A Journey into Global Stability: From Monotone to Mixed Monotone and from Autonomous to Nonautonomous

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For one-dimensional maps such as the logistic, the Ricker, and the rational (Beverton-Holt) the positive equilibria are not only locally asymptotically stable, but are also globally asymptotically stable or, as we might say, in those examples "local stability implies global stability". A natural question to ask is: under what circumstances will the local stability of an equilibrium point imply its global stability? There are two effective results that address this question

Theorem (Elaydi-Sacker, Coppel)

Assume that f is continuous on an interval I such that all orbits of the equation $x(t+1) = f(x(t))$ are bounded. Then every orbit converges to an equilibrium point if and only if there are no 2-cycles

A journey into Global Stability: One-dimensional Maps

Definition

A map $f : [a, b] \to b$, b maybe ∞, is an S-map if

(i) f is C^3-map and f' vanishes at most one point d which is a relative extremum of f.

(ii) There exists $x^* \in (a, b)$ such that $f(x) > x$ if $x < x^*$, and $f(x) < x$, if $x > x^*$,

(iii) The Schwarzian derivative $Sh(x) < 0$ for all $x \in [a, b]$, except at the critical point d, where

$$Sh(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Theorem (Allwright-Singer)

Let f be an S-map and x^* is a fixed point of f. If $|f'(x^*)| \le 1$, then x^* is globally asymptotically stable.

A journey into Global Stability: multi-dimension

E-S Theorem was extended to a special class of higher dimensional maps, called triangular maps. (applied to hierarchical models in biology and economics)

A map $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is called triangular when

$$F(\mathbf{x}) = (f_1(x_1), f_2(x_1, x_2), \ldots, f_n(x_1, x_2, \ldots, x_n))^T$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n_+$.

A hyperspace H_k of dimension $k \leq n$ is defined as

$$H_k = \{(x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0)^T : x_i > 0\}.$$

A fiber $\mathcal{F}_k$ in H_k is a one-dimensional subset of H_k, then one of the fibers is given by

$$\mathcal{F}_k = \{(x_1^*, x_2^*, \ldots, x_{k-1}^*, x_k, 0, 0, \ldots, 0)^T : x_k > 0\}$$

where $(x_1^*, x_2^*, \ldots, x_{k-1}^*, x_k, 0, 0, \ldots, 0)^T$ is a fixed point in H_{k-1}.

To state our main theorem, we will make the following assumptions.

(H_1) : All orbits of the system are bounded.

(H_2) : There are no 2-cycles on any fiber.

(H_3) : There are finitely many fixed points.

Theorem (Balreira, Elaydi, Luis)

Let $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a continuous triangular map of Kolmogorov type such that assumption (H_1), (H_2), and (H_3) hold true. Then every orbit must converge to a fixed point of the map F in $\mathbb{R}^n_+$.

A journey into Global Stability: Liapunov Method applied to Epidemic Models

The SEI Compartmental Model (with no treatment) The host population is divided into the following epidemiological classes or subgroups: susceptibles (S), exposed (E, infected but not infectious), infectious (I). $N(t) = S(t) + E(t) + I(t)$ denotes the total population. Let Λ be the recruitment rate of the population, d be the natural death rate, γ be the death rate caused by the disease, and the mean exposed period is $\frac{1}{\alpha}$ where $\alpha > 0$ is the rate of loss of latency. In nearly 5-10 % of susceptible people, latent TB may be activated due to immune evasion by *Mtb* from intracellular phagosome within the macrophage, perpetrating TB. The parameters α, γ and d,verify $0 \leq \alpha \leq 1$, $0 \leq \gamma \leq 1$, $0 \leq d \leq 1$ Assuming there is exogenous reinfection, the disease dynamics may be represented by the following system of difference equations where Λ is the recruitment, r_1 and r_2, the rate of recovering people from Exposed and Infectious, $0 \leq r_1, r_2 \leq 1$. The fraction of susceptibles that escapes the infection at time t is $\mu_1 \varphi_1(I(t)/N)$ where $\varphi_1(I/N)$ is the escape function, and μ_1, $0 \leq \mu_1 \leq 1$ is the level of infection. It is known that even after heavy exposure to TB, some individuals do not develop *M. tuberculosis* infection and innate immunity probably account for this natural protection or *early clearance* of *M. tuberculosis*. Therefore, asymptotically, this fraction is bounded below by $1 - \mu_1$. Hence in the same time interval, the fraction of susceptibles that did not escape from the infection is $\mu_1(1 - \varphi(I/N))$

$$
\begin{aligned}
S(t+1) &= \Lambda + (1-\mu_1-d)S(t) + \mu_1\varphi_1(I(t)/N(t))S(t) + r_1E(t) + r_2I(\\
E(t+1) &= \mu_1(1-\varphi_1(I(t)/N(t)))S(t) + (1-d-\alpha-r_1)E(t) \\
I(t+1) &= \alpha E(t) + (1-d-\gamma-r_2)I(t)
\end{aligned}
\tag{1}
$$

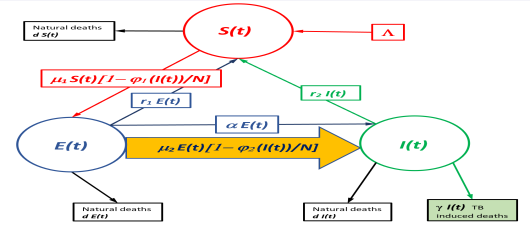

Figure 1: Chartflow of both SEI Compartment Models(with no treatment) The chart flow of the non-exogenous model is obtained by deleting the arrow whose interior is colored in orange.

Theorem (LaSalle Invariance Principle)

Consider the difference equation

$$\mathbf{x}(t+1) = F(\mathbf{x}(t)) \tag{2}$$

where $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *is continuous on a subset* G *of* $\mathbb{R}^n_+$*. Suppose there is a Liapunov function* $V : G \to \mathbb{R}$ *such that* V *is continuous on the closure* $\overline{G}$ *of* G*. Let* $E = \{\mathbf{x} : \Delta V(\mathbf{x}(t)) = 0\}$ *and* M *be the largest positively invariant subset of* E*. Assume that for every point* $\mathbf{x} \in G$*, its orbit* $O(\mathbf{x})$ *is bounded and is a subset of* G*. Then there exists* $c \in R$ *such that for every* $x \in G$*,* $\omega(x) \subset M \cap V^{-1}(c)$*.*

A journey into Global Stability: Liapunov Method applied to Epidemic Models, Next Generation Matrix Method

Let $X_0 = (E, I)^T$, $X_1 = S^T$, $X = (X_0, X_1) \in R^3_+$. Hence system may be written as

$$\begin{aligned} X_0(t+1) &= G_0(X(t)) \\ X_1(t+1) &= G_1(X(t)) \end{aligned} \quad (3)$$

where $G_0(X(t)) = \begin{pmatrix} E(t+1) \\ I(t+1) \end{pmatrix} = \mathcal{F}(t) + \mathcal{T}(t)$, and $G_1(X(t)) = S(t+1)$, where

$$\mathcal{F}(t) = \begin{pmatrix} \mu_1(1-\varphi_1(I(t)/N)S(t) \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(t) \\ \mathcal{F}_2(t) \end{pmatrix}$$

is the vector of new infections that survive in the time interval $[0, t]$, and

$$\mathcal{T}(t) = \begin{pmatrix} (1-d-\alpha-r_1)E(t) \\ \alpha E(t) + (1-d-\gamma-r_2)I(t) \end{pmatrix} = \begin{pmatrix} \mathcal{T}_1(t) \\ \mathcal{T}_2(t) \end{pmatrix}$$

is the vector of all other transitions.

A journey into Global Stability: Liapunov Method applied to Epidemic Models

Next, we compute the Jacobian matrix of $\mathcal{T}(t)$ and $\mathcal{F}(t)$ at the disease-free equilibrium (DFE) $\mathcal{E}_0 = (0, 0, S^*) = (0, 0, N)$

$$F(t)|_{(0,0,S^*)} = \begin{pmatrix} \frac{\partial \mathcal{F}_1(t)}{\partial E} & \frac{\partial \mathcal{F}_1(t)}{\partial I} \\ \frac{\partial \mathcal{F}_2(t)}{\partial E} & \frac{\partial \mathcal{F}_2(t)}{\partial I} \end{pmatrix} = \begin{pmatrix} 0 & \mu_1\beta_1\frac{S^*}{N^*} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mu_1\beta_1 \\ 0 & 0 \end{pmatrix}$$

$$T(t)|_{(0,0,S^*)} = \begin{pmatrix} \frac{\partial \mathcal{T}_1(t)}{\partial E} & \frac{\partial \mathcal{T}_1(t)}{\partial I} \\ \frac{\partial \mathcal{T}_2(t)}{\partial E} & \frac{\partial \mathcal{T}_2(t)}{\delta I} \end{pmatrix} = \begin{pmatrix} 1 - d - \alpha - r_1 & 0 \\ \alpha & 1 - d - \gamma - r_2 \end{pmatrix}.$$

A journey into Global Stability: Liapunov Method applied to Epidemic Models

Define a Liapunov function as

$$
\begin{aligned}
V(X_0, X_1) &= W^T(I - T)^{-1}X_0 \\
&= \left(\frac{1}{d+\alpha+r_1} + \frac{\alpha^2}{(d+\gamma+r_2)(d+\alpha+r_1)}, \frac{\alpha}{d+\gamma+r_2} \right) X
\end{aligned}
$$

with $X_0 \in \mathbf{R}^2_+ | \{0\}$.
Now let $\mathbf{f}(X_0, X_1) = (F+T)X_0 - G_0(X_0, X_1)$. Then $X_0(t+1) = (F+T)X_0(t) - \mathbf{f}(X_0(t), X_1(t))$ since $G_0(0, X_1^*) = 0$, $\mathbf{f}(0, X_1^*) = \mathbf{0}$.

Theorem

Assume that $\mathcal{R}_0 \leq 1$. Then $X^* = (0, X_1^*)$ is globally asymptotically stable.

A journey into Global Stability: Liapunov Method applied to the Ricker Map

Example (Baigent, Elaydi, et al. JDEA 2023)

Consider the 2-D Ricker map

$$x(t+1) = x(t)e^{\alpha - c_{11}x(t) - c_{12}y(t)}$$
$$y(t+1) = y(t)e^{\beta - c_{22}y(t) - c_{21}x(t)}.$$

Define a Liapunov function $V : \mathbb{R}_+^2 \to \mathbb{R}_+$ as

$$V(x,y) = \frac{c_{21}}{c_{11}}x^2 + \frac{c_{12}}{c_{22}}y^2 + 2\frac{c_{12}c_{21}}{c_{11}c_{22}} - 2\alpha\frac{c_{21}}{c_{11}}x - 2\beta\frac{c_{12}}{c_{22}}y.$$

Theorem

If $\alpha, \beta \in (0,2]$ and $\frac{c_{12}}{c_{22}} < \frac{\alpha}{\beta} < \frac{c_{11}}{c_{22}}$, then the interior equilibrium point (x^*, y^*) is globally asymptotically stable and each of the axial equilibrium points $(\alpha, 0)$ and $(0, \beta)$ is a saddle point with the positive half-axis as is stable manifold and the heteroclinic orbit

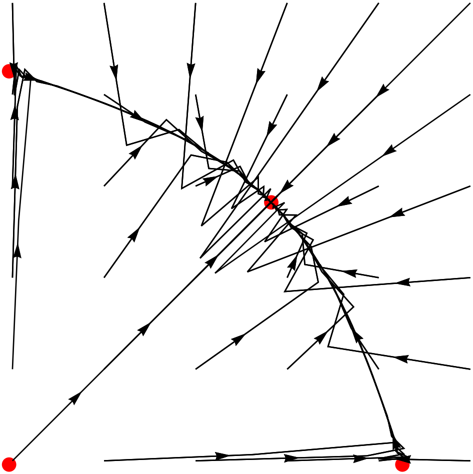

Figure 2: $\alpha = \beta = 1.25$

A journey into Global Stability: Monotone Maps

Definition

Let $F : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be a continuous map, with $F(x, y) = (f(x, y), g(x, y))^T$. Then we say that the map F is competitive (strongly competitive), if $f(x, y)$ is non-decreasing (increasing) in x and non-increasing (decreasing) in y or non-increasing (decreasing)in x and non-decreasing (increasing) in y.

Theorem (Hal Smith)

Assume that the map $F : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is C^1 and satisfies the following conditions:
(i) $\det JF(\mathbf{x}) > 0$ *for all* $\mathbf{x} \in \mathbb{R}^2_+$,
(ii) F is competitive,
(iii) F is injective.
Then if an orbit has a compact closure in $\mathbb{R}^2_+$, then it must converge to a fixed point of the map F.

An Example

Example

A general Leslie-Gower model, for 2-species is given by

$$x_1(t+1) = \frac{b_1 x_1(t)}{1 + c_{11} x_1(t) + c_{12} x_2(t)}$$
$$x_2(t+1) = \frac{b_2 x_2(t)}{1 + c_{21} x_1(t) + c_{22} x_2(t)} \tag{4}$$

where $x_i(t)$ represents the size or density of species x_i at time t, c_{ii} the intraspecific parameters for species x_i, and c_{ij} the interspecific parameter between species x_i and x_j The map representing this model is monotone and if $\frac{c_{21}}{c_{11}} < \frac{b_2 - 1}{b_1 - 1} < \frac{c_{22}}{c_{12}}$, then the interior (coexistence) equilibrium point is GAS in the interior of R^2_+

A challenging Map

In their book, Vincent and Brown introduced biological models with Darwinian evolution. Cushing and his Collaborators did focus on evolutionary discrete models and their local stabilities. Here is a single species evolutionary model of the Ricker type

$$
\begin{aligned}
x(t+1) &= x(t)e^{\alpha - \frac{u^2}{2} - c_0 x(t)} \\
u(t+1) &= (1-\sigma^2)u(t) - c_1\sigma^2 x(t)
\end{aligned}
\tag{5}
$$

where σ, $0 < \sigma^2 < 2$, is the speed of evolution, and c_1 measures the difference between the competition intensities experienced by individuals that have the population mean trait and those whose traits are slightly different from the mean.

Clearly this map is not monotone and moreover, none of the known tools can help to prove global stability

A challenging Map: Global Stability for a Special Case-the Limiting Equation

Now if we assume that the trait u is independent of the density or the size of the species, then $c_1 = 0$. In this case we can use the theory of the limiting equations.

Let $F, \ F_t : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n_+$ to be continuous functions for all $t \in \mathbb{Z}_+$. Assume that

$\mathbf{A_1}$: F_t converges uniformly to F as $t \to \infty$.

$\mathbf{A_2}$: $F_t : int(\mathbb{R}^n_+) \longrightarrow int(\mathbb{R}^n_+)$.

Theorem (D'Aniello and Elaydi)

Assume $\mathbf{A_1}$ and $\mathbf{A_2}$ and the limiting equation has an equilibrium point $\mathbf{x}^* \in \mathbb{R}^n_+$. Then if $\mathbf{x}^* \in int(\mathbb{R}^n_+)$, and if it is globally asymptotically stable on $int(\mathbb{R}^n_+)$ as an equilibrium point of limiting equation, then all solutions of the nonautonomous difference equation with $\mathbf{x(0)} \in int(\mathbb{R}^n_+)$ tend to $\mathbf{x}^*$.

Limiting Equation

Assume $c_1, c_2 = 0$, we have

$$\begin{aligned} x(t+1) &= x(t)e^{\alpha - u_1^2/2 - c_{01}x(t)} \\ y(t+1) &= y(t)e^{\beta - u_2^2/2 - c_{02}y(t)} \\ u_1(t+1) &= (1-\sigma^2)u_1(t) \\ u_2(t+1) &= (1-\sigma^2)u_2(t) \end{aligned} \tag{6}$$

Theorem

Assume $0 < \alpha, \beta < 2$, and $0 < \sigma^2 < 2$, then the equilibrium point $(x^*, y^*, 0, 0)$ is globally asymptotically stable.

Mixed Monotone Maps

Let X be an ordered metric space. A continuous map $F : X \to X$ is mixed monotone if there exits a map $f : X \times X \to X$ satisfying

(i) $F(\mathbf{x}) = f(\mathbf{x}, \mathbf{x})$ for all $\mathbf{x} \in X$;

(ii) for $\mathbf{y} \in X$ and $\mathbf{x}_1 \leq \mathbf{x}_2$ we have $f(\mathbf{x}_1, \mathbf{y}) \leq f(\mathbf{x}_2, \mathbf{y})$;

(iii) for $\mathbf{x} \in X$ and $\mathbf{y}_1 \leq \mathbf{y}_2$ we have $f(\mathbf{x}, \mathbf{y}_2) \leq f(\mathbf{x}, \mathbf{y}_1)$.

Consider again the system

$$x(t+1) = x(t)e^{\alpha - \frac{u^2}{2} - c_0 x(t)}$$
$$u(t+1) = (1-\sigma^2)u(t) - c_1\sigma^2 x(t) \qquad (7)$$

Theorem

Let $0 < \alpha < 1$, $\sigma^2 \in (1,2)$ and $c_1 > 0$. Then the map F representing the above system is mixed monotone

Sketch of the proof. Define $f((x_1,u_1),(x_2,u_2)) = F(x_1,u_2)$. Then one shows that the map f satisfies the conditions above and thus the map F is mixed monotone.

Multi-Species

The above result may be extended to multi-species.

$$\begin{cases} x(t+1) = x(t)e^{\alpha - u_1^2(t)/2 - c_{11}(0)x(t) - c_{12}y(t)} \\ y(t+1) = y(t)e^{\beta - u_2^2(t)/2 - c_{21}x(t) - c_{22}(0)y(t)} \\ u_1(t+1) = (1-\sigma_1^2)u_1(t) - \sigma_1^2 c_1 x(t) \\ u_2(t+1) = (1-\sigma_2^2)u_2(t) - \sigma_2^2 c_2 y(t) \end{cases} . \tag{8}$$

Theorem

Let $0 < \alpha, \beta < 1$, $\sigma^2 \in (1,2)$ and $c_1, c_2 > 0$. Then the map F representing the above system is mixed monotone.

Sketch of the proof. Define $f((x_1, y_1 u_1, u_2), (x_2, y_2, \hat{u}_1, \hat{u}_2)) = F(x_1, y_1, \hat{u}_1, \hat{u}_2)$. Then one shows that the map f satisfies the conditions above and thus the map F is mixed monotone.

Mixed Monotone

Next, we study the case when $c_1, c_2 < 0$. Using the concept of topological conjugacy.

Definition (Topological Conjugacy)

Let X and Y be two topological spaces. Two maps $F : X \to X$ and $G : Y \to Y$ are topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ G = F \circ h$, or equivalently, $G = h^{-1} \circ F \circ h$.

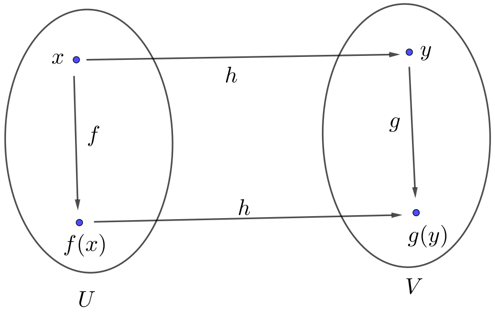

Figure 3: conjugacy of the maps F and G

Dynamical Equivalence of Conjugate Maps

Theorem (Main Theorem)

Suppose that the maps F and G are topologically conjugate, with a topological conjugacy homeomorphism h. If x^* is a globally asymptotically stable fixed point of F, then $h(x^*)$ is a globally asymptotically stable fixed point of G.

Mixed Monotone

Assume that $c_1, c_2 < 0$ and let $\hat{c}_i = -c_i > 0$, $i = 1, 2$. Then the maps

$$
\begin{aligned}
F(x, y, u_1, u_2) &= \Big(x e^{\alpha - u_1^2/2 - c_{11}(0)x - c_{12}y}, y e^{\beta - u_2^2/2 - c_{21}x - c_{22}(0)y}, \\
&\quad (1-\sigma_1^2)u_1 - \sigma_1^2 c_1 x, (1-\sigma_2^2)u_2 - \sigma_2^2 c_2 y \Big),
\end{aligned}
$$

$$
\begin{aligned}
G(x, y, u_1, u_2) &= \Big(x e^{\alpha - u_1^2/2 - c_{11}(0)x - c_{12}y}, y e^{\beta - u_2^2/2 - c_{21}x - c_{22}(0)y}, \\
&\quad (1-\sigma_1^2)u_1 - \sigma_1^2 \hat{c}_1 x, (1-\sigma_2^2)u_2 - \sigma_2^2 \hat{c}_2 y \Big)
\end{aligned}
$$

are topologically conjugate by using the homeomorphism $h(x, y, u_1, u_2) = (x, y, -u_1, -u_2)$

Theorem

Assume that $0 < \alpha, \beta < 1$, $\sigma_i^2 \in (1, 2)$ and $c_i > 0$, $i = 1, 2$. Then the map F given above is mixed monotone.

Global Stability-4D

Theorem

Under the above assumptions, the equilibrium point $(x^*, y^*, u_{1F}^*, u_{2F}^*)$ *is globally asymptotically stable.*

Global Stability-6D

Consider the following special 3-D Ricker map

$$\begin{cases} x_1(t+1) = x_1(t)e^{r-x_1(t)-ax_2(t)-ax_3(t)} \\ x_2(t+1) = x_2(t)e^{r-ax_1(t)-x_2(t)-ax_3(t)} \\ x_3(t+1) = x_3(t)e^{r-ax_1(t)-ax_2(t)-x_3(t)} \end{cases}. \tag{9}$$

Theorem (Balreira, Elaydi, Luis , Gyllenberg et al.)

The 3-D Ricker map has a globally asymptotically stable interior fixed point when $0 < r < \frac{1}{1+2a}$ *and* $0 < a < 1$.

Global Stability-6D

Consider the special case of the 3-species evolutionary model of the Ricker type

$$\begin{cases} x_1(t+1) = x_1(t)e^{r-u_1^2/2-x_1(t)-ax_2(t)-ax_3(t)} \\ x_2(t+1) = x_2(t)e^{r-u_2^2/2-ax_1(t)-x_2(t)-ax_3(t)} \\ x_3(t+1) = x_3(t)e^{r-u_3^2/2-ax_1(t)-ax_2(t)-x_3(t)} \\ u_1(t+1) = (1-\sigma_1^2)u_1(t) - \sigma_1^2 c_1 x_1(t) \\ u_2(t+1) = (1-\sigma_2^2)u_2(t) - \sigma_2^2 c_2 x_2(t) \\ u_3(t+1) = (1-\sigma_3^2)u_3(t) - \sigma_2^3 c_2 x_3(t) \end{cases} . \tag{10}$$

Theorem

Assume that $0 < r < \frac{1}{1+2a}$, $0 < a < 1$, $\sigma^2 \in (1,2)$. Then the above system has a globally asymptotically stable interior fixed point.

Periodic Mixed Monotone

Consider the non-autonomous periodic evolutionary system

$$
\begin{aligned}
x(t+1) &= x(t)e^{\alpha_t - \frac{u^2}{2} - c_0 x(t)} \\
u(t+1) &= (1-\sigma^2)u(t) - c_1\sigma^2 x(t)
\end{aligned}
\tag{11}
$$

where the parameters α_t is periodic of period p, i.e., when $\alpha_{t+p} = \alpha_t$ for all $t = 0, 1, \ldots$ and some positive integer $p > 1$. These equations maybe represented by the maps $F_t(x,u) = \left(xe^{\alpha_t - u^2/2 - x}, (1-\sigma^2)u - c_1\sigma^2 x\right)$, $\alpha_{t+p} = \alpha_t$, $t = 0, 1, \ldots$. In other words, $F_{t+p} = F_t$. It is clear that the origin is a fixed point of the $p-$periodic System since it is a fixed point of each individual map F_t. Since the Jacobian of the composition map $\Phi_p = F_{p-1} \circ \ldots \circ F_1 \circ F_0$ is the product of the individual Jacobians, it follows that

Periodic Mixed Monotone

Lemma

Let $\sigma^2 \in (0,2)$ and $0 < \alpha_t < 1$, $t = 0, 1, 2, \ldots$, with $\alpha_{t+p} = \alpha_t$. Then the origin is a saddle fixed point of the $p-$periodic System (11).

Now, if $c_1 = 0$ then the p-periodic System has a $p-$periodic cycle of the form

$$C_p = \{(\overline{x}(0), 0), (\overline{x}(1), 0), \ldots, (\overline{x}(p-1), 0)\}.$$

Hence, we have the following result.

Lemma

Let $0 < \alpha_t < 2$, $t = 0, 1, 2, \ldots$, *such that* $\alpha_{t+p} = \alpha_t$ *for some* $p > 1$. *Then the* $p-$*periodic evolutionary Ricker system*

$$\begin{cases} x(t+1) = x(t)e^{\alpha_t - u^2(t)/2 - x(t)} \\ u(t+1) = (1-\sigma^2)u(t) \end{cases},$$

has a globally asymptotically stable p-periodic cycle of the form $\{(\overline{x}(0), 0), (\overline{x}(1), 0), \ldots, (\overline{x}(p-1), 0)\}$.

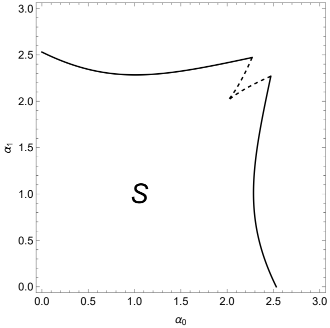

Figure 4: Region **S**, in the parameter space bifurcation diagram, where the one-dimensional 2-periodic Ricker equation has a non-trivial globally asymptotically stable 2−periodic cycle. As the parameters cross the solid curves a period-doubling bifurcation takes place while when it crosses the dashed curves a saddle-node bifurcation takes place.

Periodic Mixed Monotone

Theorem

Let $\sigma^2 \in (1,2)$, $c_1 \neq 0$ and $0 < \alpha_t < 1$, $t = 0, 1, 2, \ldots$ with $\alpha_{t+p} = \alpha_t$ for some $p > 1$. The composition of the p mixed monotone maps of evolutionary Ricker system

$$F_t(x,u) = \left(x e^{\alpha_t - u^2/2 - x}, (1-\sigma^2)u - c_1\sigma^2 x \right), \quad F_{t+p} = F_t$$

is a mixed monotone map.

Periodic Mixed Monotone-period 2

Consider the difference equation

$$\mathbf{x}(t+1) = F(\mathbf{x}(t), \alpha_j) = F_j(\mathbf{x}(t)), \tag{12}$$

where $F : U \times G \to U$ is continuous, $U \subset \mathbb{R}^2_+$, $G \subset \mathbb{R}_+$ and $JF_j(\mathbf{x}) = JF(\mathbf{x}, \alpha_j)$ (the Jacobian matrix) is continuous on $\mathbb{R}^2_+ \times G$. We start our analysis with a perturbation result that is crucial in our investigation of the global stability of the periodic 2-cycle of a 2-periodic system.

Theorem

Let $\mathbf{x}_0^* = (x_0^*, u_0^*)$ be the interior equilibrium point of $F_0(\mathbf{x})$. Assume that $(\mathbf{x}_0^*, \alpha_0) \in U \times G$ and the spectral radius $\rho(JF(\mathbf{x}_0^*, \alpha_0)) < 1$ and $\mathbf{x}_0^*$ is globally asymptotically stable hyperbolic interior equilibrium point of F_0. Then there exists $\delta > 0$ and a unique $\mathbf{x}^*(\alpha) \in U$ for $\alpha \in \mathbf{B}(\alpha_{\mathbf{0}}, \delta)$ such that $F(\mathbf{x}^*(\alpha), \alpha) = \mathbf{x}^*(\alpha)$ and $F^t(\mathbf{z}) \to \mathbf{x}^*(\alpha)$ as $t \to \infty$ for all $\mathbf{z} \in U$.

Sketch of the proof

Since $\|JF(\mathbf{x}_0^*, \alpha_0^*))\| < \rho < 1$, and $JF(x, u)$ is continuous, there exists $\delta_1 > 0$ and $\eta > 0$ such that $\|JF(\mathbf{x}, \alpha))\| < \rho < 1$ for all $\mathbf{x} \in B(\mathbf{x}_0^*, \eta)$ and $\alpha \in B(\alpha_0, \delta_1)$. Choose $\delta_0 < \delta_1$ such that $\|F(\mathbf{x}_0^*, \alpha_0) - F(\mathbf{x}_0^*, \alpha)\| < (1-\rho)\eta$, for $\alpha \in B(\alpha_0, \delta_0)$. Thus for $\mathbf{x}_1, \mathbf{x}_2 \in \overline{B(\mathbf{x}_0^*, \eta)}$, $\alpha \in B(\alpha_0, \delta_0)$. Hence by the mean value theorem

$$\|F(\mathbf{x}_1, \alpha) - F(\mathbf{x}_2, \alpha)\| \leq \int_0^1 \|JF(s\mathbf{x}_1 + (1-s)\mathbf{x}_2, \alpha) ds \|\mathbf{x}_1 - \mathbf{x}_2\| \leq \rho|\mathbf{x}_1 - \mathbf{x}_2\|$$

and
$\|F(\mathbf{x}_1, \alpha) - F(\mathbf{x}_0^*, \alpha_0)\| \leq \|F(\mathbf{x}_1, \alpha) - F(\mathbf{x}_0^*, \alpha)\| + \|F(\mathbf{x}_0^*, \alpha) - F(\mathbf{x}_0^*, \alpha_0)\| \leq \rho\|\mathbf{x}_1 - \mathbf{x}_0\| + (1-\rho)\eta$. Thus the map $F(\mathbf{x}, \alpha)$ is a uniform contraction self-map.

Periodic Mixed Monotone: perturbation

Theorem (Main Theorem 1)

Assume the conditions above, in which $\alpha = \alpha_0$. Then for sufficiently small $\delta > 0$ and letting $\alpha_1 = \alpha_0 \pm \delta$, there is a 2-periodic cycle which is globally asymptotically stable in the interior of the first quadrant if $c_1 < 0$ and in the interior of the fourth quadrant if $c_1 > 0$.

Sketch of the proof.

Proof.

Let us now consider the second iteration $F^2 = F \circ F$ of the map F We perturb F^2 and write it as the composition of two maps $G = F_1 \circ F_0$, where $F_0 = F$, in which $\alpha = \alpha_0$, and $F_1(x, u) = (xe^{\alpha_1 - u^2/2 - c_0 x}, (1 - \sigma^2)u - c_1\sigma^2 x)$ where $\alpha_1 = \alpha_0 \pm \delta$. Using The perturbation theorem, there exists an interior 2-periodic cycle which is globally asymptotically stable. □

Periodic Mixed Monotone: Example

Example

Let $c_0 = 1$, $\sigma^2 = 1.5$, $c_1 = 2$ and $\alpha_0 = 0.3$. Then $(0.210977, -0.421954)$ is a fixed point of the map F_0, where $F_0(x, u) = (xe^{0.3-u^2/2-x}, -0.5u - 2\sigma^2 x)$.
Now, the Jacobian matrix of F_0 evaluated at the fixed point $(0.271644, -0.461582)$ is given by

$$JF_0 = JF_0(0.271644, -0.461582) = \begin{pmatrix} 0.789023 & 0.0890228 \\ -3. & -0.5 \end{pmatrix}.$$

Example (cont.)

Hence, $\det JF_0 = -0.127443$ and $\operatorname{tr} JF_0 = 0.289023$. Clearly, local stability conditions are satisfied and we have local asymptotic stability. Hence, $(0.210977, -0.421954)$ is a globally asymptotically stable fixed point of F_0. Thus the mapping F_0^2 is mixed monotone. Now, letting $\alpha_1 = \alpha_0 + 0.2 = 0.5$, it follows that $C_2 = \{(\bar{x}(0), \bar{u}(0)), (\bar{x}(1), \bar{u}(1))\}$ (which is approximately $(0.271644, -0.461582), (0.251217, -0.58414))$ is a globally asymptotically stable $2-$periodic cycle of the system, where the composition map is $G = F_1 \circ F_0$ with $F_1(x, u) = (xe^{0.5-u^2/2-x}, -0.5u - 2\sigma^2 x)$.

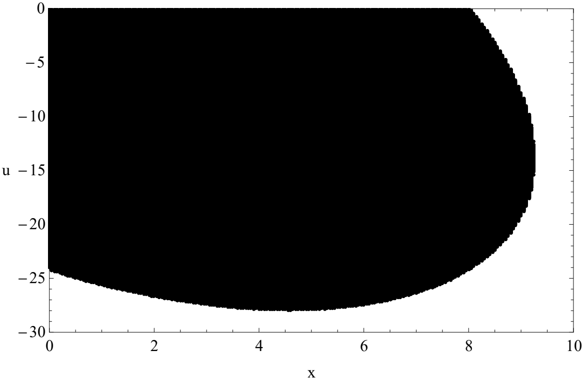

Figure 5: The absorbing region of the 2−periodic cycle when $c_1 > 0$

An Extension to 2-species

One may extend the preceding theorem to two-dimensional 2-periodic Ricker map with evolution

$$\begin{cases} x(t+1) = x(t)e^{\alpha_t - u_1^2(t)/2 - c_{11}(0)x(t) - c_{12}y(t)} \\ y(t+1) = y(t)e^{\beta_t - u_2^2/2 - c_{21}x(t) - c_{22}(0)y(t)} \\ u_1(t+1) = (1-\sigma_1^2)u_1(t) - \sigma_1^2 c_1 x(t) \\ u_2(t+1) = (1-\sigma_2^2)u_2(t) - \sigma_2^2 c_2 y(t) \end{cases}, \tag{13}$$

where $\alpha_{t+2} = \alpha(t)$ and $\beta_{t+2} = \beta(t)$

An Extension to p-periodic Systems

Here we start with a 2-periodic system $F_1 \circ F_0$, with α_0 for F_0 and $\alpha_1 = \alpha_0 \pm \delta_1$. Then we move to the 4-periodic system $F_3 \circ F_2 \circ F_1 \circ F_0$, with α_0 for F_0 and $\alpha_1 = \alpha_0 \pm \delta_1$ for F_1, $\alpha_2 = \alpha_1 \pm \delta_2$ for F_2 and $\alpha_3 = \alpha_2 \pm \delta_3$ for F_3. Now suppose that $p = 2k$. Then for F_{2k-1} we let $0 < \alpha_{2k-1} = \alpha_0 \pm \sum_{i=1}^{2k-1} \delta_i < 1$

P-Periodic Mixed Monotone: perturbation

Theorem (Main Theorem 2)

Assume that $p = 2k$ and the conditions above, in which $\alpha = \alpha_0$. Then for sufficiently small $\sum_{i=1}^{2k-1} \delta_i > 0$ and letting $\alpha_{2k-1} = \alpha_0 \pm \sum_{i=1}^{2k-1} \delta_i$, there is a p-periodic cycle which is globally asymptotically stable

Sketch of the proof.

Proof.

Using The peturbation theorem, there exists an interior p-periodic cycle which is globally asymptotically stable. □

Open Problems and Conjectures

Open Problem 1: extend the global stability results to the case $0 < \sigma^2 < 1$

Open Problem 2: Extend the application of mixed monotonicity to other types of models such as the predator-prey models, structured population models, etc.

Open Problem 3: A challenging problem is to study evolutionary epidemic models. Two research teams are at the initial stages of this work.

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Thank you for your attention

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