

Surgery of Two Liouville Tori into Two in One Integrable Problem of the Dynamics of Magnetic Vortices

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- Critical set of the momentum map
- Bifurcation diagram of the momentum map
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Constrained Problem of Three Magnetic Vortices

Free vortices at positions $r_\alpha = (x_\alpha, y_\alpha)$ are characterized by intensities Γ_α and polarities λ_α , which take values $+1$ or -1 . The topographic vortex with is fixed at the origin point.

$$\Gamma_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha}, \quad \Gamma_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}, \quad H = \frac{\Gamma_1}{\lambda_1} \ln |r_1| + \frac{\Gamma_2}{\lambda_2} \ln |r_2| + \frac{\Gamma_1 \Gamma_2}{\lambda_1 \lambda_2} \ln |r_1 - r_2|, \quad (1)$$

$$\mathcal{P}^4 = \{(r_1, r_2) \mid r_1^2 \neq 0, r_2^2 \neq 0, r_1 \neq r_2\}.$$

Lie-Poisson bracket:

$$\{f, g\} = \sum_{\alpha=1}^N \frac{1}{\Gamma_\alpha} \left(\frac{\partial f}{\partial x_\alpha} \frac{\partial g}{\partial y_\alpha} - \frac{\partial f}{\partial y_\alpha} \frac{\partial g}{\partial x_\alpha} \right). \quad (2)$$

Additional integral of motion:

$$F = \Gamma_1 r_1^2 + \Gamma_2 r_2^2. \quad (3)$$

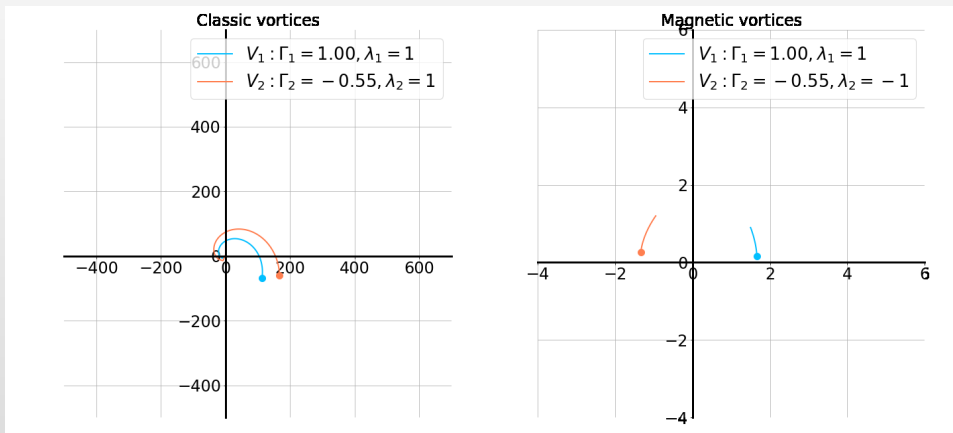
Total vortical moment:

$$M_c = \Gamma_1 (x_1 \dot{y}_1 - y_1 \dot{x}_1) + \Gamma_2 (x_2 \dot{y}_2 - y_2 \dot{x}_2) = -\frac{\Gamma_1}{\lambda_1} - \frac{\Gamma_2}{\lambda_2} - \frac{\Gamma_1 \Gamma_2}{\lambda_1 \lambda_2}.$$

Constrained Problem of Three Magnetic Vortices

$$(x_1^0, y_1^0) = (1.5; 0.9)$$

$$(x_2^0, y_2^0) = (-0.95; 1.2)$$



Liouville–Arnold Theorem

Let us consider a completely Liouville integrable system with n degrees of freedom. The common level surface

$$\mathcal{T}_f = \{\mathbf{x} \in \mathcal{P}^{2n} \mid F_i(\mathbf{x}) = f_i, i = \overline{1, n}\}$$

is a disjoint union of several connected components (fibers/leaves). The fiber is *regular* if dF_i are linearly independent at each of its points. Otherwise, the fiber is *singular*.

Theorem

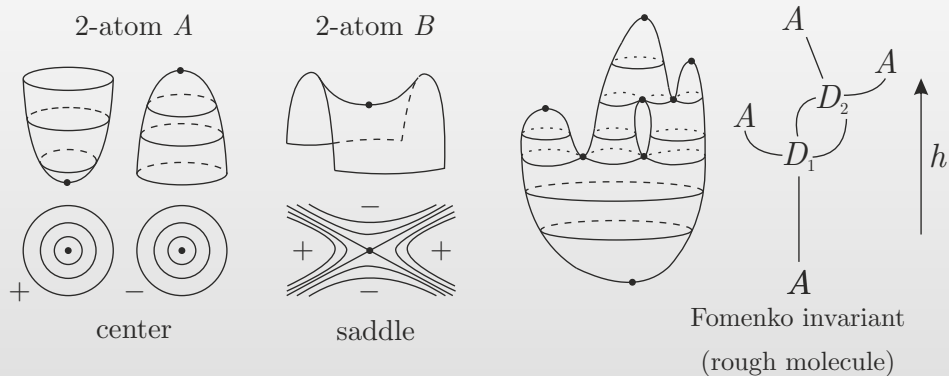
If $\mathcal{T}_f$ is regular:

- $\mathcal{T}_f$ is a smooth n -dimensional submanifold in $\mathcal{P}^{2n}$ invariant under the flows $\text{sgrad } F_1, \dots, \text{sgrad } F_n$;
- fibers are diffeomorphic to a quotient of $\mathbb{R}^n$ with respect to the lattice $\mathbb{Z}^k$;
- if the fiber is compact, it is diffeomorphic to an n -dimensional torus $\mathbb{T}^n$ (Liouville torus).

The decomposition of the phase space $\mathcal{P}^{2n}$ into connected components of common level surfaces is called a *Liouville foliation*.

Singularities of Hamiltonian Systems with 1 DOF

Connected components of common level surfaces: $\mathbb{R}^1$ (noncompact), $\mathbb{S}^1$ (compact).



Classification of compact nondegenerate singularities

Rank 1 Singularities in the Case of 2 DOF

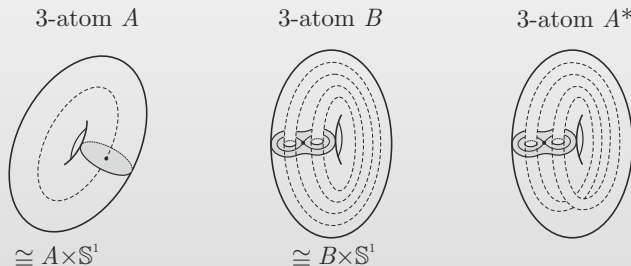
Connected components: $\mathbb{T}^2$ (compact); Cyl or $\mathbb{R}^2$ (noncompact).

Momentum (integral) map:

$$\mathcal{F}: \mathcal{P}^4 \rightarrow \mathbb{R}^2(f, h), \quad \mathcal{F}(\mathbf{x}) = (F(\mathbf{x}), H(\mathbf{x})).$$

The set of critical points of the map $\mathcal{F}$ of rank $m < 2$:

$$\mathcal{C}^m = \{\mathbf{x} \mid \text{rank } d\mathcal{F}(\mathbf{x}) = m\}.$$



V. V. Vedyushkina, A. T. Fomenko, *Integrable topological billiards and equivalent dynamical systems*, *Izvestiya: Mathematics*, **81**:4 (2017).

Critical Points of the Momentum Map

The set of critical points $\mathcal{C} = \mathcal{C}^0 \cup \mathcal{C}^1$:

$$\begin{aligned}\mathcal{C}^0 &= \emptyset, & \mathcal{C}^1 &= \{(x_1, y_1, x_2, y_2) \mid F_1 = 0, F_2 = 0\}. \\ F_1 &= x_1 y_2 - x_2 y_1, \\ F_2 &= x_1 x_2 [(x_1 - x_2)^2 + (y_1 - y_2)^2] [\lambda_2 (x_2^2 + y_2^2) - \lambda_1 (x_1^2 + y_1^2)] + \\ &\quad + (x_1^2 + y_1^2) (x_2^2 + y_2^2) (x_1 - x_2) (\Gamma_1 x_1 + \Gamma_2 x_2).\end{aligned}$$

In polar coordinates

$$x_1 = \rho_1 \cos \theta_1, \quad y_1 = \rho_1 \sin \theta_1, \quad x_2 = \rho_2 \cos \theta_2, \quad y_2 = \rho_2 \sin \theta_2,$$

$\mathcal{C}$ is given by solutions of two systems

$$\begin{aligned}\begin{cases} W(\rho_1, \rho_2) = 0, \\ \theta_1 = \pi + \theta_2, \end{cases} & \quad \begin{cases} W(\rho_1, -\rho_2) = 0, \\ \theta_1 = \theta_2, \end{cases} & (4) \\ W(\rho_1, \rho_2) &= (\rho_1 + \rho_2) (\lambda_1 \rho_1^2 - \lambda_2 \rho_2^2) + \rho_1 \rho_2 (\Gamma_1 \rho_1 - \Gamma_2 \rho_2),\end{aligned}$$

and $W(\rho_1, \rho_2)$ is a homogeneous polynomial of degree 3 in variables (ρ_1, ρ_2) .

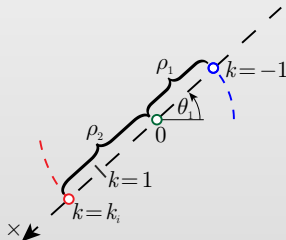
Critical Circles

Let $k = \rho_2/\rho_1$, then $W(\rho_1, k\rho_1) = 0$ can be written as

$$\lambda_2 k^3 + (\Gamma_2 + \lambda_2)k^2 - (\Gamma_1 + \lambda_1)k - \lambda_1 = 0. \quad (5)$$

The critical submanifolds can be parameterized:

$$\Pi_i(t_i): \begin{cases} x_1 = -\operatorname{sgn} k_i \cos \theta_2 t_i, & x_2 = |k_i| \cos \theta_2 t_i, \\ y_1 = -\operatorname{sgn} k_i \sin \theta_2 t_i, & y_2 = |k_i| \sin \theta_2 t_i, \end{cases} \quad \theta_2 \in [0; 2\pi], \quad t_i > 0.$$



Singular periodic motions in the case $\theta_1 = \pi + \theta_2$.

Stability of Critical Motions

Let us write equation (5) as

$$A(k) = B(k) k^2, \quad A(k) = \Gamma_1 k + \lambda_1(k+1), \quad B(k) = \Gamma_2 + \lambda_2(k+1),$$

and let

$$C_i = A_i^2 - (k_i^3 + \lambda_1 \lambda_2)^2.$$

Theorem

Singular periodic motions Π_i are of elliptical (“center”) type if $C_i > 0$ and of hyperbolic type (“saddle”) if $C_i < 0$. If $C_i = 0$, the singularity is degenerate.

Let $Q_h^3 = \{\mathbf{x} \in \mathcal{P}^4 \mid H(\mathbf{x}) = h\}$ be an isoenergetic surface.

Theorem

The quadratic form defined by the Hessian of the function $F|_{Q_h^3}(\mathbf{x}_0)$, $\mathbf{x}_0 \in \mathcal{C}^1$, has the following eigenvalues:

$$\mu_1 = 0, \quad \mu_2 = \frac{2\Gamma_1\Gamma_2}{B_i} \frac{k_i^2 + 1}{k_i(k_i + 1)}, \quad \mu_3 = \frac{2\Gamma_1\Gamma_2}{B_i} \frac{k_i + 1}{k_i^3} C_i.$$

Bifurcation Diagram of the Momentum Map

The *bifurcation diagram* is the set of critical *values*:

$$\Sigma = \mathcal{F}(\mathcal{C} \cap \mathcal{P}^4) \subset \mathbb{R}^2(f, h), \quad \Sigma = \bigcup_i \Sigma_i, \quad \Sigma_i = \mathcal{F}(\Pi_i).$$

If $M_c \neq 0$

$$\Sigma_i : h_i(f) = \frac{M_c}{2} \ln \left[\frac{\operatorname{sgn} m_i}{f} \right] + \frac{\eta_i}{2},$$

$$m_i = \Gamma_2 k_i^2 + \Gamma_1, \quad \eta_i = \frac{\Gamma_1 \Gamma_2}{\lambda_1 \lambda_2} \ln(k_i + 1)^2 + \frac{\Gamma_2}{\lambda_2} \ln k_i^2 + M_c \ln |m_i|,$$

Let $\mathcal{D}$ be the discriminant of equation (5). If $M_c = 0$, then

$$\mathcal{D} < 0 \Rightarrow \Sigma = \Sigma_1 : \left\{ (f, h) \mid f \operatorname{sgn} m_1 > 0, h = \frac{\eta_1}{2} \right\},$$

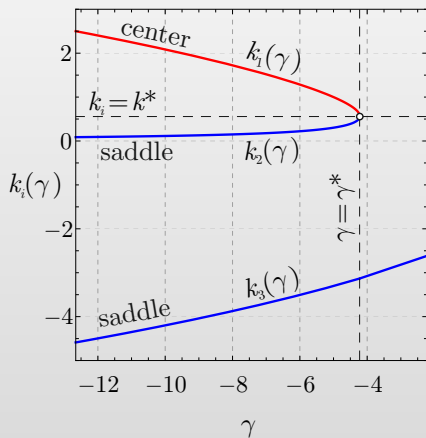
$$\mathcal{D} > 0 \Rightarrow \Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3, \quad \Sigma_2 = \left(0, \frac{\eta_2}{2} \right), \quad \Sigma_3 = \left(0, \frac{\eta_3}{2} \right),$$

$$\mathcal{D} = 0 \Rightarrow \Sigma = (0, \ln 16).$$

Coincidence of Images of Critical Circles

Consider the system with the following parameter values: $\Gamma_1 = \gamma$, $\Gamma_2 = -1$, $\lambda_1 = 1$, $\lambda_2 = -1$, where $\gamma < \gamma^*$,

$$\gamma^* = -\frac{1}{6} \left(4 + \sqrt[3]{1261 - 57\sqrt{57}} + \sqrt[3]{1261 + 57\sqrt{57}} \right) \approx -4.219136248742.$$



Coincidence of Images of Critical Circles

The bifurcation diagram of the momentum map Σ consists of a union of images $\Sigma_i = \mathcal{F}(\Pi_i)$ of three critical circles Π_i , i. e.

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3.$$

$$\Sigma_i : h_i(f) = \frac{2\gamma + 1}{2} \ln(-f) + \frac{\eta_i}{2},$$

$$\eta_i = \ln k_i^2(\gamma) + \gamma \ln [k_i(\gamma) + 1]^2 - (2\gamma + 1) \ln [k_i^2(\gamma) - \gamma].$$

Additional first integral

$$F = \gamma r_1^2 - r_2^2 < 0,$$

thus the range of the momentum map has the boundary

$$\Sigma_0 : f = 0,$$

and *augmented* bifurcation diagram

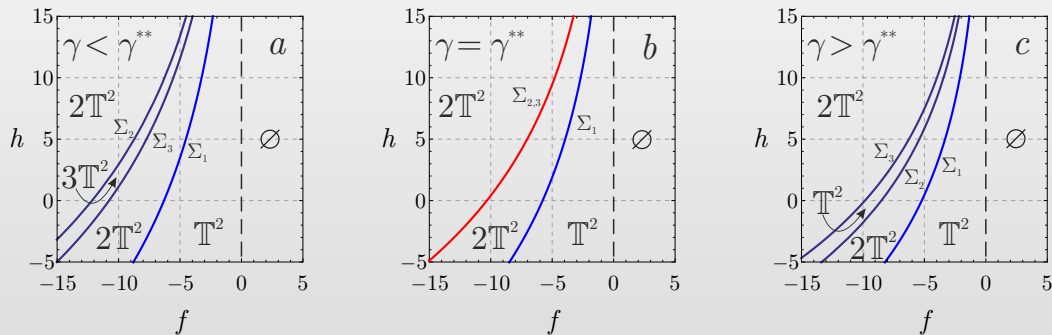
$$\Sigma' = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3.$$

Coincidence of Images of Critical Circles

Lemma

Images Σ_2 and Σ_3 of critical circles Π_2 and Π_3 coincide if $\Gamma_1 = \gamma^{**}$, $\Gamma_2 = -1$, $\lambda_1 = 1$, $\lambda_2 = -1$, where

$$\gamma^{**} \approx -13.552\ 76761\ 49683\ 66593\ 90987\ 91141\ 37627.$$



Augmented bifurcation diagram Σ' for $\gamma < \gamma^*$.

Hamiltonian System with 1 DOF

Coordinate transformation:

$$\begin{aligned}x_1 &= \frac{\operatorname{sgn}(\Gamma_1 \Gamma_2)}{\sqrt{|\Gamma_1|}} [u \cos(\alpha) - v \sin(\alpha)], & x_2 &= \frac{\cos(\alpha)}{\sqrt{|\Gamma_2|}} \sqrt{W(u, v)}, \\y_1 &= \frac{\operatorname{sgn}(\Gamma_1 \Gamma_2)}{\sqrt{|\Gamma_1|}} [u \sin(\alpha) + v \cos(\alpha)], & y_2 &= \frac{\sin(\alpha)}{\sqrt{|\Gamma_2|}} \sqrt{W(u, v)}, \\W(u, v) &= \operatorname{sgn}(\Gamma_1 \Gamma_2) (f \operatorname{sgn} \Gamma_1 - u^2 - v^2).\end{aligned}$$

Backward substitution

$$U = \frac{\sqrt{|\Gamma_1|}(x_1 x_2 + y_1 y_2)}{\operatorname{sgn}(\Gamma_1 \Gamma_2) \sqrt{x_2^2 + y_2^2}}, \quad V = \frac{\sqrt{|\Gamma_1|}(x_2 y_1 - x_1 y_2)}{\operatorname{sgn}(\Gamma_1 \Gamma_2) \sqrt{x_2^2 + y_2^2}}$$

leads to canonical variables with respect to the Lie–Poisson bracket (2):

$$\{U, V\} = -\{V, U\} = \operatorname{sgn} \Gamma_1, \quad \{U, U\} = \{V, V\} = 0.$$

Hamiltonian System with 1 DOF

The reduced system is Hamiltonian in terms of the variables (u, v) :

$$\begin{aligned} \dot{u} &= \frac{\partial H_r}{\partial v} \operatorname{sgn} \Gamma_1, & \dot{v} &= -\frac{\partial H_r}{\partial u} \operatorname{sgn} \Gamma_1, \\ H_r &= \frac{1}{2} \left\{ \frac{\Gamma_1}{\lambda_1} \ln \left[\frac{u^2 + v^2}{|\Gamma_1|} \right] + \frac{\Gamma_2}{\lambda_2} \ln \left[\frac{W(u, v)}{|\Gamma_2|} \right] + \right. \\ &\quad \left. + \frac{\Gamma_1 \Gamma_2}{\lambda_1 \lambda_2} \ln \left[\frac{|\Gamma_2| v^2 + \left(\sqrt{|\Gamma_1| W(u, v)} - u \operatorname{sgn} (\Gamma_1 \Gamma_2) \sqrt{|\Gamma_2|} \right)^2}{|\Gamma_1 \Gamma_2|} \right] \right\}. \end{aligned}$$

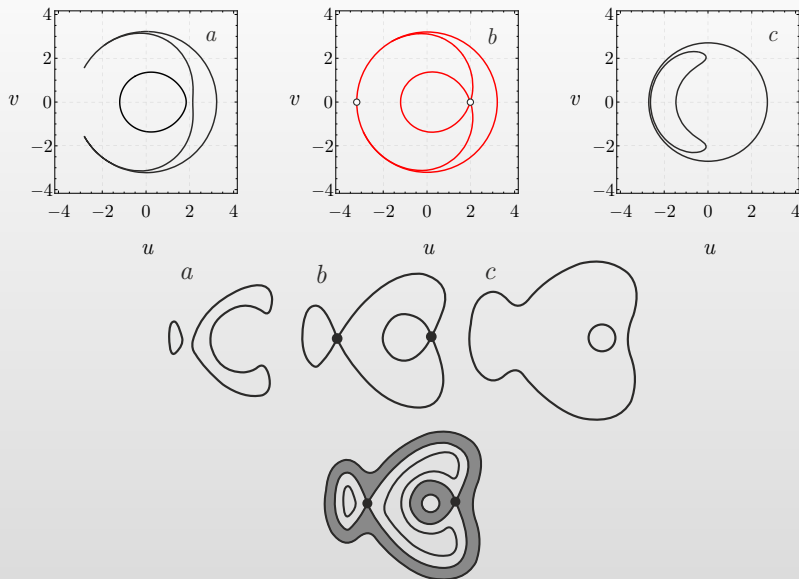
Theorem

Let $\mathcal{T}_{u,v} = \{(u, v) \mid H_r = h\}$, then $\mathcal{T}_{x,y} \simeq \mathcal{T}_{u,v} \times \mathbb{S}^1$. Critical circles Π_i are transformed into stationary points of the reduced system:

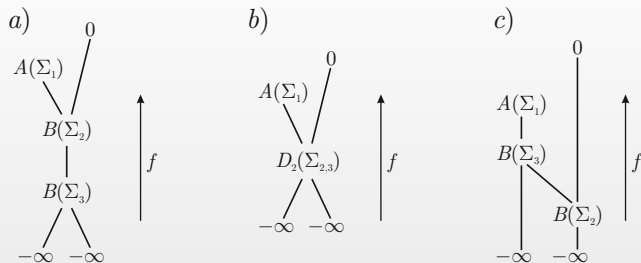
$$(u_i^*, v_i^*) = \left(-\operatorname{sgn} (\Gamma_1 \Gamma_2 k_i) \sqrt{|\Gamma_1|} t_i, 0 \right).$$

The stationary points of the reduced system determine the set of critical points $\mathcal{C}$.

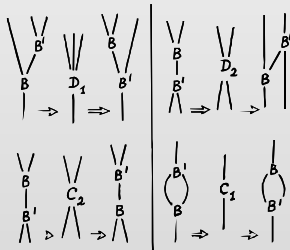
Surgery of Two Liouville Tori into Two



Surgery of Two Liouville Tori into Two



Rough molecules for a) $\gamma > \gamma^{**}$, b) $\gamma = \gamma^{**}$, c) $\gamma < \gamma^{**}$.



Degenerate Singularity

At the point $\gamma = \gamma^*$, there is a degenerate surgery $\mathbb{T}^2 \rightarrow \mathbb{T}^2$
(parabolic singularity).

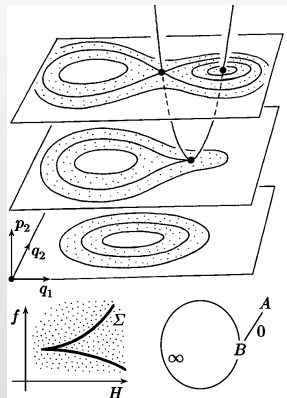
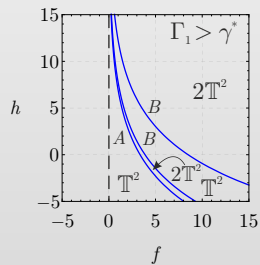
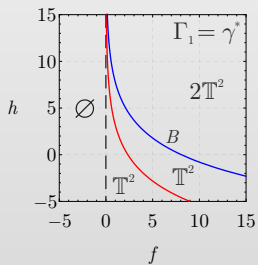
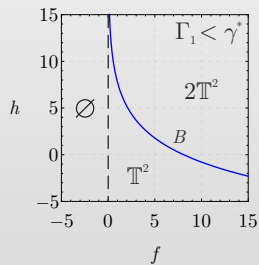
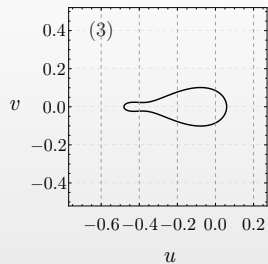
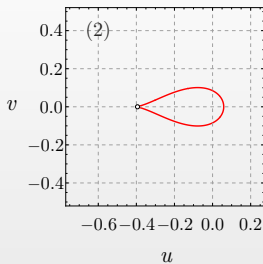
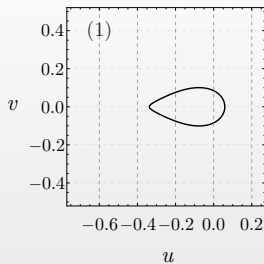


Fig. 10.13, p. 396: A.V. Bolsinov, A.T. Fomenko, *Integrable Hamiltonian Systems. Geometry, Topology, Classification*, 2004.

Degenerate Singularity



THANK YOU