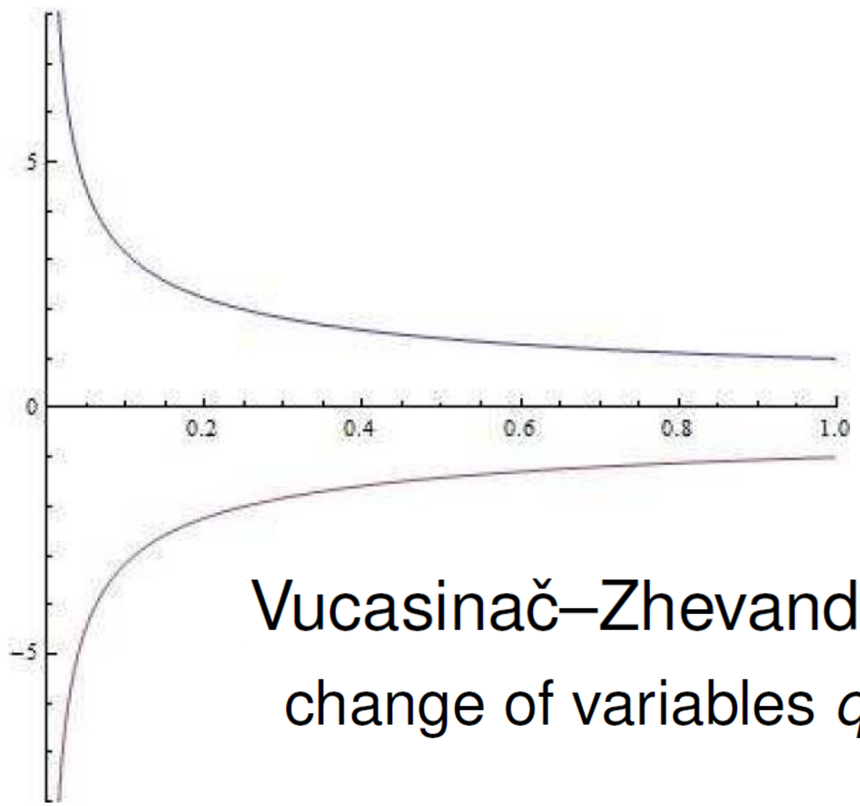
**Hamiltonian**
$$H(x,p) = D(x)p^2$$

**<u>Difficulty</u>**: the Lagrangian manifold is singular (improper projection onto the base) What to do?




Singular Lagrangian manifold $\Lambda_0$ (blue line) and the bottom profile (yellow, dashed)

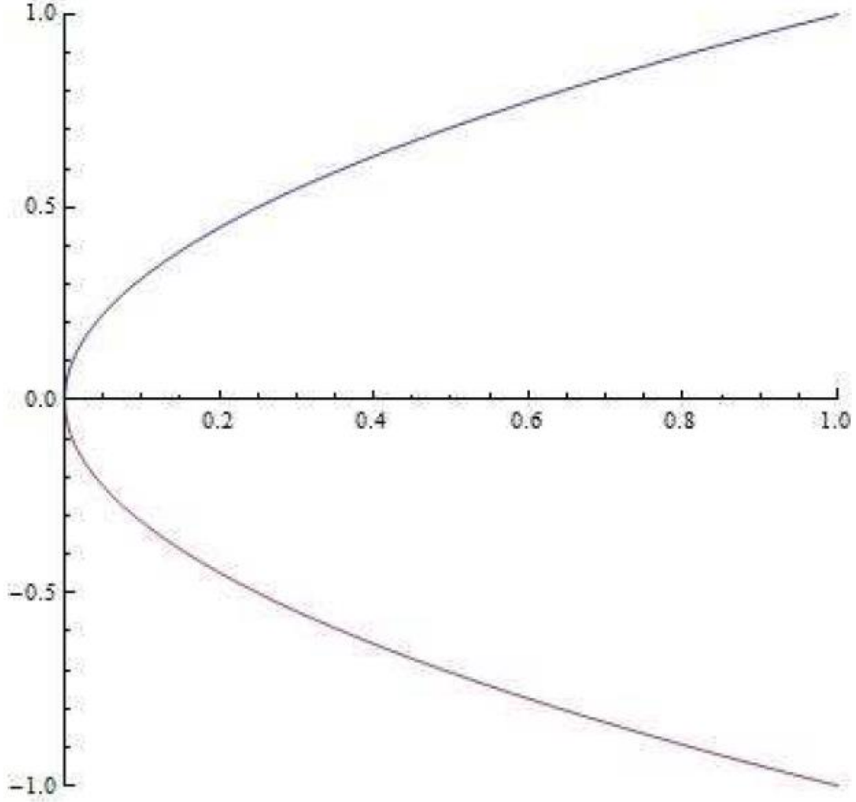
$p$ Left end $(x=0)$ $q$




$x$

Vucasinač–Zhevandrov 2002

change of variables $q = 1/p$




$x$

It remained to do one simple step…


S. Yu. Dobrokhotov, V. E. Nazaikinskii, B. Tirozzi, Algebra i Analiz, 22:6, 67-90 (2010).

V. E. Nazaikinskii, Math. Notes, 89:5, 749-753 (2011).

We have the canonical transformation $g:(x,p)\mapsto(\theta,q)$, $\begin{cases}\theta = p^2 x, \\ q = -\dfrac{1}{p}.\end{cases}$

This transformation regularizes the problem

It can be defined by the formulas $\theta = \dfrac{\partial\Phi}{\partial q}(x,q)$, $p = \dfrac{\partial\Phi}{\partial x}(x,q)$ with generating function $\Phi(x,q) = -\dfrac{x}{q}$

The Fock quantized canonical transformation is

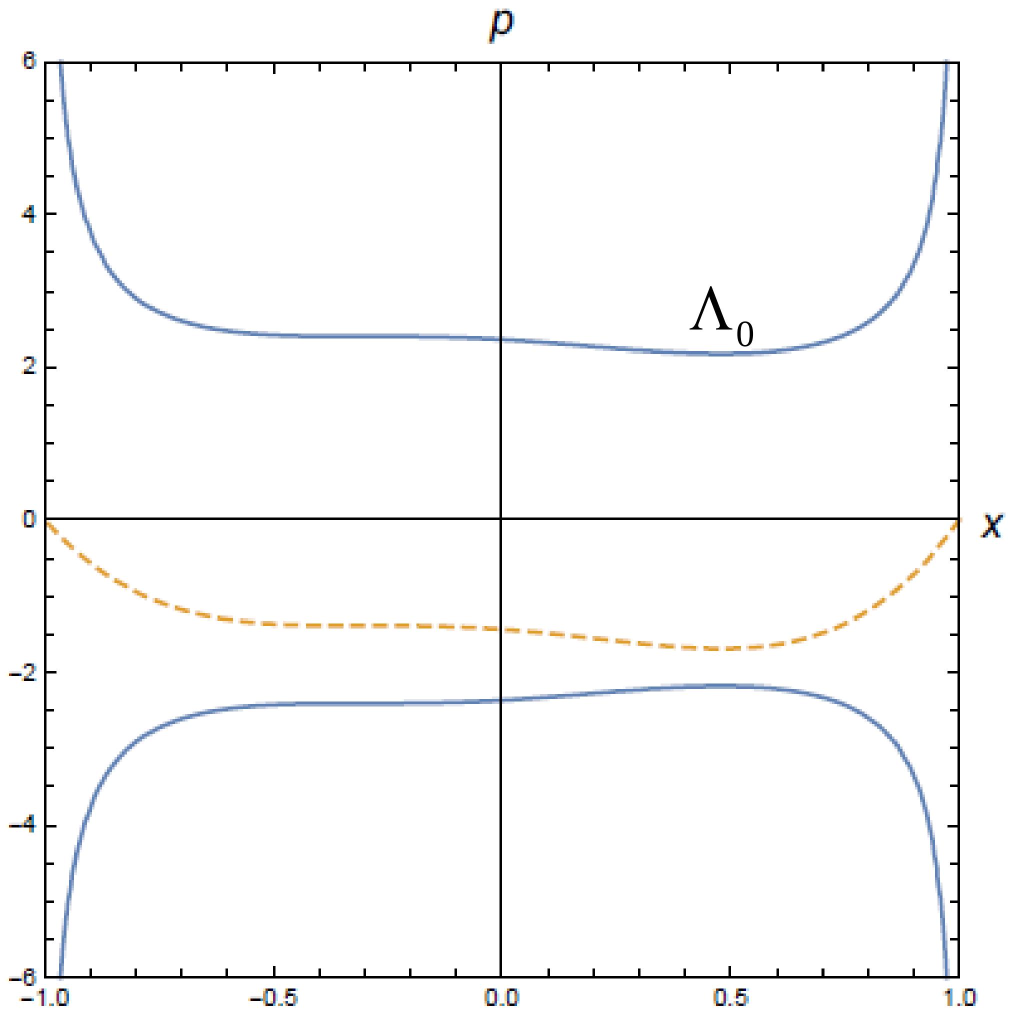
$$[T(g)u](x) = \int K(x,\theta)u(\theta)\,d\theta$$

$J_0$ is the Bessel function

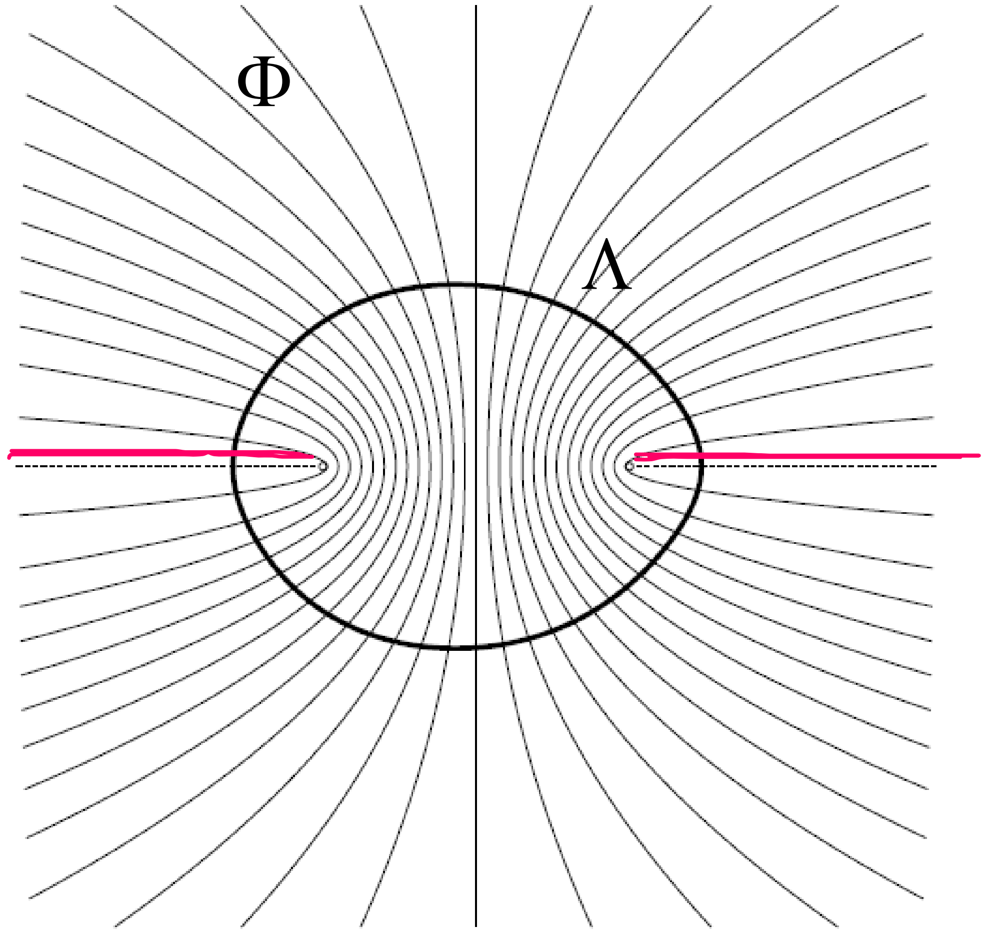
$$K(x,\theta) = \frac{1}{2\pi h}\int_{-\infty}^{\infty} e^{-\frac{i}{h}\left(\frac{x}{q}+\theta q\right)}\frac{dq}{q} = -\frac{i}{h}J_0\left(\frac{2\sqrt{x\theta}}{h}\right).$$

Thus, we have the Hankel transform instead of the usual Fourier transform

# Extended phase space and the Lagrangian manifold




$T^*(-1,1)$




$\Phi \simeq \mathbb{R}^2 \setminus \{(-1,0),(1,0)\}$

# Example

$X = [-1,1], \qquad M = \mathbb{S}^2$

$L = -\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}$ Eigenfunctions=Legendre polynomials $P_m(x)$

Canonical operator gives the asymptotics

$$P_m(\cos\theta) = \sqrt{\frac{\theta}{\sin\theta}} J_0((m+1/2)\theta) + O(1/m)$$

E. Hilb, Über die Laplacesche Reihe, M. Zeitschrift, v. 5, pp. 17–25. (1919)

# Example: Bessel functions of integer index

Reminder:
Equation

$$r^2 \frac{d^2 v(r)}{dr^2} + r \frac{dv(r)}{dr} + (r^2 - n^2) v(r) = 0$$

Integral representation

$$\mathbf{J}_n(r) = \frac{1}{2\pi} \oint e^{i(r \sin\theta - n\theta)} \, d\theta$$

Let us pretend we do not know the integral representation and try to obtain the asymptotics as $n \to \infty$ and/or $r \to \infty$

$n = 0$ the case considered above; $n > 0$ more general case

# General Oscillating Integrals

- Nondegenerate phase function $\Phi(x,\theta)$ defined on $V \subset \mathbb{R}^n_x \times \mathbb{R}^m_\theta$ : the differentials $d(\Phi_{\theta_1}),\ldots,d(\Phi_{\theta_m})$ are linearly independent on

$$C_\Phi = \{(x,\theta) \in V : \Phi_\theta(x,\theta) = 0\}$$

- Lagrangian manifold $\Lambda_\Phi = \jmath_\Phi(C_\Phi)$,

$$\jmath_\Phi : C_\Phi \ \to \ \mathbb{R}^{2n}_{(x,p)}, \ (x,\theta) \ \mapsto \ (x, \Phi_x(x,\theta))$$

- Oscillating integral

$$[K^\mu_\Phi A](x,\mu) = \frac{e^{i\pi\frac{m}{4}}}{(2\pi\mu)^{m/2}} \int_{\mathbb{R}^m} e^{\frac{i}{\mu}\Phi(x,\theta)} B(x,\theta,\mu)\, d\theta_1 \cdots d\theta_m,$$

$$B(x,\theta,\mu) = |F_{(\Phi,d\sigma)}(x,\theta)|^{1/2} A(\jmath(x,\theta),\mu)$$

$$(x,\theta) \in C_\Phi$$

$$F_{(\Phi,d\sigma)} = \frac{\jmath_\Phi(d\sigma) \wedge d(\Phi_{\theta_1}) \wedge \cdots \wedge d(\Phi_{\theta_m})}{dx_1 \wedge \cdots \wedge dx_m \wedge d\theta_1 \wedge \ldots d\theta_m}$$

L. Hörmander (1971)
Fourier integral operators

# Universal Phase Function and the Main Formula

- Lagrangian manifold $\Lambda \subset \mathbb{R}^{2n}_{(x,p)}$, $x = X(\alpha), p = P(\alpha), \alpha = (\alpha_1,\dots,\alpha_n)$ loc. coord.
- Action $\tau(\alpha): d\tau = \langle P, dX \rangle$, measure (volume form) $d\sigma = \sigma(\alpha)\, d\alpha_1 \wedge \dots \wedge d\alpha_n$

**Universal phase function**

$$S(x,\alpha) = \tau(\alpha) + \langle P(\alpha), x - X(\alpha) \rangle$$

$$\langle a,b \rangle = \sum_{j=1}^{n} a_j b_j$$

- Local canonical operator:

$$\alpha_0 \in \Lambda, \operatorname{rank} X_\alpha(\alpha_0) = k \quad \Rightarrow \quad \alpha = (\phi,\theta) \equiv (\phi_1,\dots,\phi_k,\theta_1,\dots,\theta_{n-k}), \ \operatorname{rank} X_\phi = k$$

$\Pi(\phi,\theta) X_\phi(\phi,\theta) = E_{k\times k}$ (left inverse); $\quad \langle \Pi, x - X \rangle = 0 \ \Rightarrow \phi = \phi(x,\theta)$

$$\Phi(x,\theta) = S(x, \phi(x,\theta), \theta)$$

$$[K^\mu_\Lambda A](x) = \frac{e^{i\pi m/4}}{(2\pi\mu)^{m/2}} \int e^{\frac{i}{\mu}\Phi(x,\theta)} ((\sigma \det M)^{1/2} A)(\phi(x,\theta),\theta)\, d\theta$$

$$M(\phi,\theta) = \left( \Pi^*(\phi,\theta) \quad P_\phi(\phi,\theta)\Pi(\phi,\theta) X_\theta(\phi,\theta) \right)$$

$x = (x_1, x_2) \in \mathbb{R}^2,$ $(r, \varphi)$ polar coordinates

Consider the function $u(x,\mu) = e^{in\varphi} \mathbf{J}_n\left(\frac{r}{\mu}\right),$ $n = \frac{\gamma}{\mu}$

$$\hat{H}_1 u \equiv -\mu^2 \Delta u = u$$

$$H_1(x,p) = p^2 \equiv p_1^2 + p_2^2$$

$$\hat{H}_2 u = x_1\left(-i\mu\frac{\partial}{\partial x_2}\right)u - x_2\left(-i\mu\frac{\partial}{\partial x_1}\right)u = \gamma u$$

$$H_2(x,p) = x_1 p_2 - x_2 p_1$$

Hamiltonians in involution; “Liouville” Lagrangian manifold (their common level)

Lagrangian manifold

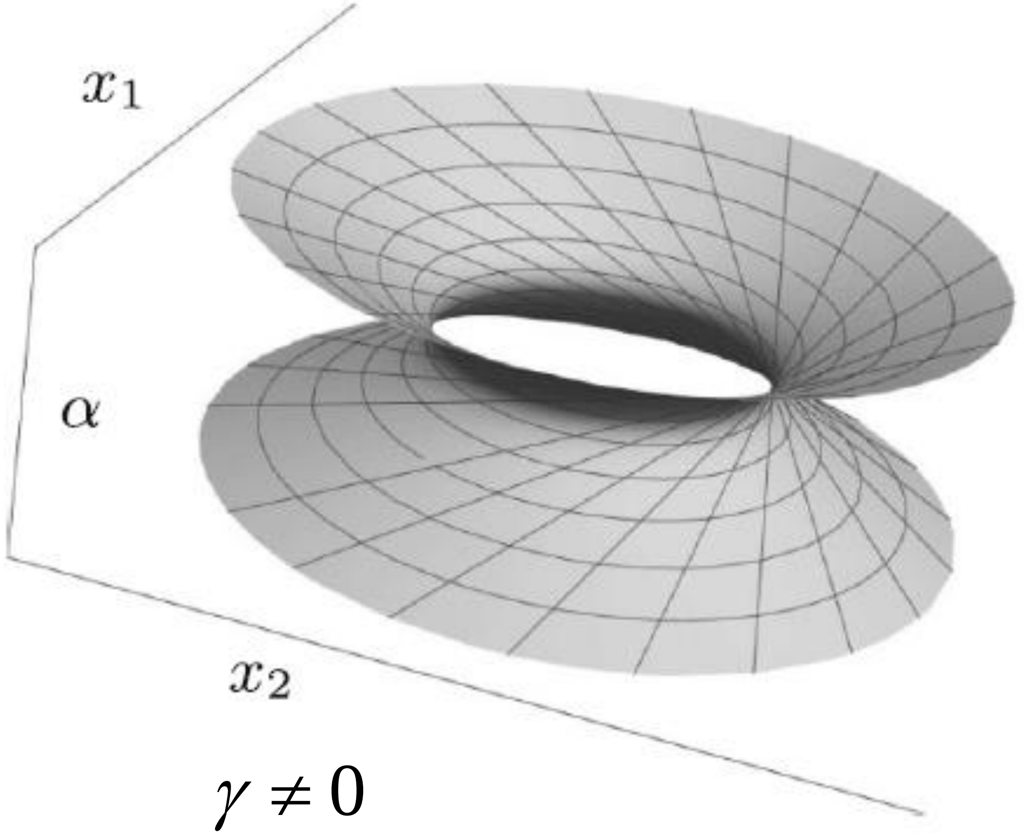
$$\Lambda_\gamma = \{(x,p) \in \mathbb{R}^4 : p^2 = 1,\ x_1 p_2 - x_2 p_1 = \gamma\}$$

Parametric description:

$$x = X(\alpha,\psi) \equiv \alpha \mathbf{n}(\psi) - \gamma \mathbf{n}'(\psi),$$

$$p = P(\alpha,\psi) \equiv \mathbf{n}(\psi),$$

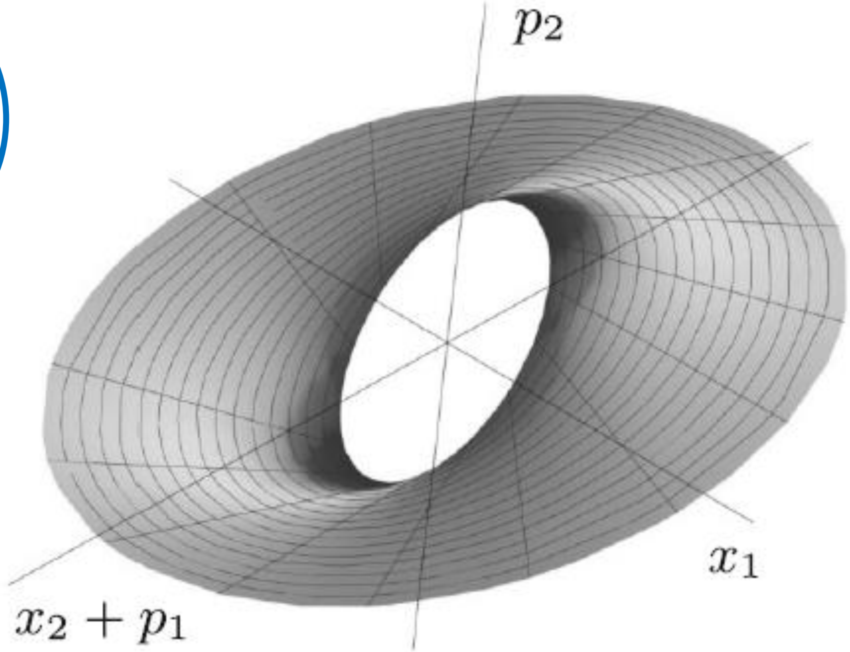
$$\alpha \in \mathbb{R}, \quad \psi \in \mathbb{S}^1$$




$\gamma \neq 0$

$\alpha = 0$

focal points




$\gamma = 0, \alpha > 0$

# Canonical Operator in Eikonal Coordinates

Assume that on a Lagrangian manifold $\Lambda$ one has $d\tau = \langle P, dX \rangle \neq 0.$

Then there exist functions $\psi = (\psi_1, \ldots, \psi_{n-1})$ on $\Lambda$ such that $(\tau, \psi)$ are local coordinates on $\Lambda$

Eikonal coordinates

$$x = X(\tau,\psi),\ p = P(\tau,\psi)$$

Define $\tau = \tau(x,\psi)$:

$$\langle P(\tau,\psi), x - X(\tau,\psi) \rangle = 0$$

Assume that $\det\begin{pmatrix} P & P_\psi \end{pmatrix} \neq 0.$

Let $\sigma = \sigma(\tau,\psi)$ be the density of the measure $d\sigma = \sigma\, d\tau \wedge d\psi_1 \wedge \cdots \wedge d\psi_{n-1}$

$$[K_{(\Lambda,d\sigma)}A](x) = \left(\frac{i}{2\pi\mu}\right)^{(n-1)/2} \int e^{i\frac{\tau}{\mu}} A(\tau,\psi)\sqrt{\sigma \det(P\ P_\psi)}\,\Big|_{\tau=\tau(x,\psi)} d\psi_1 \cdots d\psi_{n-1}$$

$$x = X(\alpha, \psi) \equiv \alpha \mathbf{n}(\psi) - \gamma \mathbf{n}'(\psi),$$

$$p = P(\alpha, \psi) \equiv \mathbf{n}(\psi),$$

$$\alpha \in \mathbb{R}, \quad \psi \in \mathbb{S}^1$$

$$\mathbf{n}(\psi) = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$$

Action (eikonal) $\tau(\alpha,\psi) = \alpha + \gamma\psi$

$(\tau,\psi)$ eikonal coordinates

$$\det(P\ P_\psi) = 1$$

$d\sigma = d\alpha \wedge d\psi = d\tau \wedge d\psi$ invariant measure

$$\langle P, x - X \rangle = 0 \quad \Rightarrow \quad \tau(x,\psi) = \langle \mathbf{n}(\psi), x \rangle + \gamma\psi$$

$A = 1$ solution of the transport equation

$$[K_{(\Lambda,d\sigma)}A](x) = \left(\frac{i}{2\pi\mu}\right)^{(n-1)/2} \int e^{i\frac{\tau}{\mu}} A(\tau,\psi)\sqrt{\sigma \det(P\ P_\psi)}\,\Big|_{\tau=\tau(x,\psi)} d\psi_1 \cdots d\psi_{n-1}$$

$$[K_{(\Lambda_\gamma,d\sigma)}1](x) = \left(\frac{i}{2\pi\mu}\right)^{1/2} \int e^{\frac{i}{\mu}\langle \mathbf{n}(\psi), x \rangle + \gamma\psi}\, d\psi \Rightarrow \quad \mathbf{J}_n(r) = \frac{1}{2\pi}\oint e^{i(r\sin\theta - n\theta)}\, d\theta$$

# Some literature


- S. Yu. Dobrokhotov, V. E. Nazaikinskii, B. Tirozzi, Russ. J. Math. Phys. 17:4, 434-447 (2010).
- S. Yu. Dobrokhotov, V. E. Nazaikinskii, B. Tirozzi, St. Petersburg Math. J., 22:6, 895-911 (2011).
- V. E. Nazaikinskii, Math. Notes, 89:5, 749-753 (2011).
- V. E. Nazaikinskii, Math. Notes, 92:1, 144-148 (2012).
- S. Yu. Dobrokhotov, V. E. Nazaikinskii, B. Tirozzi, Russ. J. Math. Phys. 20:4, 389-401 (2013).
- V. E. Nazaikinskii, Math. Notes, 96:1, 99–109 (2014).
- V. E. Nazaikinskii, Math. Notes, 96:2, 248-260 (2014).
- V. E. Nazaikinskii, Russ. J. Math. Phys. 21:2, 289--290 (2014).
- A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, Math. Notes, 104:4, 471-488 (2018).
- A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, Zh. Mat. Fiz. Anal. Geom., 14:4, 393–405 (2018).
- S. Yu. Dobrokhotov, V. E. Nazaikinskii, Math. Notes, 107:5 (2020), 847–853.
- A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. A. Tolchennikov, J. Math. Sci., 270:4 (2023), 507–530.

Thank you!