On Stability of Solitons for a Rotating Charge with a Fixed Center of Mass in the Maxwell Field

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A rotating charge with a fixed center of mass in the Maxwell field

Consider the system which describes a rotating charged particle at rest interacting with the Maxwell electromagnetic field [1],

$$E(-x,t) = -E(x,t), \;\; B(-x,t) = B(x,t), \tag{1}$$

$$\dot{E}(x,t) = \nabla \wedge B(x,t) - (\omega(t)\wedge x)\rho(x), \;\; \dot{B}(x,t) = -\nabla \wedge E(x,t), \tag{2}$$

$$\nabla \cdot E(x,t) = \rho(x), \;\; \nabla \cdot B(x,t) = 0, \tag{3}$$

$$I\dot{\omega}(t) = \int x \wedge [E(x,t) + (\omega(t)\wedge x)\wedge B(x,t)]\rho(x)\,dx, \tag{4}$$

$x \in \mathbb{R}^3$, $t \in \mathbb{R}$. Here E, B are electric and magnetic fields respectively, ω is the angular velocity of the rotating particle centered at the origin, ρ is a distribution of the mass and the charge of the particle which we assume to be proportional, we also set the mechanical mass of the particle to be equal to 1.

1. Imaykin, V., Komech, A., Spohn, H. *Rotating charge coupled to the Maxwell field: scattering theory and adiabatic limit*, Monatshefte für Mathematik, Vol. 142. No 1-2. 143-156, 2004.

A rotating charge with a fixed center of mass in the Maxwell field

The symmetry and regularity conditions for ρ are the following:

$$\rho \in C_0^\infty(\mathbb{R}^3), \quad \rho(x) = \rho_r(|x|), \quad \rho(x) = 0 \ \text{ for } \ |x| > R_\rho > 0. \tag{5}$$

I is the moment of inertia of the particle defined, in the considered spherical symmetry case as

$$I = \frac{2}{3} \int x^2 \rho(x)\, dx. \tag{6}$$

Then $\hat{\rho}(k) = \rho_1(r), r = |k|$, $\partial_{k_j} \hat{\rho}(k) = \rho_1' k_j / r$, put $\tilde{\rho} := \rho_1'/r$.

Existence of Dynamics, Conservation Laws

Set

$$L = (\mathbb{R}^3, L^2(\mathbb{R}^3;\mathbb{R}^3), L^2(\mathbb{R}^3;\mathbb{R}^3))$$

Define the phase space $\mathcal{M}$ for the system (1)–(4) as the non-linear manifold of states $(\omega, E(x), B(x)) \in L$ satisfying (1) and (3).
Let ρ satisfy (5). Then for initial data in $\mathcal{M}$ there exists a dynamics for the system (1)–(4) in $C(\mathbb{R};\mathcal{M})$. Besides, the energy

$$H(t) := \frac{1}{2} I\omega(t)^2 + \frac{1}{2}\int |E(x,t)|^2 + |B(x,t)|^2 dx \equiv H(0), \quad t \in \mathbb{R} \tag{7}$$

is conserved. Note that the total momentum $P(t) := \int E(x,t) \wedge B(x,t)\, dx \equiv 0$ by (1).

Solitons

The system (1)-(4) is rotational invariant and admits soliton-type solutions (solitons) $E(x,t) = E_\omega(x)$, $B(x,t) = B_\omega(x)$, $\omega(t) = \omega = const \in \mathbb{R}^3$. The solitons satisfy the stationary equations

$$E_\omega(-x) = -E_\omega(x), \;\; B_\omega(-x) = B_\omega(x), \tag{8}$$

$$\nabla \wedge B_\omega(x) - (\omega \wedge x)\rho(x) = 0, \;\; \nabla \wedge E_\omega(x) = 0, \tag{9}$$

$$\nabla \cdot E_\omega(x) = \rho(x), \;\; \nabla \cdot B_\omega(x) = 0, \tag{10}$$

$$\int x \wedge [E_\omega(x) + (\omega \wedge x) \wedge B_\omega(x)]\rho \, dx = 0. \tag{11}$$

In Fourier space, the soliton fields are expressed by

$$\hat{E}_\omega(k) = \frac{-ik\hat{\rho}(k)}{k^2}, \;\; \hat{B}_\omega(k) = -\frac{k \wedge (\omega \wedge \nabla_k \hat{\rho}(k))}{k^2}. \tag{12}$$

Previous Results on Stability of Solitons

Proposition 1 [2]. *For the system (1)-(4), in the phase space* $\mathcal{M}$*, the zero soliton* $(E = E_0(x), B = 0, \omega(t) \equiv 0)$ *is Lyapunov stable (and as well orbital stable) but is not asymptotically stable.*

2. Imaykin, V. *On stability of zero soliton for a charged particle at rest in the Maxwell field*, Proceedings of the International Conference "Analytical and numerical methods for solving of hydrodynamics, mathematical physics and biology problems", dedicated to the 100-th anniversary of K.I. Babenko, 26-29 August 2019, Pushchino, Moscow Region, P. 78-80.

Stability of a Nonzero Soliton under Perturbations of Uniformly Compact Support

Theorem 2 [3]. *Let us fix a non-zero* $\omega \in \mathbb{R}^3$ *and* $R > 0$. *Consider solutions to the Cauchy problem for the system (1)-(4) with initial data*
$\omega + \Omega_0,\ E_\omega(x) + e_0(x),\ B_\omega(x) + b_0(x)$.
The soliton $(\omega, E_\omega(x), B_\omega(x))$ *is Lyapunov stable and as well orbital stable with respect to perturbations of uniformly compact support*
i.e. $\forall \varepsilon > 0$ *and* $\forall R > 0\ \exists \delta > 0$ *such than for any solution* $(\omega(t), E(t), B(t))$ *with the initial data* $\omega_0 = \omega + \Omega_0, E_0 = E_\omega + e_0, B_0 = B_\omega + b_0$ *from the condition* $\|(\Omega_0, e_0, b_0)\|_L < \delta$ *it follows that* $\|(\omega(t) - \omega, E(t) - E_\omega, B(t) - B_\omega)\|_L < \varepsilon$
$\forall t \in \mathbb{R}\ \forall(\Omega_0, e_0(x), b_0(x))$ *such that* $\operatorname{supp} e_0 \subset \{|x| \leq R\}$ *and* $\operatorname{supp} b_0 \subset \{|x| \leq R\}$.

3. Imaykin, V. *Stability of solitons for a rotating charge with a fixed center of mass in the Maxwell field*, Journal of Mathematical Sciences, Vol. 255, No 5, June 2021, P. 653-663.

Sketch of the Proof. Equations for Perturbations

Put

$$E(x,t) = E_\omega(x) + e(x,t), \quad B(x,t) = B_\omega(x) + b(x,t), \quad \omega(t) = \omega + \Omega(t) \tag{13}$$

with

$$\nabla \cdot e = 0, \;\; \nabla \cdot b = 0, \;\; e(-x,t) = -e(x,t), \quad b(-x,t) = b(x,t). \tag{14}$$

Insert (13) into (2), take the stationary equations (9), (11) into account and obtain the following system for the perturbations (e, b, Ω):

$$\dot{e} = \nabla \wedge b - (\Omega \wedge x)\rho, \quad \dot{b} = -\nabla \wedge e, \tag{15}$$

$$I\dot{\Omega} = \int x \wedge [e + (\Omega \wedge x) \wedge B_\omega + (\omega \wedge x) \wedge b + (\Omega \wedge x) \wedge b]\rho \, dx. \tag{16}$$

Important remark. By the energy conservation (7) and (13),

$$(\Omega(t), e(x,t), b(x,t)) \text{ is bounded in } L \text{ uniformly in } t. \tag{17}$$

Stability of the Zero Soliton

In the case $\omega = 0$ (the zero soliton) the equation (10) by (6) reduces to

$$I\dot{\Omega} = \int x \wedge [e + (\Omega \wedge x) \wedge b]\rho \, dx. \tag{18}$$

The system (15), (16), (17) is almost identical to the initial system (1)-(4). The difference is that one has $\nabla \cdot E = \rho$ in the initial system but $\nabla \cdot e = 0$ in the latter system. Anyway, by a straightforward computation one can check that the energy

$$h = h(\Omega, e, b) := (1/2)I\Omega^2 + (1/2)\int (|e|^2 + |b|^2)dx \tag{19}$$

is conserved along the solutions to the system (15), (16), (17). This fact implies Proposition 1.

Solution of the System for Perturbation

$$\hat{e} = \frac{d}{dt}\hat{K}_t\hat{e}_0 + im\hat{K}_t\hat{b}_0 - \int_0^t \frac{d}{dt}\hat{K}|_{t-s}\hat{j}(s)\,ds, \tag{20}$$

$$\hat{b} = -im\hat{K}_t\hat{e}_0 + \frac{d}{dt}\hat{K}_t\hat{b}_0 + \int_0^t im\hat{K}_{t-s}\hat{j}(s)\,ds, \tag{21}$$

$$m := k\wedge, \quad \hat{K}_t(k) := \frac{\sin(|k|t)}{|k|}, \quad j(x,s) = (\Omega(s)\wedge x)\rho(x).$$

For Ω we obtain the closed equation

$$I\dot{\Omega} = \int x\wedge[e + (\Omega\wedge x)\wedge B_\omega + (\omega\wedge x)\wedge b + (\Omega\wedge x)\wedge b]\rho\,dx, \tag{22}$$

where e, b are given by (20), (21).

The structure of the equation (22)

We rewrite the equation (22) as

$$I\dot{\Omega} = \int x \wedge [(\Omega \wedge x) \wedge B_\omega + e + (\omega(t) \wedge x) \wedge b]\rho \, dx. \tag{23}$$

In (23), we consider $\omega(t)$ as the known function, the first component of the solution to the system (1) – (4).
The right-hand side of (23) reads $T_1 + T_2 + T_3$ with

$$T_1 := \int x \wedge [(\Omega \wedge x) \wedge B_\omega]\rho \, dx, \quad T_2 := \int (x \wedge e)\rho \, dx, \quad T_3 := \int x \wedge [(\omega(t) \wedge x) \wedge b]\rho \, dx.$$

For the first term we have

$$T_1 = \int (\Omega \wedge x)(x \cdot B_\omega)\rho \, dx = K \wedge \Omega,$$

where

$$K := -\int x(x \cdot B_\omega)\rho \, dx = -\frac{2}{3}\omega \int \tilde{\rho}\Delta_k \hat{\rho} \, dk. \tag{24}$$

The structure of the equation (22)

$$T_2(t) = -\int (e \wedge x)\rho\, dx = -\int (\hat{e} \wedge i\nabla_k)\hat{\rho}\, dk = T_{21}(t) + T_{22}(t),$$

$$T_{21}(t) = \int dk\, \tilde{\rho}\left(i\cos|k|t(k \wedge \hat{e}_0) + |k|\sin|k|t\, \hat{b}_0 \right), \tag{25}$$

$$T_{22}(t) = \frac{2}{3}\int dk\, \tilde{\rho}^{\,2} k^2 \int_0^t ds \cos|k|(t-s)\Omega(s). \tag{26}$$

$$T_3(t) = \int x \wedge [(\omega(t) \wedge x) \wedge b]\rho\, dx = \omega(t) \wedge \int (x \cdot b)\, x\rho\, dx T_{31}(t) + T_{32}(t),$$

$$T_{31}(t) = \omega(t) \wedge \int dk \left[i\frac{\sin|k|t}{|k|} k \wedge \hat{e}_0 - \cos|k|t\, \hat{b}_0 \right] \tilde{\rho}, \tag{27}$$

$$T_{32}(t) = \omega(t) \wedge \frac{2}{3}\int dk\, \tilde{\rho}^2 k^2 \int_0^t ds\, \frac{\sin|k|(t-s)}{|k|}\Omega(s). \tag{28}$$

The stability of solutions to the equation (22)

Finally, (22) reads

$$\dot{\Omega} = M\Omega + T(t), \tag{29}$$

where M is the skew-adjoint matrix of $m := (K/I)\wedge$ and

$$T(t) := (1/I)(T_{21}(t) + T_{31}(t) + T_{22}(t) + T_{32}(t)). \tag{30}$$

The equation (29) is an integro-differential equation of the type which is considered in [4]; the problems of existence and stability of solutions are studied. In particular, for any finite $T \in \mathbb{R}$ the solution $\Omega(\cdot)$ depends continuously in $C(0, T; \mathbb{R}^3)$ on initial datum Ω_0:

$$\Omega(\cdot) \to 0 \ \ \mathrm{C(0,T;\mathbb{R}^3)} \ \ \text{as} \ \ \Omega_0 \to 0 \ \ \text{in} \ \ \mathbb{R}^3 \ \ \text{in particular,} \ \ \Omega(T) \to 0. \tag{31}$$

By (31), (20), (21)

$$(\Omega(T), e(\cdot, T), b(\cdot, T)) \to (0,0,0) \ \ \text{in} \ \ L \ \ \text{as} \ \ (\Omega_0, e_0, b_0) \to (0,0,0) \ \ \text{in} \ \ L. \tag{32}$$

4. L. Liberman, *On stability of solutions to integro-differential equations*, Izvestiya Vysshyh Uchebnyh Zavedeniy, Mathemetics, No. 3(4), 142-151 (1958).

x-representation

In x-representation (20), (21) read as

$$\begin{pmatrix} e(x,t) \\ b(x,t) \end{pmatrix} = U(t) \begin{pmatrix} e_0(x) \\ b_0(x) \end{pmatrix} - \int_0^t U(t-s) \begin{pmatrix} j(x,s) \\ 0 \end{pmatrix} ds, \tag{33}$$

where $U(t)$ is the group of the free Maxwell equation. Note that the group is isometric in the space $[L^2(\mathbb{R}^3;\mathbb{R}^3)]^2$ by the corresponding energy conservation law for free Maxwell equations.

$$\begin{pmatrix} e_{(0)}(x,t) \\ b_{(0)}(x,t) \end{pmatrix} := U(t) \begin{pmatrix} e_0(x) \\ b_0(x) \end{pmatrix}. \tag{34}$$

Then, in x-representation,

$$T_{21}(t) = \int (x \wedge e_{(0)}(x,t))\rho(x)\, dx, \quad T_{31}(t) = \omega(t) \wedge \int (x \cdot b_{(0)}(x,t)) x \rho(x)\, dx. \tag{35}$$

The special case of zero initial fields perturbation

Let us consider the special case of initial data $e_0(x) = 0$, $b_0(x) = 0$, and Ω_0 is arbitrary. In this case one has $e_{(0)}(x,t) = 0$, $b_{(0)}(x,t) = 0$ and hence, $T_{21}(t) = 0$ and $T_{31}(t) = 0$ by (35). Then the equation (29) becomes a linear homogeneous integro-differential equation and its solution reads

$$\Omega(t) = A(t)\Omega_0, \tag{36}$$

where $A(t)$ is a 3×3-matrix and $\|A(t)\|$ is bounded uniformly in $t \in \mathbb{R}$ by (17). By (33) and (36),

$$\begin{pmatrix} e(x,t) \\ b(x,t) \end{pmatrix} = W(x,t)\Omega_0\,, \tag{37}$$

where $W(x,t)$ is a 6×3 matrix. The components $w_{ij}(t,x)$, $i = 1, ..., 6$, $j = 1, ..., 3$ of the matrix are functions bounded in L^2 uniformly in $t \geq 0$ due to (17). Then we obtain the following preliminary result on stability:

Proposition 2. *For the system (2)-(4), in the phase space $\mathcal{M}$, the soliton $(\omega, E_\omega, B_\omega)$ is Lyapunov stable (and as well orbital stable) with respect to perturbations of type $(\Omega_0, e_0 = 0, b_0 = 0)$.*

Completing the proof of Theorem 2

Consider initial perturbations (e_0, b_0, Ω_0) such that

$$\operatorname{supp} e_0 \subset \{|x| \le R\}, \ \operatorname{supp} b_0 \subset \{|x| \le R\};$$

with a fixed $R > 0$.
By the strong Huygens principle for the group of the free Maxwell equations the supports of $e_{(0)}(x,t)$ and of $b_{(0)}(x,t)$ are subsets of the region $\{|x| > t - R\}$.
Then, since ρ is compact supported, there is a $\overline{T} = \overline{T}(R, R_\rho)$ such that $T_{21}(t) = 0$ and $T_{31}(t) = 0$ for $t \ge \overline{T}$ by (35). Then for $t \ge \overline{T}$ the equation (29) reads

$$\dot{\Omega} = M\Omega + \frac{1}{I}(T_{22}(t) + T_{32}(t)) \text{ with the initial condition } \overline{\Omega} := \Omega(\overline{T}), \quad (38)$$

where $\Omega(t)$ is the solution to (29) for $0 \le t \le \overline{T}$. The equation (38) is a linear homogeneous integro-differential equation w.r.t. Ω.

Completing the proof of Theorem 2

By (32)

$$\overline{\Omega} \to 0 \text{ in } \mathbb{R}^3 \text{ as } (\Omega_0, e_0, b_0) \to 0 \text{ in } L. \tag{39}$$

Further, for $t \geq \overline{T}$ the solution in (33) reads

$$\begin{pmatrix} e(x,t) \\ b(x,t) \end{pmatrix} = U(t) \begin{pmatrix} e_0(x) \\ b_0(x) \end{pmatrix} -$$

$$\int_0^{\overline{T}} U(t-s) \begin{pmatrix} j(x,s) \\ 0 \end{pmatrix} ds - \int_{\overline{T}}^{t} U(t-s) \begin{pmatrix} j(x,s) \\ 0 \end{pmatrix} ds. \tag{40}$$

i) In the right-hand side of (40), for the first term $(e_{(0)}, b_{(0)})$ we have

$$\|e_{(0)}(\cdot,t)\|^2_{L^2} + \|b_{(0)}(\cdot,t)\|^2_{L^2} = \|e_0\|^2_{L^2} + \|b_0\|^2_{L^2}, \tag{41}$$

since the group $U(t)$ is unitary.

Completing the proof of Theorem 2

ii) For the second term let us observe that $s \in [0; \overline{T}]$ with a fixed $\overline{T}$, the group $U(t-s)$ is unitary, and $j(x,s) = (\Omega(s) \wedge x)\rho(x)$. Then, by continuous dependence, see (31), (32), this term is bounded in $[L^2(\mathbb{R}^3;\mathbb{R}^3)]^2$ uniformly in $t \in \mathbb{R}$ and tends to zero as $(\Omega_0, e_0, b_0) \to 0$.
iii) For the third term it follows from i), ii), and (17) that it is also bounded in $[L^2(\mathbb{R}^3;\mathbb{R}^3)]^2$ uniformly in $t \in \mathbb{R}$. Further, the current reads

$$j(x,s) = (\Omega(s)\overline{\Omega} \wedge x)\rho, \ \ s \geq \overline{T},$$

and after integrating the term becomes $\overline{W(x,t)}\,\overline{\Omega}$, where the components

$$\overline{w}_{ij}(x,t) \ \text{of the matrix} \ \overline{W}(x,t) \ \text{are bounded in} \ L^2 \ \text{uniformly in} \ t, \tag{42}$$

because of the uniform boundness (17). By (39) this term tends to zero in L as $(\Omega_0, e_0, b_0) \to (0,0,0)$ in L.
iv) Finally, for $\Omega(t)$ itself we have $\Omega(t) \to 0$ in $\mathbb{R}^3$ as $(\Omega_0, e_0, b_0) \to (0,0,0)$ in L by (31) for $t \leq \overline{T}$ and by (36) for $t \geq \overline{T}$.
Then the conclusion of Theorem 2 follows from i) to iv). The proof is complete.

Recent Results on Stability

5. A.I. Komech and E.A. Kopylova, *On the Stability of Solitons for the Maxwell-Lorentz Equations with Rotating Particle*, Milan J. Math. Vol. 91 (2023) 155-173.

$$I_{\rm eff} := I + \frac{2}{3}\int \frac{|\nabla\hat\rho(r)|^2}{k^2}dk \gg I.$$

6. A.I. Komech and E.A. Kopylova, *On the Stability of Solitons for 3D Maxwell-Lorentz Equations with Spinning Particle*, Mathematics, Physics, Life, Conference dedicated to the 85-th anniversary of Vadim A. Malyshev, Moscow State University, June 26-30, 2023. (29.06.2023)

Thank you for your attention!