Material balance equation as a method of sewing analytical and numerical solutions for the three regimes of the flows

A. Zhaglova, D. Anikeev, I. Indrupskiy, E. Zakirov, A. Ibraguimov,

TTU and OGRI of Russian Academy of Science, Moscow. Work was funded by the grant : 122022800272-4

III International Conference "Mathematical Physics, Dynamical Systems, Infinite-Dimensional Analysis",Section of Mathematical Physics (July 5-13, 2023, Moscow Region, Dolgoprudny)

July 13, 2023

Goal of the project. Material Balance, Symmetric and Isotropic Case

The goal of the project is to find R_0 which can be assigned to match calculated pressure in the block containing well to actual pressure at each point of the flow near well. Material Balance (MB) in general for transient flow has a form

$$-4K \cdot (p_0(s) - p_1(s)) = -\frac{q}{h} + \phi \cdot c_p \cdot \frac{V_0}{h} \cdot \frac{p_0(s+\tau) - p_0(s)}{\tau} \qquad (1.1)$$

We consider three scenario:

1. Steady State (SS) (This case was considered for Linear Darcy flow by Peaceman)
2. Pseudo Steady State (PSS)
3. Boundary Dominated (BD)

Domain of Flow and Discretization on the Five Spot Grid

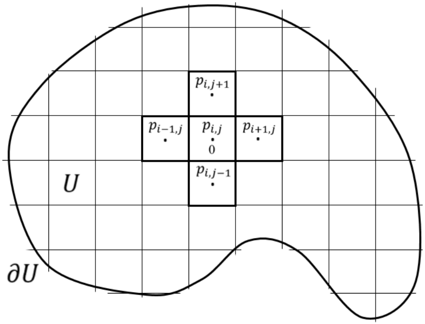

Figure 1: Domain of the Flow with source at 0

Material Balance on the Five Spot Grid

We divide all area of flow into $M \times M$ blocks. For all blocks(see fig.1), $0 \le i \le M$, $0 \le j \le M$. For the block of interest $Q_{i,j}$ for the Darcy flow can be reduced to the form:

$$Kh\frac{\Delta y}{\Delta x} \cdot (p_{i+1,j} - 2p_{i,j} + p_{i-1,j}) + \\ + Kh\frac{\Delta x}{\Delta y} \cdot (p_{i,j+1} - 2p_{i,j} + p_{i,j-1}) = q_{i,j} \tag{1.2}$$

In the above equation, $q_{i,j} = 0$ if $i \neq 0$ or $j \neq 0$ ($q_{i,j} = q \cdot \delta_{i,j}$ – Kronecker symbol). Evidently size of the block in x and y direction are correspondingly Δx and Δy and are converging to 0 as $M \to \infty$. Let us denote $2M \times 2M$ matrix P_M

$$P_M = \left(p_{i,j}\right)_{((-M \le i \le M);(-M \le j \le M))} \tag{1.3}$$

Consider BVP in the bounded domain U containing source point 0 (see fig.1):

$$-\nabla\left(Kh\cdot\nabla p\right)=q\cdot\delta(x) \text{ in the domain } U \quad (1.4)$$

$$p(x)=0 \text{ on the boundary } \partial U \quad (1.5)$$

Here $x=(x,y)$ and h thickness of the domain of flow. Elements of the matrix P_M represent values of the solution of the discrete Poisson equation with RHS localised at center $(0,0)$ stock/source. Let upgrade system by boundary conditions.

Using classical machinery for the Green function construction using Wiener's approximation of the generalised solution we expect that following Conjecture can be proved.

Conjecture 1

Let $(x, y) \neq (0, 0)$ *fixed point of the domain* U*. This point belongs to one of the elements of the grid* U_M *which approximates domain* U*. Let* $p_M(x, y)$ *is solution of the system* $2M \times 2M$*, extended to be* $C^2(U) \cap C^0(\bar{U})$*. Then as* $M \to \infty$ *function* $p_M(x, y) \to U(x, y)$*, where*

$$U(x, y) = \frac{1}{2\pi} \ln \frac{1}{r} + \varphi(x, y), \ r = \sqrt{x^2 + y^2}, \tag{1.6}$$

is the Green function in the domain U *for Laplace operator.*

The goal is to compute the Green function for the Laplace equation in the domain Ω for homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\Delta U(x,x_0) &= \delta(x-x_0)\ ,\ x \in U\ ,\ x_0 \in U\ ; \\ U(x,x_0) &= 0 \text{ on } \Gamma = \partial U. \end{aligned} \tag{1.7}$$

Setting

$$U(x,x_0) = G(x-x_0) + \varphi(x,x_0), \tag{1.8}$$

where G is the fundamental solution

$$G(x-x_0) = -\frac{1}{2\pi} \ln |x-x_0|\ . \tag{1.9}$$

It follows that the corrector φ is the solution of

$$\begin{aligned} -\Delta\varphi(x,x_0) &= 0\ ,\ x \in U\ ,\ x_0 \in U\ ; \\ \varphi(x,x_0) &= -G(x-x_0) \text{ on } \Gamma = \partial U. \end{aligned} \tag{1.10}$$

This equation is homogeneous, and hence $\varphi(\cdot,x_0) \in C^\infty(U)$. Moreover, since we assume that $x_0 \in U$, the boundary data in (1.9) is a smooth function.

Green Function approximation

The idea of singularity correction is to solve the corrector equation with a standard numerical method, such as the usual five point finite difference approximation of the Laplacian. We denote by $\varphi_h(x, x_0)$ the approximation of $\varphi(x, x_0)$ at the grid points $x \in U_h$, where h is the spacing. Then the following convergence result is well known

Theorem 2

If $\varphi \in C^4(U)$ and R_h is the restriction to the grid U_h then

$$|\varphi_h - R_h\varphi|_\infty \leq \frac{h^2}{48}|\varphi|_{C^4(U)}.$$

For a set A the oscillation of a function is defined as

$$osc_A f = \sup_{x\in A} f - \inf_{x\in A} f.$$

Property of the corrector and fundamental solution

Theorem 3

Suppose $R > 0$ is such that $B(x_0, R) \subset U$, then there exist $C > 0$ depending on R only such that for any $r < R$

$$osc_{B(x_0,r)}\varphi(x, x_0) \leq C \cdot r \tag{1.11}$$

This follows from the smoothness of φ in the interior domain.

Theorem 4

For any $r_0 < r < R$ if

$$\ln \frac{r}{r_0} = \frac{\pi}{2} \tag{1.12}$$

then

$$4 \cdot \left(G(x, x_0)|_{x \in \partial B(x_0,r)} - G(y, x_0)|_{y \in \partial B(x_0,r_0)} \right) = 1 \tag{1.13}$$

Main qualitative property of the Green Function

From Theorems 3 and 4 follows

Theorem 5

Let r and r_0 the same as in Theorem 4, then

$$4 \cdot \left(U(x, x_0)|_{x \in \partial B(x_0, r)} - U(y, x_0)|_{y \in \partial B(x_0, r_0)} \right) = 1 + O(r) \qquad (1.14)$$

Back to Material Balance as Sewing Machinery

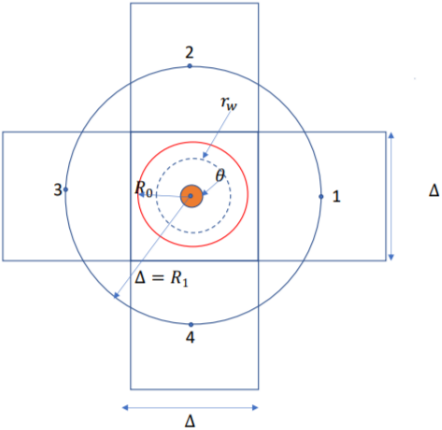

Figure 2: Five spot Grid of size Δ and well at a center, and auxiliary $U(0, R_w, R_1)$ and $U(0, R_w, R_0)$

Peaceman Well Block Radius and Fundamentals in Finite Difference Solution

$$p_1 - p_0 = \frac{\alpha}{4} q, \text{ here } \alpha = \frac{\mu}{kh}. \tag{1.15}$$

Analytical solution

$$p(r) = \alpha \frac{q}{2\pi} \ln \frac{r}{R_1} + p(R_1). \tag{1.16}$$

Peaceman Well-Posedness can be stated as

Problem 6

Let value of p_1 and p_0 relate by material balance (1.15).*Let $\theta < R_0 < \Delta$. Find R_0 s.t.*

$$p(\theta) = \alpha \frac{q}{2\pi} \ln \frac{\theta}{R_0} + p_0, \tag{1.17}$$

and

$$p(\theta) = \alpha \frac{q}{2\pi} \ln \frac{\theta}{\Delta} + p_1. \tag{1.18}$$

Peaceman Well Block Radius

Theorem 7

Assume that total rate of the production q and size of the grid Δ are given. Assume that single fully penetrated well located at the center of the numerical block $[-\frac{\Delta}{2}, \frac{\Delta}{2}]^2$. Let R_w is such that $\ln \frac{\Delta}{\theta} > \frac{\pi}{2}$. Then necessary and sufficient conditions that guarantee Peaceman well posedness is

$$\ln \frac{\Delta}{R_0} = \frac{\pi}{2}. \quad (1.19)$$

Interpretation in this section of the Peaceman paper is made directly without significant modification. But it is already clear that main aim is to sew numerical and analytical solutions formulated on the different scale to provide needed information for tuning procedure between numerical solution and observed data through relation

$$p(r) = \alpha \frac{q}{2\pi} \ln \frac{r}{R_0} + p_0 \quad (1.20)$$

Another view on the problem

$$p_1 - p_0 = \frac{\alpha}{4} q \tag{1.21}$$

$$p_1 - p_w = \frac{\alpha}{2\pi} q \ln \frac{\Delta}{R_w} \tag{1.22}$$

$$p_0 - p_w = \frac{\alpha}{2\pi} q \ln \frac{R_0}{R_w} \tag{1.23}$$

Theorem 8

Assume that q and p_w solve equation for given p_1

$$q = \frac{2\pi k}{\mu} \frac{p_1 - p_w}{\ln \frac{\Delta}{R_w}} = 2\pi\alpha^{-1} \frac{p_1 - p_w}{\ln \frac{\Delta}{R_w}}. \tag{1.24}$$

Then, if R_0 satisfy equation

$$R_0 = \Delta \cdot e^{-\frac{\pi}{2}} \tag{1.25}$$

system (1.21)-(1.23) *has a solution for any mutually related p_1, p_0, and p_w.*

Forchheimer Flow

$$-\frac{\partial p}{\partial r} = \alpha_1 v_r + \beta v_r |v_r| \qquad (1.26)$$

In (1.26) if $\beta = 0$ one can get classical Darcy equation. Due to 1-D continuity equation radial velocity

$$v_r = -\frac{q}{2\pi r} \qquad (1.27)$$

for any $r > 0$ if total rate (over well) q is fixed. From (1.26) and(1.27) follows

$$\frac{\partial p}{\partial r} = \alpha_1 \frac{q}{2\pi r} + \beta \frac{q}{4\pi^2 r^2} \qquad (1.28)$$

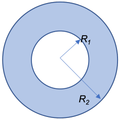

Figure 3: General Annual Domain U

Forchheimer Continue

Consider flow from $\partial B(0,R_2)$ to $\partial B(0,R_1)$ in the annular domain U (see Fig.3):

$$\begin{cases} U = B(0,R_2)\setminus B(0,R_1), \text{ with fixed total rate } q = \int_S v(r)ds, \\ \text{Given pressure on one of the boundaries } \partial B(0,R_i): \\ p(r)\big|_{r=R_i} = f_i \text{ for } i = 1 \text{ or } 2. \end{cases} \quad (1.29)$$

From basic integration follows explicit formula for generic solution for two terms Forchheimer law:

$$p(r) = \frac{\alpha_1 q}{2\pi}\ln r - \beta\frac{q}{4\pi^2}\frac{1}{r} + constant \quad (1.30)$$

Then using boundary conditions in (1.29) one can get a generic formula for pressure depletion between two contours($\partial B(0,R_i)$) of the boundary of annular domain $U(0,R_1,R_2) = B(0,R_2)\setminus B(0,R_1)$.

$$p\big|_{r=R_2} - p\big|_{r=R_1} = f_2 - f_1 = \frac{\alpha_1 q}{2\pi}\ln\frac{R_2}{R_1} + \beta\frac{q}{4\pi^2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad (1.31)$$

Forchheimer Continue

We will hypothesise that on coarse greed material balance is still linear, whether near well correction is due to Forchheimer type of non-linearity. From Linear Material Balance Equation (1.21) and (1.31) with $\beta \neq 0$ follows the system of 3 equations:

$$p_1 - p_0 = \frac{\alpha}{4} q \tag{1.32}$$

$$p_1 - p_w = \frac{\alpha}{2\pi} q \ln \frac{\Delta}{R_w} + \beta \frac{q^2}{4\pi^2} \left(\frac{1}{R_w} - \frac{1}{\Delta} \right) \tag{1.33}$$

$$p_0 - p_w = \frac{\alpha}{2\pi} q \ln \frac{R_0}{R_w} + \beta \frac{q^2}{4\pi^2} \left(\frac{1}{R_w} - \frac{1}{R_0} \right) \tag{1.34}$$

Peaceman analogue of Well block radius for Non-linear Flow

Theorem 9

Assume that q and p_w solves quadratic equation

$$p_1 - p_w = \frac{\alpha}{2\pi} q \ln \frac{\Delta}{R_w} + \beta \frac{q^2}{4\pi^2} \left(\frac{1}{R_w} - \frac{1}{\Delta} \right) \tag{1.35}$$

Then, if R_0 satisfy equation

$$R_0 = \Delta \cdot e^{-\delta \frac{\pi}{2}} \tag{1.36}$$

system (1.32)-(1.34) *has a solution for any mutually related $p_1, \; p_0,$ and p_w if δ satisfies equation*

$$\delta + \beta \frac{q}{\alpha \pi^2} \left(\frac{e^{\delta \frac{\pi}{2}}}{\Delta} - \frac{1}{\Delta} \right) = 1 \tag{1.37}$$

Difference between a Classical and an Adjusted Equations of Non-linear Flow

$$p_0 - p_w = \frac{\alpha}{2\pi} q \ln \frac{R_{0_{Darcy}}}{R_w} + \beta \frac{q^2}{4\pi^2} \frac{1}{R_w} \tag{1.38}$$

This formula is used to relate well flow rate q to the difference of grid block pressure p_0 and well pressure p_w. The key problem is that R_0 is computed just as in the linear case.

$$p_0 - p_w = \frac{\alpha}{2\pi} q \ln \frac{R_{0_{Darcy}}}{R_w} + \beta \frac{q^2}{4\pi^2} \left(\frac{1}{R_w} - \frac{1}{\Delta} \right). \tag{1.39}$$

This is a formula derived using the correction. The difference is in the summand with $\frac{1}{\Delta}$.

Table 1: Sets of the problems for transient flows

	$\frac{\phi c_p}{K} p_t = \Delta p$		
	SS	PSS	BD
BC	$p\big\|_{\Gamma_w} = p_w$, $p\big\|_{\Gamma_e} = p_e$	$\frac{\partial p}{\partial \nu}\big\|_{\Gamma_w} = \frac{q}{\|\Gamma_w\|}$; $\frac{\partial p}{\partial \nu}\big\|_{\Gamma_e} = 0$	$p\big\|_{\Gamma_w} = p_w$; $\frac{\partial p}{\partial \nu}\big\|_{\Gamma_e} = 0$

For each of the transient regimes **Goal is to find its own "Peaceman" radius** but such that corresponding $R_0^{'}$ to be time independent. For this reason we use the following sets of solutions

$$SS : p_{SS}(x,t) = W_{SS}(x)\ ;\ \Delta W_{SS} = 0\ ;\ W_{SS}(x)\big|_{\Gamma_w} = 0\ ;\ W_{SS}(x)\big|_{\Gamma_e} = 1$$

$$PSS : p_{PSS}(x,t) = W_{PSS}(x) + At; \Delta W_{PSS} = A; W_{PSS}\big|_{\Gamma_w} = 0\ ;\ \frac{\partial W_{PSS}}{\partial \nu}\Big|_{\Gamma_e} = 0$$

$$BD : p_{BD}(x,t) = \varphi(x)e^{-\lambda t}; -\Delta\varphi = \lambda\varphi; \varphi\Big|_{\Gamma_w} = 0\ ;\ \frac{\partial \varphi}{\partial \nu}\Big|_{\Gamma_e} = 0$$

Peaceman well block radius R_0^{ss} for Steady State (SS) MB, case $C_p = 0$

Let $p_{an}^{ss}(r)$ is pressure distribution of Steady state Problem, then depending on "size" of the grid (Δ) R_0^{ss} can be obtained on, such that function p_{an}^{ss} obey Steady state material balance (SS-MB) namely

$$p_1 = p_{an}^{ss}(r)\Big|_{|r|=\Delta} \tag{1.40}$$

$$p_0 = p_{an}^{ss}(r)\Big|_{|x|=R_0^{ss}} \tag{1.41}$$

Here p_1 and p_0, obtained from numerical simulation of the process on the grid of size Δ and R_0^{ss} to be found. It was proven the that

$$\boxed{R_0^{Peaceman} = R_0^{ss} = e^{-\frac{\pi}{2}} \cdot \Delta} \tag{1.42}$$

R_0^{ss} does not depend on rate of the production and external radius of the reservoir R_e and well radius r_w.

This R_0^{ss} can be used

1. **To interpret numerically calculated P_0 for inverse problem**
2. **To forecast value of the well pressure for direct problem**

Theorem 10

In order analytical PSS solution (Column 2) to satisfy material balance (1.43)

$$-4K\cdot(p_0(s)-p_1(s)) = -\frac{q}{h} + \phi\cdot c_p\cdot\frac{V_0}{h}\cdot\frac{p_0(s+\tau)-p_0(s)}{\tau} \tag{1.43}$$

with constant production rate q it is sufficient

$$\boxed{-\pi + \frac{\left(R_0^{pss}\right)^2}{r_e^2} = -2\cdot\left(\ln\frac{\Delta}{R_0^{pss}}\right)} \tag{1.44}$$

Corollary 11

Due to constraints that $A \approx \frac{1}{r_e^2}$ *PSS solution is not "trivial" only for bounded domain but from formula* (1.44) *one can easily conclude that*

$$\boxed{R_0^{pss} \to R_0^{Peaceman} \text{ as } r_e \to \infty.} \tag{1.45}$$

Theorem 12

In order analytical BD solution to satisfy material balance with constant pressure value on the well and non-permeable external boundary it is sufficient that R_0^{bd} satisfies transcendent equation

$$4 \cdot \left(\varphi_0(\lambda_0 \Delta) - \varphi_0(\lambda_0 R_0^{bd}) \right) = \tag{1.46}$$

$$= \frac{\partial \varphi_0(\lambda_0 r)}{\partial r} \bigg|_{r=r_w} \cdot 2\pi r_w \cdot \Delta + \varphi_0(\lambda_0 \cdot R_0^{bd}) \cdot \frac{\phi c_p}{K} \cdot \frac{e^{-\lambda_0^2 \tau} - 1}{\tau}$$

Conjecture 13

Exist τ_0 and $\beta_0 = r_e/r_w > 1$ exist R_0^{bd}, which solves equation (1.46) for all $\tau < \tau_0$ and $r_e/r_w > \beta_0$. Meaning that Peaceman problem is well-posed for R_0^{bd} be function of τ_0 and β_0.

Remark 1

For "small" τ, $\tau << 1$, and big r_e, $r_e/r_w >> 1$ above formula can be well approximated by equation (1.47). Equation (1.46) is analogue of Peaceman formula for boundary dominated regime of the flow. By finding R_0^{bd} from this equation we provide correct value to calculate Peaceman well radius.

If parameter τ is negligible ($\tau << 1$) then equation (1.46) can be well approximated by equation

$$\boxed{4 \cdot \left(\varphi_0(\lambda_0 \Delta) - \varphi_0(\lambda_0 R_0^{bd})\right) = \frac{\partial \varphi_0(\lambda_0 r)}{\partial r}\Big|_{r=r_w} \cdot 2\pi r_w \cdot \Delta} \tag{1.47}$$

Here φ_0 is eigenfunction based on Bessel composition

$$\varphi_0(\lambda_0 r) = J_0(\lambda_0 r_w) \cdot N_0(\lambda_0 r) - J_0(\lambda_0 r) \cdot N_0(\lambda_0 r_w) \;, \tag{1.48}$$

and λ_0 root of the transcendent equation

$$\boxed{0 = J_0(\lambda_0 r_w) \cdot \frac{\partial N_0(\lambda_0 r)}{\partial r}\Big|_{r=r_e} - \frac{\partial J_0(\lambda_0 r)}{\partial r}\Big|_{r=r_e} \cdot N_0(\lambda_0 r_w)} \tag{1.49}$$

and $r_e,$ and r_w are exterior reservoir and well radius

Theorem 14

In case of BD boundary condition as $\tau \to 0$

$$\boxed{R_0^{bd} \to R_0^{Peaceman} \quad as \quad r_e \to \infty} \tag{1.50}$$

We are planning to generalised results for PSS and BD Peaceman formula for Non-Linear Forchheimer type of the flows

The End

THANK YOU!