Unitary representation of random transformations of a Hilbert space and limit theorems

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Plan

1. Measures on a real separable Hilbert space what are invariant with respect to Hamiltonian flows.

2. Representastion of groups in the space of quadratically integrable with respect to invariant measure functions.

3. Random shifts operator and law of large numbers.

4. Central limit type theorem and generalized convergence of measures.

A. Weyl theorem

Theorem. *If a topological group G is not locally compact then there is no nontrivial σ-additive σ-finite locally finite Borel measure on the group G which is left-invariant.*

Hence there is no nontrivial σ-additive σ-finite locally finite Borel shift-invariant measure on an infinite dimensional normed linear space.

Shift-invariant measures on a Hilbert space

Rectangles in a real separable Hilbert space E:
A set $\Pi \subset E$ is called rectangle if there are ONB $\{e_j\} \equiv \mathcal{E}$ and $\exists\ a, b \in l_\infty$ such that

$$\Pi = \{x \in E :\ (x, e_j) \in [a_j, b_j)\ \forall\ j \in \mathbf{N}\}. \tag{1}$$

A rectangle (1) is called measurable if it eigther empty set or

$$\sum_{j=1}^{\infty} \max\{0, \ln(b_j - a_j)\} < +\infty. \tag{2}$$

Let $\lambda(\Pi) = 0$ if $\Pi = \emptyset$ and let

$$\lambda(\Pi) = \exp(\sum_{j=1}^{\infty} \ln(b_j - a_j)) = \prod_{j=1}^{\infty}(b_j - a_j) \tag{3}$$

if $\Pi \neq \emptyset$.

The ring generated by measurable rectangles

For any ONB $\mathcal{E}$ the symbol $K_1(\mathcal{E})$ notes the set of measurable rectangles which edges collinear to a vector of ONB $\mathcal{E}$.

Let $r_{\mathcal{E}}$ be the ring -generated by the class of sets $K_1(\mathcal{E})$.

A function $m: \ \mathcal{R} \to R$ is called a measure on the space E if $\mathcal{R}$ is a ring of subsets of the space E and the function m is additive.

Lemma 1.1. *Let $\mathcal{E}$ be an ONB in the space E. Then the function of a set (3)*

$$\lambda_{\mathcal{E}}: \ K_1(\mathcal{E}) \to [0, +\infty)$$

is additive. Moreover, it has the unique additive extension to the ring $r_{\mathcal{E}}$. This extension is the finite additive measure on the space E

$$\lambda_{\mathcal{E}}: \ r_{\mathcal{E}} \to [0, +\infty)$$

Properties of the measure $\lambda_{\mathcal{E}}$

Let $\mathcal{E}$ be an ONB in E. The outer and interior measure of a set $A \subset E$ are given by equalities

$$\bar{\lambda}_{\mathcal{E}}(A) = \inf_{B \in r_{\mathcal{E}}:\ B \supset A} \lambda_{\mathcal{E}}(B),\ \underline{\lambda}_{\mathcal{E}}(A) = \sup_{B \in r_{\mathcal{E}}:\ B \subset A} \lambda_{\mathcal{E}}(B).$$

$$\mathcal{R}_{\mathcal{E}} = \{A \in E:\ \bar{\lambda}_{\mathcal{E}}(A) = \underline{\lambda}_{\mathcal{E}}(A) \in [0, +\infty)\}$$

Lemma 1.2. *Let B_R be a ball in the space E of radius R.*
If $R < \frac{1}{\sqrt{2}}$, then $\underline{\lambda}_{\mathcal{E}}(B_R) = \bar{\lambda}_{\mathcal{E}}(B_R) = \lambda_{\mathcal{E}}(B_R) = 0$;
If $R > \frac{1}{\sqrt{2}}$, then $\bar{\lambda}_{\mathcal{E}}(B_R) = +\infty$ и $\underline{\lambda}_{\mathcal{E}}(B_R) = 0$.

Theorem 1.1. *A family of sets $\mathcal{R}_{\mathcal{E}}$ is the ring. The measure $\lambda_{\mathcal{E}}:\ \mathcal{R}_{\mathcal{E}} \to [0, +\infty)$ has following properties*
1) is invariant with respect to a shift on any vector of the space E;
2) complete, locally finite and σ-finite;
3) is not σ-additive;
4) its continuation by Lebesgue-Caratheodory scheme vanishes on the space E.

Hilbert space $L_2(E, \mathcal{R}_\mathcal{E}, \lambda_\mathcal{E}, \mathbb{C})$

Let $S(\mathcal{R}_\mathcal{E})$ be the space of finite linear combinations over the field $\mathbb{C}$ of indicator functions of sets from the ring $\mathcal{R}_\mathcal{E}$. Let introduce non-negative hermitian sesquilinear form: for arbitrary functions $f,\ g \in S(\mathcal{R}_\mathcal{E})$, where

$$f(x) = \sum_{k=1}^{s} \alpha_k \chi_{A_k}(x) \quad \text{and} \quad g(x) = \sum_{l=1}^{p} \beta_l \chi_{B_l}(x)$$

we have

$$(f,\ g) = \sum_{k=1}^{s} \sum_{l=1}^{p} \alpha_k \overline{\beta}_l \lambda(A_k \cap B_l).$$

We call functions $f,\ g \in S(\mathcal{R}_\mathcal{E})$ *equivalent* if $(f-g,\ f-g) = 0$. So the linear space of classes of equivalence of functions from $S(\mathcal{R}_E)$ is pre-Hilbert, and after the procedure of completion obtain $\mathcal{H}_\mathcal{E} = L_2(E, \mathcal{R}_\mathcal{E}, \lambda_\mathcal{E}, \mathbf{C})$.

Lemma 1.3. Hilbert space $\mathcal{H}_\mathcal{E}$ is not separable.

The space E as the group with respect to summation operation has the representation in the space $\mathcal{H}_{\mathcal{E}}$ by the abelian unitary group of shift operators $\mathcal{S}=\{\mathbf{S}_h,\ h\in E\}$ acting by the rule

$$\mathbf{S}_h u(x)=u(x-h),\ x\in E.$$

The subgroup $\mathbf{S}_{th},\ t\in\mathbb{R}$ of the group $\mathcal{S}=\{\mathbf{S}_a,\ a\in E\}$.
Theorem 2.1. *Let $\mathcal{E}$ be an ONB in the space E and $h\in E$. Then, the one-parameter unitary group $\mathbf{S}_{th},\ t\in\mathbb{R}$ is continuous in the strong operator topology of the space $B(\mathcal{H}_{\mathcal{E}})$ iff $\{(h,e_k)\}\in l_1$.*
Corollary 2.2. Let $\mathcal{E}$ be an ONB in the space E. If the group $\mathcal{S}$ is equipped with the strong operator topology τ_{sot} of the space $B(\mathcal{H}_{\mathcal{E}})$, then the topological group $(\mathcal{S},\ \tau_{sot})$ is the unitary representation of the abelian topological group $(E,\ \|\cdot\|_E)$ in the space $\mathcal{H}_{\mathcal{E}}$. But this representation is not continuous.

Random shift operators in the space $\mathcal{H}_{\mathcal{E}}$

Let $h_n \in E,\ n \in \mathbb{N}$ be a sequence of random vectors. Then $\mathbf{S}_{h_n}$: $\mathbf{S}_{h_n} u(x) = u(x + h_n)$, $x \in E$, $\forall\ u \in \mathcal{H}_{\mathcal{E}}$, $\forall\ n \in \mathbb{N}$, be a sequence of unitary operators.

Definition 2.4.

Law of Large Numbers for asequence of random vectors $\{h_k\}$:

$$\lim_{n\to\infty} P(\| \frac{h_n}{n} + ... + \frac{h_1}{n} - \mathrm{M}(\frac{h_n}{n} + ... + \frac{h_1}{n})\|_E > \epsilon) = 0\ \forall\ \epsilon > 0.$$

LLN in SOT for a sequence of random shift operators $\{\mathbf{S}_{th_k}\}$:

$$\lim_{n\to\infty} P(\|(\mathbf{S}_{h_n}^{\frac{1}{n}} \circ ... \circ \mathbf{S}_{h_1}^{\frac{1}{n}} - \mathrm{M}[\mathbf{S}_{h_n}^{\frac{1}{n}} \circ ... \circ \mathbf{S}_{h_1}^{\frac{1}{n}}])x\|_{\mathcal{H}_{\mathcal{E}}} > \epsilon)\ = 0 \quad (4)$$

$\forall\, \epsilon > 0\ \ \forall\, x \in \mathcal{H}_{\mathcal{E}}$.

LLN for vectors and argument shift operators

Theorem 2.5 *Let $\{h_n\}$ be a sequence of IID random vectors in a Hilbert space E. Let λ be a shift-invariant measure on the space E. Let $\{\mathbf{S}_{h_n}\}$ be a sequence of argument shift operators on vectors h_n respectively, acting in the Hilbert space $\mathcal{H} = L_2(E, \lambda, \mathbb{C})$.*

If a function $u \in \mathcal{H}$ has the property of continuity in the mean quadratic with respect to shift, then LLN for the vectors $\{h_n\}$ implies LLN in the strong operator topology for operators $\{\mathbf{S}_{h_n}\}$.

If $\exists\ B \subset E$ such that $\lambda(B) \geqslant 0$ and

$$c\|h\|_E \leqslant \|\mathbf{S}_h \chi_B - \chi_B\|_{\mathcal{H}} \quad \forall \quad h : \ \|h\|_E \leqslant 1$$

with some $c > 0$, then LLN in the strong operator topology for operators $\{\mathbf{S}_{h_n}\}$ implies LLN for the vectors $\{h_n\}$.

If $E = \mathbb{R}^d$ with Lebesgue measure then LLN for $\{h_n\}$ is equivalent to LLN for $\{\mathbf{S}_{h_n}\}$.
If infinite dimansional space E is equipped with the measure $\lambda_{\mathcal{E}}$ then LLN for $\{\mathbf{S}_{h_n}\}$ is stronger than LLN for $\{h_n\}$.

Diffusion semigroup in the space $\mathbb{H}_{\mathcal{E}}$

Theorem 3.1. *Let $\nu_{\mathbf{D}}$ be a – σ-additive Gaussian measure on the Hilbert space E with trace-class covariation operator $\mathbf{D}$; let $\mathcal{E}$ be the ONB of eigenvectors of the operator $\mathbf{D}$. Then one-parametric family of linear operators $\mathbf{U_D}(t)$, $t \geqslant 0$, acting by the equality*

$$\mathbf{U_D}(t)u = \int_E \mathbf{S}_h u d\nu_{t\mathbf{D}}(h), \ \ t \geqslant 0,$$

(here $(v, \mathbf{U_D}(t)u)_{\mathbb{H}_{\mathcal{E}}} = \int_E (v, \mathbf{S}_h u)_{\mathbb{H}_{\mathcal{E}}} d\nu_{t\mathbf{D}} \ \forall \ u, v \in \mathbb{H}_{\mathcal{E}}$)

is the semigroup of self-adjoint contractions in the space $\mathbb{H}_{\mathcal{E}}$.

Theorem 3.2. *Let $\mathbf{U}$ be a semigroup of contractive linear operators in a Hilbert space $\mathbb{H}$. There are two subspaces $\mathbb{H}^0, \mathbb{H}^1$: $\mathbb{H} = \mathbb{H}^0 \oplus \mathbb{H}^1$; $\mathbb{H}^0, \mathbb{H}^1$ are invariant with respect to semigroup $\mathbf{U}$,*

$$\mathbf{U_D}(t)|_{\mathbb{H}^0} = \mathbf{I}_{\mathbb{H}^0} \chi_{\{0\}}(t), \ t \geqslant 0,$$

where χ_A is the indicator function of a set $A \subset R$;
$\mathbf{U_D}(t)|_{\mathbb{H}^1}$, $t \geqslant 0$ is strong continuous semigroup in $\mathbb{H}^1$.

Sobolev Spaces and semigroup generators

Let $\mathbf{D}$ be a nonnegative trace class operator in the space E, let $\mathcal{E} = \{e_k\}$ be its eigen ONB such that $\mathbf{D}e_k = d_k e_k$.

$$W^l_{2,\mathbf{D}} = \{u \in \mathcal{H}_{\mathcal{E}} : \ \partial^l_{e_j} u \in \mathcal{H}_{\mathcal{E}} : \ \sum_{j=1}^{\infty} d_j \|\partial^l_{e_j} u\|^2_{\mathcal{H}_{\mathcal{E}}} < +\infty\}$$

where $\partial_{e_j} u = v \in \mathcal{H}_{\mathcal{E}} \ \Leftrightarrow \ \lim_{s\to 0}(\|\frac{1}{s}(\mathbf{S}_{se_j} - \mathbf{I})u - v\|_{\mathcal{H}_{\mathcal{E}}}) = 0.$

Theorem 3.3. *Let $\nu_{\mathbf{D}}$ be a centered Gaussian measure on the space E with a trace class covariation operator $\mathbf{D}$. Let $\mathcal{E}$ be an ONB such that $\mathbf{D}e_k = d_k e_k$. Then the semigroup $\mathbf{U}_{\mathbf{D}}$ is strongly continuous in the space $\mathcal{H}_{\mathcal{E}}$ iff $\mathbf{D}^{1/2}$ is trace class.*
In this case the space $W^2_{2,\mathbf{D}}$ is the domain of the generator $\mathbf{\Delta}_{\mathbf{D}}$ of the semigroup $\mathbf{U}_{\mathbf{D}}$

$$\mathbf{\Delta}_{\mathbf{D}} u = \sum_{j=1}^{\infty} d_j \partial^2_{e_j} u, \quad u \in W^2_{2,\mathbf{D}}.$$

Weak convergence of measures

To study a CLT-type theorem we should consider the notion of weak convergence of measures. By means of operator approach to this notion we can extend the class of topologies to studying the convergence of measures.

Let E be a separable real Hilbert space, $\mathcal{B}(E)$ be the Borel σ-algebra of the space E. Let $ca(E, \mathcal{B}(E))$ be the Banach space of Borel measures on the space E with bounded variation.

A sequence $\{\mu_n\} : \mathbb{N} \to ca(E, \mathcal{B}(E))$ is called weakly converging to the measure $\mu \in ca(E, \mathcal{B}(E))$ if the equality

$$\lim_{n\to\infty} \int\limits_E f(x) d\mu_n(x) = \int\limits_E f(x) d\mu(x)$$

holds for any $f \in C_b(E)$.

Weak convergence of measures in the operator form

Let $X = C_b(E)$ be a locally convex space of bounded continuous functions equipped with the topology τ_p of pointwise convergence. Let $\mathcal{L}(X)$ be the locally convex space of linear mapping of the space X into itself equipped with the topology of pointwise convergence. Let $rca(E, \mathcal{B}(E))$ be a space of regular Borel mesures with bounded variations.

Theorem 4.1. *The weak convergence of the sequence* $\{\mu_n\} : \mathbb{N} \to rca(E, \mathcal{B}(E))$ *to the measure* $\mu \in rca(E, \mathcal{B}(E))$ *is equivalent to the pointwise convergence of the sequence of convolution operators* $\{\Phi_{\mu_n}\}$ *to the operator* Φ_μ *in the locally convex space* $\mathcal{L}(X)$*. Here*

$$\Phi_{\mu_n} u(x) = \int\limits_E u(x-y)\mu_n(dy).$$

$$\Phi_\mu u(x) = \int\limits_E u(x-y)\mu(dy).$$

Let $\mathcal{A}$ be an algebra of subsets of the space E. Let X, Y be a locally convex space of $\mathcal{A}$-measurable functions $u: E \to \mathbb{C}$. Let $\mathcal{L}(X)$ be a locally convex space of linear operators $X \to X$.

Definition 4.2. *A sequence $\{\mu_n\}: \mathbb{N} \to ca(E, \mathcal{A})$ is called weakly converging to the measure $\mu \in ca(E, \mathcal{A})$ with respect to the space $\mathcal{L}(X)$ if the sequence of operators Φ_{μ_n} converges in the space $\mathcal{L}(X)$ to the operator Φ_μ. Here*

$$\Phi_{\mu_n} u(x) = \int_E u(x-y) d\mu_n(y), \ u \in X, \ x \in E,$$

and

$$\Phi_\mu u(x) = \int_E u(x-y) d\mu(y), \ u \in X, \ x \in E.$$

Definition 4.3. A sequence of $(E, \mathcal{A}(E))$-valued random variables ξ_n is called weakly converging in distribution to the random variable ξ with respect to the space $\mathcal{L}(X)$ if the sequence $\{\mu_n\}$ converges $\mathcal{L}(X)$-weakly to the measure μ. Here these measures are defined by the equalities
$\mu_n(A) = P(\xi_n^{-1}(A)),\ A \in \mathcal{A}(E),\ n \in \mathbb{N}$, and
$\mu(A) = P(\xi^{-1}(A)),\ A \in \mathcal{A}(E)$.

CLT-type generalized convergence of random processes

Theorem 4.4. *Let E be a Hilbert space. Let the E-valued random process ξ is given by the equality $\xi(t) = \sqrt{t}h$, $t \geqslant 0$.*
Here h is a random vector in the space E with $\mathrm{M}(h) = 0$, with covariation operator $\mathbf{D} \in B(E)$ and $\mathrm{M}(\|h\|_E^3) < +\infty$.
Let $\{\xi_n\}$ be a sequence of independent E-valued random processes such that the distribution of any of them coincides with the distribution of the process ξ.
If $\{\eta_n\}$ is the sequence of random processes

$$\eta_n(t) = \sum_{k=1}^{n} \xi_k(\tfrac{t}{n}),\ t \geqslant 0,\ n \in \mathbb{N},$$

then the sequence $\{\eta_n\}$ converges weakly with respect to the space $(B(L_2(E)), \tau_{sot})$ to Gaussian random process with normal distribution $N(0, t\mathbf{D})$, $t \geqslant 0$, uniformly on any segment of the semiaxe $\mathbb{R}_+$.

The generalization of CLT-type convergence for composition of shifts operators along random iid Hamiltonian fields.

Real Hilbert space with the symplectic structure

Symplectic structure for studying a random Hamiltonian flows

Shift-invariant symplectic form ω on the space E in nondegenered skew-symmetric bilinear form on E.
There is an ONB $\mathcal{E} = \{e_k\}$ in the space E such that
$\omega(e_{2j}, e_n) = \delta_{2j-1,n}\ \forall\ j \in \mathbb{N}$.
$E = P \oplus Q,\ P = Q = l_2$.
$\mathcal{F} = \{f_j\} = \{e_{2j-1}\}$ – ONB in P;
$\mathcal{G} = \{g_j\} = \{e_{2j}\}$ – ONB in Q.
$\mathbf{J}$ is linear operator in E associated with the symplectic form

$$\omega(x, y) = (x, \mathbf{J}y)_E$$

$$\mathbf{J}(g_j) = -f_j,\ \mathbf{J}(f_j) = g_j.$$

$$\mathbf{J}^2 = -\mathbf{I},\ \mathbf{J}^* = -\mathbf{J}.$$

Symplectic rectangles

Definition 5. A set $\Pi \subset E$ is called absolutely measurable symplectic rectangle in the space E if there are the decomposition $E = Q \oplus P$ and ON Bases $\{f_j\}$, $\{g_k\}$ in the spaces Q, P such that

$$\Pi = \{z \in E : \ ((z, f_i), (z, g_i)) \in B_i, \ i \in \mathbb{N}\}, \tag{5}$$

where B_i are Lebesgue-measurable sets in the plane $\mathbb{R}^2$ such that

$$\sum_{j=1}^{\infty} \max\{\ln(\lambda_2(B_j)), 0\} < +\infty$$

(here λ_2 is Lebesgue measure on the plane $\mathbb{R}^2$).

Two-dimensional structure

Let $\mathcal{K}_2(E)$ be the set of absolutely measurable symplectic rectangles.
Let $\mathcal{K}_{\mathcal{F},\mathcal{G}}(E)$ be the set of absolutely measurable symplectic rectangles which have the form (5) for given pair of ONB $\{f_j\}$, $\{g_k\}$ in the spaces Q, P.
Let the function $\lambda_{\mathcal{K}_{\mathcal{F},\mathcal{G}}} : \mathcal{K}_{\mathcal{F},\mathcal{G}}(E) \to [0, +\infty)$ be defined by the equality

$$\lambda_{\mathcal{K}_{\mathcal{F},\mathcal{G}}}(\Pi) = \prod_{j=1}^{\infty} \lambda_2(B_j) = \exp(\sum_{j=1}^{\infty} \ln(\lambda_2(B_j))), \ \Pi \in \mathcal{K}_{\mathcal{F},\mathcal{G}}(E), \quad (6)$$

in the case $\Pi \neq \oslash$;

$\lambda_{\mathcal{K}_{\mathcal{F},\mathcal{G}}}(\Pi) = 0$ in the case $\Pi = \oslash$.

Extension of the measure $\lambda_{\mathcal{E}}$ to symplectic-invariant measure

Lemma 5.1. *The function of a set* $\lambda: \mathcal{K}_{\mathcal{F},\mathcal{G}}(E) \to [0,+\infty)$ *is additive and shift-invariant.*

Let $r_{\mathcal{F},\mathcal{G}}$ be a ring generated by the collection of sets $\mathcal{K}_{\mathcal{F},\mathcal{G}}(E)$.

Theorem 5.2. *Additive function of a set* $\lambda: \mathcal{K}_{\mathcal{F},\mathcal{G}}(E) \to [0,+\infty)$ *has the unique additive extension on the ring* $r_{\mathcal{F},\mathcal{G}}$. *The completion of the measure* $\lambda: r_{\mathcal{F},\mathcal{G}} \to [0,+\infty)$ *is the complete measure* $\lambda_{\mathcal{F},\mathcal{G}}: \mathcal{R}_{\mathcal{F},\mathcal{G}} \to [0,+\infty)$, *which is invariant with respect to a smooth symplectorphism* $\Phi: E \to E$ *such that* Φ *preserves two-dimensional symplectic subspaces* $E_k = \mathrm{span}(f_k, g_k)$, $k \in \mathbb{N}$ *of the decomposition* $E = \oplus_{k=1}^{\infty} E_k$.

Let $\mathcal{H}_{\mathcal{F},\mathcal{G}} = L_2(E, \mathcal{R}_{\mathcal{F},\mathcal{G}}, \lambda_{\mathcal{F},\mathcal{G}}, \mathbb{C})$.

Sobolev spaces and diffusion semigroup

Let $\mathbf{D} \geqslant 0$ be a trace class operator in the space with eigen ONB $\mathcal{E} = \{e_k\}: \ \mathbf{D}e_k = d_k e_k$. Let $\mathbf{J}$ be a symplectic operator on the space E such than $(\mathbf{J}e_k, e_{2j-1}) = \delta_{k,2j}$, $j \in \mathbb{N}$. Let $\mathcal{F} = \{e_{2j-1}\}, \ \mathcal{G} = \{e_{2j}\}$.

$\mathcal{W}^l_{2,\mathbf{D}} = \{u \in \mathcal{H}_{\mathcal{F},\mathcal{G}}: \ \partial^l_{e_j} u \in \mathcal{H}_{\mathcal{F},\mathcal{G}}: \ \sum\limits_{j=1}^{\infty} d_j \|\partial^l_{e_j} u\|^2_{\mathcal{H}_{\mathcal{F},\mathcal{G}}} < +\infty\}$

where $\partial_{e_j} u = v \in \mathcal{H}_{\mathcal{F},\mathcal{G}} \ \Leftrightarrow \ \lim\limits_{s\to 0}(\|\frac{1}{s}(\mathbf{S}_{se_j} - \mathbf{I})u - v\|_{\mathcal{H}_{\mathcal{F},\mathcal{G}}}) = 0.$

Let $\mathcal{H}_{\mathbf{D}}$ be the cloasure of the space $\mathcal{W}^2_{2,\mathbf{D}}$ in $\mathcal{H}_{\mathcal{F},\mathcal{G}}$-norm.

Theorem 5.3. Let $\nu_{\mathbf{D}}$ be a centered Gaussian measure in E with trace-class covariation operator with ONB $\mathcal{E}$, $\mathbf{D}e_k = d_k e_k$. Let $\mathbf{U}_{\mathbf{D}}(t) = \int\limits_E \mathbf{S}_{\mathbf{h}} d\nu_{t\mathbf{D}}(h), \ t \geqslant 0$. Then $\mathcal{H}_{\mathbf{D}} \subset \mathbb{H}^1_{\mathcal{F},\mathcal{G}}$ and $\mathcal{W}^2_{2,\mathbf{D}} = Dom(\mathbf{\Delta}_{\mathbf{D}})$ is the domain of the generator $\mathbf{\Delta}_{\mathbf{D}}$ of the semigroup $\mathbf{U}_{\mathbf{D}}|_{\mathbb{H}_{\mathbf{D}}}$, such that $\mathbf{\Delta}_{\mathbf{D}} u = \sum\limits_{j=1}^{\infty} d_j \partial^2_{e_j} u \ \forall \ u \in Dom(\mathbf{\Delta}_{\mathbf{D}})$.

Phase flow preserving symplectic measure in the space E

Let $\mathbf{a}: \ E \to E$ be a vector field satisfying the condition (**red**):
There is the sequence $\{\mathbf{a}_k\}: \ E_k \to E_k$ *where*
$E_k = \mathrm{span}(\mathbf{e}_k, \mathbf{f}_k)$, $k \in \mathbb{N}$, *such that*
1. $\mathbf{P}_{E_k}\mathbf{a} = \mathbf{a}_k$ *for all* $k \in \mathbb{N}$ *where* $\mathbf{P}_{E_k}$ *is the orthogonal projector on the subspace* E_k*;*
2. For any $k \in \mathbb{N}$ *the field* $\mathbf{a}_k: \ \mathbb{R}^2 \to \mathbb{R}^2$ *is continuously differentiable and has zero divergence.*
Let $\mathbf{X}_k^t$, $t \in \mathbb{R}$, be a group of shifts along the vector field $\mathbf{a}_k$:

$$\frac{d}{dt}(\mathbf{X}_k^t(x_k)) = \mathbf{a}_k(\mathbf{X}_k^t(x_k)), \ t \in \mathbb{R}, \ x_k \in E_k.$$

Let $\mathbf{X}^t$, $t \in \mathbb{R}$, be a group of shifts along the vector field $\mathbf{a}$:

$$\frac{d}{dt}(\mathbf{X}^t(x)) = \mathbf{a}(\mathbf{X}^t(x)), \ t \in \mathbb{R}, \ x \in E.$$

Then $\mathbf{X}^t$, $t \in \mathbb{R}$ is the Hamiltonian flow in the space E.

Koopman flow in $\mathbb{H}_{\mathcal{F},\mathcal{G}} = L_2(E, \mathcal{R}_{\mathcal{F},\mathcal{G}}, \lambda_{\mathcal{F},\mathcal{G}}, \mathbb{C})$

The family of linear operators $\mathbf{S}_{\mathbf{a}}^t$, $t \in \mathbb{R}$:

$$\mathbf{S}_{\mathbf{a}}^t u(x) = u(\mathbf{X}^{-t}(x)), \, x \in E, \; t \in \mathbb{R}, \; u \in \mathbb{H}_{\mathcal{F},\mathcal{G}},$$

is the unitary group.

Theorem 5.4. Let $h \in E$. One-parametric group of unitary operators $\mathbf{S}_{\mathbf{a}}^t$, $t \in \mathbb{R}$, is strong continuous group of linear operators in the subspace $\mathbb{H}_{\mathcal{F},\mathcal{G}}$
iff
$\exists \; N \in \mathbb{N} : \; \mathbf{a}_k = 0 \, \forall \; k > N.$

Random walks along a random vector field

Theorem 5.5. *Let* $\mathbf{a}: \ \Omega \to C_b^3(E, E)$ *be a random vector field such that* $\mathbf{a}(\omega)$ *satisfies the condition (red)* $\forall \ k \in \mathbb{N}$ *the random vector field* $\mathbf{a}_k = \mathbf{P}_{E_k}\mathbf{a}$ *satisfies conditions*

1) $\int\limits_{\Omega} \mathbf{a}_k(x, \omega) dP(\omega) = 0, \ \ x \in E$;

2) $\exists \ c > 1: \ \ c^{-1} d_k \mathbf{P}_{E_k} \leqslant \mathbf{D}^{(k)}(x) \leqslant c d_k \mathbf{P}_{E_k} \ \forall \ x \in E, \ \forall k \in \mathbb{N}$,
$\mathbf{D}^{(k)}_{i,j}(x) = \frac{1}{2}\mathrm{M}((\mathbf{a}_k)_i(x)(\mathbf{a}_k)_j(x)), \ x \in E, \ k \in \mathbb{N}, \ i, j \in \{1, 2\}$;

3) $\|\mathbf{a}_k(\cdot, \omega)\|_{C_b^3(E_k, E_k)} \leqslant \sqrt{d_k}\rho$ *for some* $\rho > 0$ *and every* $\omega \in \Omega$.

Let $\{\mathbf{a}^{(k)}\}$ *be a sequence of iid random vector field such that the distribution of every of them coincides with the distribution of the random vector field* $\mathbf{a}$. *Then for every* $u \in \mathcal{H}_{\mathbf{D}}$ *we have*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|\mathrm{M}[\mathbf{S}_{\mathbf{a}^{(n)}}(\sqrt{\frac{t}{n}}) \circ \ldots \circ \mathbf{S}_{\mathbf{a}^{(1)}}(\sqrt{\frac{t}{n}}) u(x)] - \mathbf{U}_{\mathbf{a}}(t) u(x)\|_{\mathcal{H}_{\mathbf{D}}} = 0,$$

where

Random walks along a random vector field

$$\mathbf{U}_{\mathbf{a}}(t)(\otimes_{k=1}^{\infty}\chi_{B_k}) = \otimes_{k=1}^{\infty}(e^{\mathbf{L}_{\mathbf{a}_k}t}\chi_{B_k}),\ t \geqslant 0, \quad \forall\ \otimes_{k=1}^{\infty}\chi_{B_k} \in \mathcal{H}_{\mathcal{F},\mathcal{G}},$$

$$\mathbf{L}_{\mathbf{a}_k} = b_j(x)\partial_{e_j} + D_{j,k}(x)\partial_{e_j}\partial_{e_k}$$

$$b_i(x) = \frac{1}{2}\mathrm{M}[\frac{\partial^2 \mathbf{a}_i}{\partial x_j\,\partial x_k}(x)\mathbf{a}_j(x)\mathbf{a}_k(x) + \frac{\partial \mathbf{a}_i}{\partial x_j}(x)\frac{\partial \mathbf{a}_j}{\partial x_k}(x)\mathbf{a}_k(x)].$$

$$\mathbf{L}_{\mathbf{a}}v = \sum_{k=1}^{\infty}\mathbf{L}_{\mathbf{a}_k}v, \quad v \in W_{2,\mathbf{D}}^2.$$

Thank you!