On complete controllability of some three- and four-level closed quantum systems

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Problem statement

- Dynamics of a closed quantum system under a single coherent control (the Schrödinger equation):

$$\frac{d}{dt}U_t^f = -i(H_0 + f(t)V)U_t^f, \qquad U_{t=0}^f = \mathbb{I}_N$$

where

1. U_t^f - unitary evolution operator;
2. H_0 and V - free and interaction Hamiltonians ($[H_0, V] \neq 0$);
3. $f(t) \in L^2([0, T], \mathbb{R})$ - single coherent control.

Key definitions

Complete controllability

A closed quantum system is called *completely controllable* if for a unitary operator $U \in \mathrm{U}(N)$ there exists a finite final time $T > 0$ and a control $f(t) \in L^2([0, T], \mathbb{R})$ such that the solution of the Schrödinger equation satisfies

$$U = e^{i\alpha} U_T^f$$

where $\alpha \in \mathbb{R}$ is a (physically non-relevant) phase.

Dynamical algebra

Dynamical algebra of a quantum system is a Lie algebra produced by all commutators of the free and interaction Hamiltonians, multiplied by i:

$$iH_0,\ iV,\ [iH_0,\ iV],\ [iH_0,\ [iH_0, iV]],\ \ldots(etc.)$$

Key theorems about complete controllability

Necessary condition for complete controllability[1]

If a considered system is completely controllable, then a non-zero traceless skew-Hermitian operator which commutes with the dynamical algebra of this system does not exist.

Complete controllability criteria[2]

A quantum system with a free Hamiltonian H_0 and an interaction Hamiltonian V is completely controllable if and only if its dynamical algebra contains $\mathfrak{su}(N)$.

[1] *Polack T., Suchowski H., Tannor D.J.* Uncontrollable quantum systems: A classification scheme based on Lie subalgebras // Phys. Rev. A. 2009. 79, 5. 053403.

[2] *Albertini F., D'Alessandro D.* Notions of controllability for bilinear multilevel quantum systems // IEEE Trans. Automat. Control. 2003. 48, 8. 1399–1403.

What is this work exactly about?

1. Having conditions obtained in terms of H_0 and V matrix representations

$$H_0 = \mathrm{diag}(E_1,\, E_2,\, E_3,\, \ldots,\, E_N) = \sum_{k=1}^{N} E_k \left(-iB_k^{\mathrm{D}}\right),$$

$$V = \begin{pmatrix} 0 & v_{12} & v_{13} & \ldots & v_{1N} \\ v_{12}^* & 0 & v_{23} & \ldots & v_{2N} \\ v_{13}^* & v_{23}^* & 0 & \ldots & v_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{1N}^* & v_{2N}^* & v_{3N}^* & \ldots & 0 \end{pmatrix} = \sum_{i<j} v_{ij}^{\mathrm{Im}} \left(-iB_{ij}^{\mathrm{Re}}\right) + v_{ij}^{\mathrm{Re}} \left(-iB_{ij}^{\mathrm{Im}}\right)$$

where

$$B_{ij}^{\mathrm{Re}} = \left(\mu_{\beta\alpha} - \mu_{\alpha\beta}\right), \quad B_{\alpha\beta}^{\mathrm{Im}} = \left(\mu_{\alpha\beta} + \mu_{\beta\alpha}\right) i, \quad B_k^{\mathrm{D}} = \mu_{kk}\, i$$

and the matrices μ_{ij} defined with $\mu_{ij}^{(\alpha\beta)} = \delta_{\alpha i}\,\delta_{\beta j}$;

What is this work exactly about?

3. Taking arbitrary coefficients

$$v_{ij} = v_{ij}^{\mathrm{Re}} + i v_{ij}^{\mathrm{Im}} \in \mathbb{C}, \quad E_k \in \mathbb{R};$$

4. Considering *level degeneracies*

$$E_{k_1} = E_{k_2} \, [= \ldots = E_{k_m}, \; m < N]$$

and *level-spacing degeneracies*

$$(E_{k_1} - E_{k_2}) = (E_{k_2} - E_{k_3}) \text{ or } (E_{k_3} - E_{k_4}) \, [= \ldots] \neq 0$$

as well as their combinations.

What is this work exactly about?

Useful feature of dynamical algebra [3]

If a dynamical algebra $\mathcal{L}$ contains all elements $B^{\cdot}_{k(k+1)}$, then it contains $\mathfrak{su}(n)$.

So, a system is completely controllable if and only if

- $N = 3$: $B^{\mathrm{Re}}_{12}, B^{\mathrm{Im}}_{12}, B^{\mathrm{Re}}_{23}, B^{\mathrm{Im}}_{23} \in \mathcal{L}$;
- $N = 4$: $B^{\mathrm{Re}}_{12}, B^{\mathrm{Im}}_{12}, B^{\mathrm{Re}}_{23}, B^{\mathrm{Im}}_{23}, B^{\mathrm{Re}}_{34}, B^{\mathrm{Im}}_{34} \in \mathcal{L}$.

[3] *Schirmer S.G., Fu H., Solomon A.I.* Complete controllability of quantum systems // Phys. Rev. A. 2001. 63, 6. 063410.

Results: three-level systems

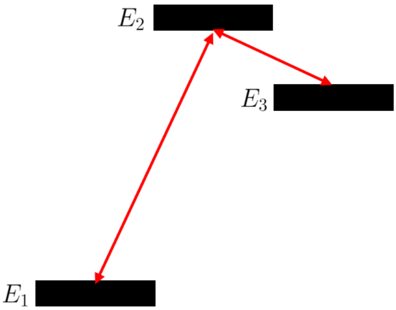
- **Systems with two allowed transitions**
  - *Hamiltonians*:

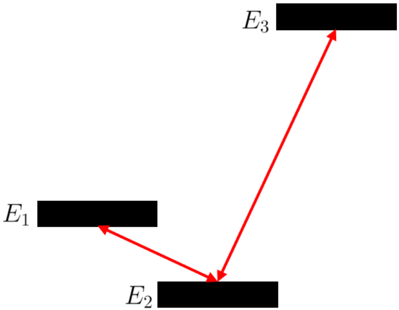
$$H_0 = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v_{12} & 0 \\ v_{12}^* & 0 & v_{23} \\ 0 & v_{23}^* & 0 \end{pmatrix}$$

  - *What has been previously known?*
    1. Systems with $E_1 = E_2$ or $E_2 = E_3$ are completely controllable for arbitrary $v_{ij} \in \mathbb{R} \setminus \{0\}$ [3];
    2. There is a number of such systems with level-spacing degeneracy which are not completely controllable (estimated for some exact real values of v_{ij})[1; 4].

[4] *Turinici G., Rabitz H.* Quantum wavefunction controllability // Chem. Phys. 2001. 267. 1–9.

Results: three-level systems





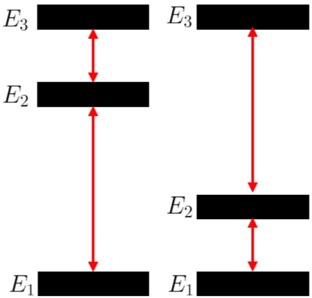
Λ-atom,
no level-spacing degeneracy

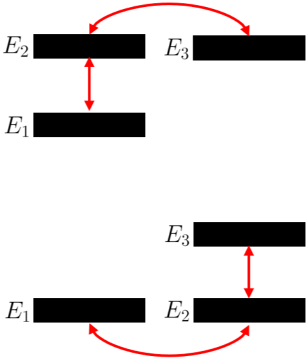
$E_1 < E_2, \; E_3 < E_2, \; E_1 \neq E_3;$

V-atom,
no level-spacing degeneracy

$E_1 > E_2, \; E_3 > E_2, \; E_1 \neq E_3;$

Results: three-level systems





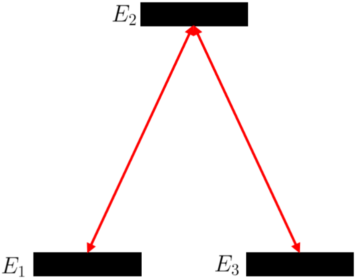
Sequential levels structure,
no level-spacing degeneracy

$E_1 < E_2 < E_3$ or $E_1 > E_2 > E_3$,
and $E_2 - E_1 \neq E_3 - E_2$;

Level degenerancy,
no level-spacing degeneracy

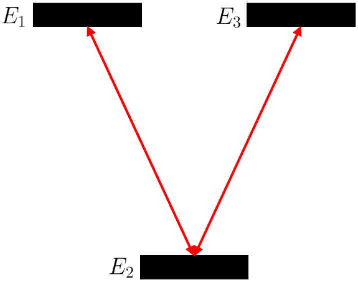
$E_1 = E_2 \neq E_3$ or $E_1 \neq E_2 = E_3$;

Results: three-level systems



Λ-atom, level-spacing degeneracy

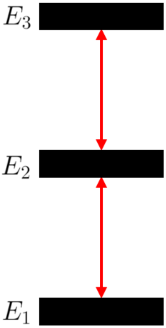
$$E_1 < E_2,\ E_3 < E_2,\ E_1 = E_3;$$



V-atom, level-spacing degeneracy

$$E_1 > E_2,\ E_3 > E_2,\ E_1 = E_3;$$

Results: three-level systems



System with level-spacing degeneracy and sequential level structure

$E_1 < E_2 < E_3$ or $E_1 > E_2 > E_3$, and $E_2 - E_1 = E_3 - E_2$.

Results: three-level systems

Controllabilty of three-level systems with two allowed transitions [5]

Let H_0 and V have the presented structure, $v_{ij} \in \mathbb{C} \setminus \{0\}$. If a system has no level-spacing degeneracy, then it is completely controllable. Otherwise, it is completely controllable if and only if $E_2 - E_1 = E_3 - E_2$ and $|v_{12}| \neq |v_{23}|$.

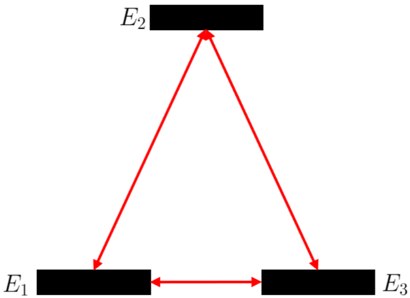
[5] *Kuznetsov S.A., Pechen A.N.* On Controllability of Λ- and V-Atoms and Other Three-Level Systems with Two Allowed Transitions // Lobachevskii J. Math. 2023 (accepted)

- **Systems with level and level-spacing degeneracies and three allowed transitions**
  - *Hamiltonians*:

$$H_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v_{12} & v_{13} \\ v_{12}^* & 0 & v_{23} \\ v_{13}^* & v_{23}^* & 0 \end{pmatrix}$$

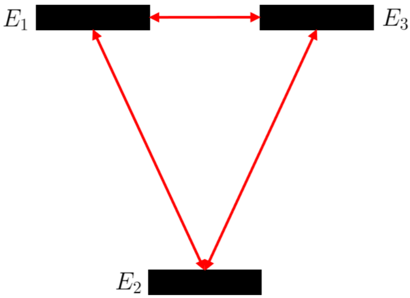
  - *Have not been approached before.*

Results: three-level systems



Modified Λ-atom
with level-spacing degeneracy

$E_1 < E_2,\ E_3 < E_2,\ E_1 = E_3;$



Modified V-atom
with level-spacing degeneracy

$E_1 > E_2,\ E_3 > E_2,\ E_1 = E_3;$

Controllability of three-level systems with level and level-spacing degeneracies and three allowed transitions

Let H_0 and V have the presented structure, $v_{ij} \in \mathbb{C} \setminus \{0\}$. If $|v_{12}| \neq |v_{23}|$, then the system is completely controllable. Otherwise, it is completely controllable if and only if $\arg(v_{13}) \neq \arg(v_{12}) + \arg(v_{23}) + \pi n$.

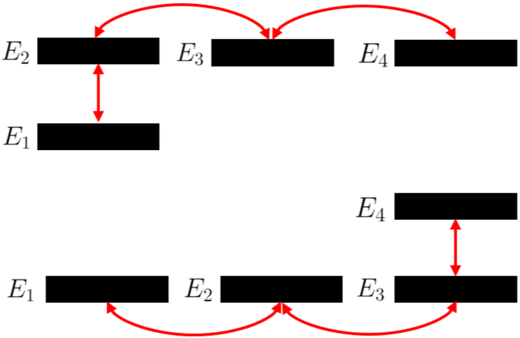
Results: four-level systems

- **Some systems with the «chained» interaction Hamiltonian**
  - *Hamiltonians*:

$$H_0 = \begin{pmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & E_3 & 0 \\ 0 & 0 & 0 & E_4 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v_{12} & 0 & 0 \\ v_{12}^* & 0 & v_{23} & 0 \\ 0 & v_{23}^* & 0 & v_{34} \\ 0 & 0 & v_{34}^* & 0 \end{pmatrix}$$

  - *What has been previously known?*
    1. Systems with $E_1 = E_2 = E_3$ or $E_2 = E_3 = E_4$ are completely controllable for arbitrary $v_{ij} \in \mathbb{R} \setminus \{0\}$ [3].

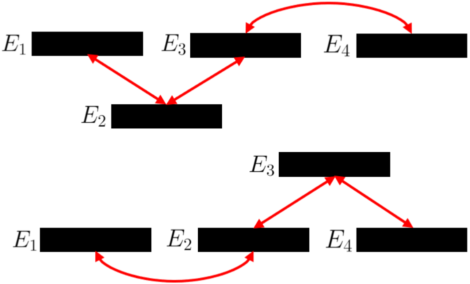
Results: four-level systems



Systems with three sequential degenerate levels

$E_1 = E_2 = E_3 \, (\neq E_4)$
or $(E_1 \neq) \, E_2 = E_3 = E_4$;

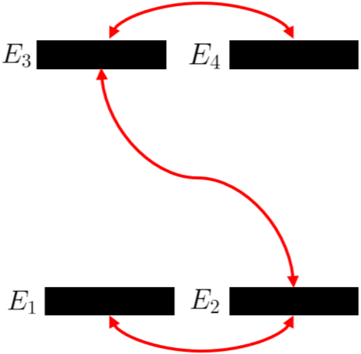
Results: four-level systems



Systems with three non-sequential degenerate levels

$E_1 = E_3 = E_4 \, (\neq E_2)$
or $(E_3 \neq) \, E_1 = E_2 = E_4$;

Results: four-level systems



«2+2» systems

$$E_1 = E_2 \,(\neq)\, E_3 = E_4.$$

Results: four-level systems

Controllability of four-level systems with sequential degenerate levels [6]

Let H_0 and V have the presented structure, $v_{ij} \in \mathbb{C} \setminus \{0\}$ and $E_1 = E_2 = E_3$ or $E_2 = E_3 = E_4$. Then the system is completely controllable.

Controllability of four-level systems with non-sequential degenerate levels

Let H_0 and V have the presented structure, $v_{ij} \in \mathbb{C} \setminus \{0\}$ and $E_1 = E_3 = E_4$ or $E_1 = E_2 = E_4$. Then the system is completely controllable.

Controllability of four-level systems with «2+2» structure (sufficient)

Let H_0 and V have the presented structure, $v_{ij} \in \mathbb{C} \setminus \{0\}$ and $E_1 = E_2$, $E_3 = E_4$. If $|v_{12}| \neq |v_{34}|$, then the system is completely controllable.

[6] *Kuznetsov S.A., Pechen A.N.* On controllability of a highly degenerate four-level quantum system with a “chained” coupling Hamiltonian // Lobachevskii J. Math. 2022. 43, 7. 1683-–1692.

Short results summary

- **Results**:

  We have obtained conditions of complete controllability in terms of H_0 and V matrix representation for

  1. All three-level systems with two allowed transitions;
  2. Three-level systems with level and level-spacing degeneracies and three allowed transitions;
  3. Some four-level systems with the «chained» interaction Hamiltonian.

- **Papers**:

  1. *Kuznetsov S.A., Pechen A.N.*, Lobachevskii J. Math. 2022. 43, 7. 1683-–1692.
  2. *Kuznetsov S.A., Pechen A.N.*, Lobachevskii J. Math. 2023 (accepted)

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