Integrable cases on e(3) и so(4)

Vladimir V. Sokolov

Kharkevich Institute for Information Transmission Problems of the Russian Academy of Sciences, Moscow

vsokolov@landau.ac.ru

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We consider the following family of Poisson brackets

$$\begin{aligned} \{M_i, M_j\} &= \varepsilon_{ijk}\, M_k, \\ \{M_i, \gamma_j\} &= \varepsilon_{ijk}\, \gamma_k, \\ \{\gamma_i, \gamma_j\} &= \kappa\, \varepsilon_{ijk}\, M_k. \end{aligned} \tag{1}$$

Here M_i and γ_i are components of 3-dimensional vectors $\mathbf{M}$ and $\mathbf{\Gamma}$, ε_{ijk} is the totally skew-symmetric tensor, κ is a parameter.

It is well-known that any linear Poisson bracket is defined by an appropriate Lie algebra. The cases $\kappa = 0$, $\kappa > 0$ and $\kappa < 0$ correspond to the Lie algebras $e(3)$, $so(4)$ and $so(3,1)$.

Bracket (1) has the two Casimir functions

$$J_1 = (\mathbf{M}, \mathbf{\Gamma}), \qquad J_2 = \kappa|\mathbf{M}|^{\mathbf{2}} + |\mathbf{\Gamma}|^{\mathbf{2}},$$

where $(\cdot,\cdot)$ stands for the standard scalar product in $\mathbb{R}^3$.

Using the linear transformation

$$U_i = \frac{1}{2}\,(M_i + \kappa^{-1/2}\gamma_i), \qquad V_i = \frac{1}{2}\,(M_i - \kappa^{-1/2}\gamma_i),$$

we can rewrite bracket (1) as

$$\{U_i,\, U_j\} = \varepsilon_{ijk}\, U_k, \qquad \{V_i,\, V_j\} = \varepsilon_{ijk}\, V_k, \qquad \{U_i,\, V_j\} = 0. \tag{2}$$

The Casimir functions for (2) are given by

$$J_1 = (U,\, U), \qquad J_2 = (V,\, V) \tag{3}$$

The canonical transformations have the form

$$\bar{U} = T_1\, U, \qquad \bar{V} = T_2\, V,$$

where T_i are orthogonal matrices.

For the Liouville integrability of the equations of motion only one additional integral functionally independent of the Hamiltonian and the Casimir functions is necessary.

A popular class of Hamiltonians is given by

$$H = (\mathbf{U}, A\mathbf{U}) + 2(\mathbf{U}, B\mathbf{V}) + (\mathbf{V}, C\mathbf{V}), \tag{4}$$

where $A = \mathrm{diag}(a_1, a_1, a_3),\ B = \mathrm{diag}(b_1, b_1, b_3),\ C = \mathrm{diag}(c_1, c_1, c_3)$.

In the particular case $C = A$ such a Hamiltonian has an additional quadratic integral I iff

$$b_1^2(a_2 - a_3) + b_2^2(a_3 - a_1) + b_3^2(a_1 - a_2) + (a_1 - a_2)(a_2 - a_3)(a_3 - a_1) = 0.$$

The integral has the form

$$I = 2(\mathbf{U}, S\mathbf{V}), \qquad S = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3), \tag{5}$$

Without loss of generality one can choose

$$A = \mathrm{diag}(\alpha_1^2, \alpha_2^2, \alpha_3^2), \qquad B = \mathrm{diag}(-\alpha_2\alpha_3,\ -\alpha_3\alpha_1,\ -\alpha_1\alpha_2).$$

Let us change the form of $so(4)$–Poisson bracket from (2) to (1). Now the quadratic homogeneous Hamiltonians have the form

$$H = (\mathbf{M},\, A\mathbf{M}) + (\mathbf{M},\, B\mathbf{\Gamma}) + (\mathbf{\Gamma},\, C\mathbf{\Gamma}), \tag{6}$$

where A, B and C are constant 3×3-matrices.

Integrable $e(3)$-Hamiltonians

The Euler-Poinsot equations for the rotation of rigid body around a fixed point is defined by the Hamiltonian of the form

$$H = aM_1^2 + bM_2^2 + cM_3^2 + 2x\gamma_1 + 2y\gamma_2 + 2z\gamma_3.$$

The famous Kowalewski case corresponds to $a = b = 1,\ c = 2$ and $z = 0$.

The Kirchhoff equations describing the motion of a rigid body in an ideal fluid is defined by the Hamiltonian of the form (6).

The canonical transformations for $e(3)$-brackets with $\kappa = 0$ form a six parameter Lie group consisting of

- orthogonal transformations $\hat{M} = S(M)$, $\hat{\Gamma} = S(\Gamma)$, where $SS^T = E$;
- transformations of the form $\hat{\gamma}_i = \gamma_i$,

$$\begin{aligned} \hat{M}_1 &= M_1 - \mu_1\gamma_2 + \mu_2\gamma_3, \\ \hat{M}_2 &= M_2 + \mu_3\gamma_3 + \mu_1\gamma_1, \\ \hat{M}_3 &= M_3 - \mu_2\gamma_1 - \mu_3\gamma_2, \end{aligned} \tag{7}$$

where μ_i are arbitrary parameters.

Using the orthogonal transformations one can bring the matrix A to the diagonal form:

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}.$$

With the help of (7) we can transform the matrix B to the symmetric (or to the upper triangular) form.

There are classical integrable cases found by Kirchhoff, Clebsch and Steklov-Lyapunov. For all these cases the matrices B and C are **diagonal** and the Hamiltonian is of the form

$$H = a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 +$$

$$2b_1\, M_1\gamma_1 + 2b_2\, M_2\gamma_2 + 2b_3\, M_3\gamma_3 +$$

$$c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2.$$

The **Kirchhoff case** is described by the relations

$$a_1 = a_2, \qquad b_1 = b_2, \qquad c_1 = c_2.$$

This is the only case with a linear additional integral $I_4 = M_3$.

For the **Clebsch** case the coefficients a_i are arbitrary and the remaining parameters satisfy the following conditions

$$b_1 = b_2 = b_3,$$

$$\frac{c_1 - c_2}{a_3} + \frac{c_3 - c_1}{a_2} + \frac{c_2 - c_3}{a_1} = 0.$$

In the **Steklov-Lyapunov** case a_i are arbitrary and

$$\frac{b_1 - b_2}{a_3} + \frac{b_3 - b_1}{a_2} + \frac{b_2 - b_3}{a_1} = 0,$$

$$c_1 - \frac{(b_2 - b_3)^2}{a_1} = c_2 - \frac{(b_3 - b_1)^2}{a_2} = c_3 - \frac{(b_1 - b_2)^2}{a_3}.$$

For both the Clebsch and Steklov-Lyapunov cases there exists an additional quadratic integral.

In 2001 I found a new integrable case with the Hamiltonian

$$\begin{aligned} H = M_1^2 + M_2^2 + 2\, M_3^2 + \\ 2\,(\mu_1\gamma_1 + \mu_2\gamma_2)\, M_3 - (\mu_1^2 + \mu_2^2)\, \gamma_3^2. \end{aligned} \tag{8}$$

The additional integral is of degree four.

Moreover, there exists the following remarkable non-homogeneous integrable combination of the Kowalewski gyrostat and the above Hamiltonian (8):

$$\tilde{H} = M_1^2 + M_2^2 + 2M_3^2 + 2a_1 M_3 - 2c_2\gamma_1 M_3 - c_2^2\,\gamma_3^2 - 2a_1 c_2\,\gamma_1 - 2c_1\,\gamma_2\,,$$

where c_1, c_2 and a_1 are arbitrary constants. If $c_2 = a_1 = 0$, then the Hamiltonian just reduces to the famous Kowalewski Hamiltonian. The case $c_2 = 0$ corresponds to the Kowalewski Hamiltonian with the additional gyrostatic term. If $a_1 = c_1$, we get the Hamiltonian function for the integrable case (8).

A Lax pair for this model was found by A.Tsiganov and VS. This is a deformation of known Lax representation

$$\frac{d}{dt} L_{kow} = [M_{kow}, L_{kow}]$$

for the Kowalewski gyrostat. The corresponding Lax matrices L_{kow} and M_{kow} are given by

$$L_{kow}(\lambda) = \lambda A + B + c_1 \lambda^{-1} C \,, \qquad M_{kow}(\lambda) = -2\lambda A + D, \qquad (9)$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & M_3 & -M_2 & 0 & 0 \\ -M_3 & 0 & M_1 & 0 & 0 \\ M_2 & -M_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -M_3 - a_1 \\ 0 & 0 & 0 & M_3 + a_1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & \gamma_1 \\ 0 & 0 & 0 & 0 & \gamma_2 \\ 0 & 0 & 0 & 0 & \gamma_3 \\ 0 & 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & -4M_3 - 2a_1 & 2M_2 & 0 & 0 \\ 4M_3 + 2a_1 & 0 & -2M_1 & 0 & 0 \\ -2M_2 & 2M_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The deformation of this pair with respect to c_2 is given by

$$L(\lambda, \mu) = L_1(\lambda) + \mu \cdot L_2(\lambda), \tag{10}$$

where

$$L_1(\lambda) = L_{kow}(\lambda) + c_2 X, \qquad L_2(\lambda) = -Id + c_2\lambda^{-1}Y,$$

and

$$X = \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma_1 \\ 0 & 0 & 0 & \gamma_1 & 0 \end{array}\right), \qquad Y = \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & \gamma_1 \\ 0 & 0 & 0 & 0 & \gamma_2 \\ 0 & 0 & 0 & 0 & \gamma_3 \\ 0 & 0 & 0 & 0 & 0 \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & 0 & 0 \end{array}\right).$$

Obviously, if $c_2 = 0$ then this matrix $L(\lambda, \mu)$ coincides with $L_{kow}(\lambda) - \mu \cdot Id$.

Integrable $so(4)$-Hamiltonians

Consider diagonal Hamiltonians of the form

$$H = (\mathbf{U}, A\mathbf{U}) + 2(\mathbf{U}, B\mathbf{V}) + (\mathbf{V}, C\mathbf{V}), \qquad (11)$$

where $A = \operatorname{diag}(a_1, a_1, a_3),\ B = \operatorname{diag}(b_1, b_1, b_3),\ C = \operatorname{diag}(c_1, c_1, c_3)$.

If such a Hamiltonian has additional polynomial integral then the following relations

$$b_1^2(a_2 - a_3) + b_2^2(a_3 - a_1) + b_3^2(a_1 - a_2) + q^2(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) = 0$$

and

$$b_1^2(c_2 - c_3) + b_2^2(c_3 - c_1) + b_3^2(c_1 - c_2) + p^2(c_1 - c_2)(c_2 - c_3)(c_3 - c_1) = 0$$

hold for some odd integers p and q.

The following classical diagonal cases are known: The Poincare case with

$$A = \mathrm{diag}(a_1, a_1, a_3),\ B = \mathrm{diag}(b_1, b_1, b_3),\ C = \mathrm{diag}(c_1, c_1, c_3).$$

Frahm- Schottky-Manakov case.

Steklov case

$$\begin{aligned} H = &\sum_i \alpha_j^2 \alpha_k^2 \, U_i^2 + 2 \sum_i \alpha_j \alpha_k (\alpha_j^2 + \alpha_k^2) \, U_i V_i + \\ &+ \sum_i (\alpha_j^4 + \alpha_k^4 + \alpha_j^2 \alpha_k^2) \, V_i^2, \end{aligned} \tag{12}$$

Adler-van Moerbeke-Reyman-Semenov-Tian-Shansky case

$$\begin{aligned} H = &-9 \sum_i \alpha_j^2 \alpha_k^2 \, U_i^2 + 6 \sum_i \alpha_j \alpha_k (\alpha_i - \alpha_j)(\alpha_i - \alpha_k) \, U_i V_i + \\ &+ \sum_i \alpha_j \alpha_k (4\alpha_i^2 - \alpha_j \alpha_k) \, V_i^2, \end{aligned}$$

where $\alpha_2 + \alpha_2 + \alpha_3 = 0$. Here $q = 1$, $p = 3$.

Non-diagonal integrable $so(4)$-Hamiltonians

Let us change the $so(4)$-brackets (2) to (1). My goal was a systematic investigation of integrable Hamiltonians of the form

$$H = (\mathbf{M},\, A\mathbf{M}) + (\mathbf{b},\, \mathbf{M}\times\mathbf{\Gamma}) \tag{13}$$

with $\mathbf{b}\neq 0$ and their inhomogeneous generalizations

$$H = (\mathbf{M},\, A\mathbf{M}) + (\mathbf{b},\, \mathbf{M}\times\mathbf{\Gamma}) + (\mathbf{k},\, \mathbf{M}) + (\mathbf{n},\, \mathbf{\Gamma}),$$

where $\times$ is the inner product, $\mathbf{k}$ and $\mathbf{n}$ are constant vectors.

1. The generalization of $e(3)$-integrable Sokolov case (8) can be written as

$$H = \frac{1}{2}|\mathbf{u}|^2|\mathbf{M}|^2 + \frac{1}{2}\Big(\mathbf{u},\, \mathbf{M}\Big)^2 - \frac{\kappa}{2}\Big(\mathbf{v},\, \mathbf{M}\Big)^2 + \Big(\mathbf{u}\times\mathbf{v},\, \mathbf{M}\times\mathbf{\Gamma}\Big),$$

where $\mathbf{u}\perp\mathbf{v}$. It has fourth degree integral.

2. The following Hamiltonian

$$H = 2\eta\Big(\mathbf{v},\, \mathbf{M}\Big)\Big(\mathbf{z},\, \mathbf{M}\Big) + \Big(\mathbf{v}\times\mathbf{z},\, \mathbf{M}\times\mathbf{\Gamma}\Big),$$

where $\kappa = \eta^2$, $\mathbf{v}$ and $\mathbf{z}$ are arbitrary constant vectors, has an integral of degree 4.

Let us consider the family of Hamiltonians

$$H = \Big(\mathbf{a},\, \mathbf{b}\Big)|\mathbf{M}|^2 + \mu\Big(\mathbf{a},\, \mathbf{M}\Big)\Big(\mathbf{b},\, \mathbf{M}\Big) + \Big(\mathbf{b},\, \mathbf{M}\times\mathbf{\Gamma}\Big).$$

The eigenvalues of the matrix A are equal to

$$\lambda_1 = \Big(\mathbf{a},\, \mathbf{b}\Big), \qquad \lambda_{2,3} = (1+\frac{\mu}{2})\Big(\mathbf{a},\, \mathbf{b}\Big) \pm \frac{\mu}{2}|\mathbf{a}||\mathbf{b}|.$$

3. If $\mathbf{a} = \mathbf{b}$ the there exists a linear integral.

4. Consider the $so(3,1)$-version $\kappa < 0$ of bracket (1). The Hamiltonian

$$H = \Big(\mathbf{a},\, \mathbf{b}\Big)|\mathbf{M}|^2 - \Big(\mathbf{a},\, \mathbf{M}\Big)\Big(\mathbf{b},\, \mathbf{M}\Big) + \Big(\mathbf{b},\, \mathbf{M}\times\mathbf{\Gamma}\Big),$$

where the vector $\mathbf{b}$ is arbitrary and the length of the vector $\mathbf{a} = (a_1, a_2, a_3)$ is related to the Poisson bracket parameter κ by

$$a_1^2 + a_2^2 + a_3^2 = -\kappa, \tag{14}$$

possesses the additional quartic integral.

5. The Hamiltonian

$$H = \Big(\mathbf{a},\, \mathbf{b}\Big)|\mathbf{M}|^2 - 2\Big(\mathbf{a},\, \mathbf{M}\Big)\Big(\mathbf{b},\, \mathbf{M}\Big) + \Big(\mathbf{b},\, \mathbf{M}\times\mathbf{\Gamma}\Big),$$

has under condition (14) the additional cubic integral.

6. The Hamiltonian

$$H = \Big(\mathbf{a},\, \mathbf{b}\Big)|\mathbf{M}|^2 - \frac{1}{2}\Big(\mathbf{a},\, \mathbf{M}\Big)\Big(\mathbf{b},\, \mathbf{M}\Big) + \Big(\mathbf{b},\, \mathbf{M}\times\mathbf{\Gamma}\Big),$$

under condition (14) has the additional sixth degree integral.

Suppose that all Kowalewski exponents for such Hamiltonian do not depend on the angle between $\mathbf{a}$ and $\mathbf{b}$ then

$$\mu = -2,\ -1, -\frac{1}{2},\ 1.$$

We find for integrable homogeneous Hamiltonians H above described possible linear terms

$$T = (\mathbf{k},\, \mathbf{M}) + (\mathbf{n},\, \boldsymbol{\Gamma}), \tag{15}$$

where $\mathbf{k}$ and $\mathbf{n}$ are constant vectors, such that the Hamiltonian $\tilde{H} = H + T$ has an additional integral I of the same degree as H.

Proposition. *The following linear terms are admissible :*

Case 1(deg I=4): $T = p_1(\mathbf{u},\, \mathbf{M}) + p_2(\mathbf{u} \times \mathbf{v},\, \boldsymbol{\Gamma})$;

Case 2(deg I=4): $T = (\mathbf{k},\, \mathbf{M}) + p_1(\mathbf{v} \times \mathbf{z},\, \boldsymbol{\Gamma})$;

Case 3(deg I=1): $T = p_1(\mathbf{b},\, \mathbf{M}) + p_2(\mathbf{b},\, \boldsymbol{\Gamma})$;

Case 4(deg I=4): $T = (p_1\mathbf{a} + p_2\mathbf{a} \times \mathbf{b}, \mathbf{M}) + p_3(\mathbf{b}, \boldsymbol{\Gamma})$;

Case 5(deg I=3): $T = (\mathbf{k},\, \mathbf{M}) + p_1(\mathbf{b},\, \boldsymbol{\Gamma})$;

Case 6(deg I=6): $T = p_1(\mathbf{a} \times \mathbf{b},\, \mathbf{M})$,

where $\mathbf{k}$ *is arbitrary vector and* p_1, p_2, p_3 *are arbitrary constants.*

Conjecture. It is very likely that all real integrable Hamiltonians of the form

$$H = (\mathbf{M},\, A\mathbf{M}) + (\mathbf{b},\, \mathbf{M} \times \mathbf{\Gamma}) + (\mathbf{k},\, \mathbf{M}) + (\mathbf{n},\, \mathbf{\Gamma}), \tag{16}$$

with $\mathbf{b} \neq 0$ are exhausted by the examples presented above.

Lemma. Suppose a Hamiltonian of this form with real coefficients has an additional polynomial integral of degree from 1 to 7; then the Hamiltonian belongs to above six families.

Scheme of computation. Using the orthogonal transformations, one can reduce any such (real) Hamiltonian to

$$H = a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 + a_4 M_1 M_3 + a_5 M_2 M_3$$

$$+ M_1 \gamma_2 - M_2 \gamma_1 + (\mathbf{k},\, \mathbf{M}) + (\mathbf{n},\, \mathbf{\Gamma}).$$

In this canonical form the vector $\mathbf{b}$ is normalized. Note that the alternative idea of bringing the matrix A to the diagonal form is extremely unsuccessful from the computational point of view.

Given the degree m of the additional integral I, we form the general m-th degree polynomial of the six variables M_i, γ_i with undetermined coefficients. The condition $\{I,\, H\} = 0$ gives rise to a bi-linear system of algebraic equations for the coefficients of H and I.

Conjecture. All this integrable inhomogeneous Hamiltonians (16) have integrable $U(so(4))$-analogs.

Kowalewski-Lyapunov test

Solutions of the form $\mathbf{X}_0 = \frac{\mathbf{K}}{t}$ for a system

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}), \qquad \mathbf{X} = (x_1, \ldots, x_N), \tag{17}$$

where $\mathbf{F} = (f_1, \ldots f_N)$ and f_i are homogeneous quadratic polynomials of $\mathbf{X}$, are called *Kowalewski solutions*.

The linearization $\mathbf{X} = \mathbf{X_0} + \varepsilon \mathbf{\Psi}$ of system (17) on a Kowalwski solution $\mathbf{X_0}$ satisfies

$$\frac{d\mathbf{\Psi}}{dt} = \frac{1}{t} S(\mathbf{\Psi}), \tag{18}$$

where S is a constant $N \times N$-matrix depending on $\mathbf{K}$.

Solutions of (18) have the form $\mathbf{\Psi} = \mathbf{v}\, t^k$, where k is an eigenvalue and $\mathbf{v}$ is an eigenvector of the matrix S. The number k is called *Kowalewski exponent*.

According to the Kowalewski-Lyapunov test, system (17) is "integrable"if for any Kowalewski solution all corresponding Kowalewski exponents are integers.