Functional of eigenvalues on the manifold of potentials

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The space of of self-adjoint periodic eigenvalue and eigenfunction boundary-value problems

We consider the family

$$-y'' + p(x)y = \lambda y; \quad y(0) - y(2\pi) = y'(0) - y'(2\pi) = 0 \quad (1)$$

$$of \quad p \in P := \{L_2(0, 2\pi) : \int_0^{2\pi} p(x)dx = 0\}$$

as a functional parameter.

For a fixed potential $p \in P$ the spectrum consists of real eigenvalues, which have multiplicity at most 2:

$$\lambda_0(p) < \lambda_1^-(p) \leq \lambda_1^+(p) < \ldots < \lambda_n^-(p) \leq \lambda_n^+(p) < \ldots$$

Eigenfunctions corresponding to eigenvalues with subscript n have precisely $2n$ nondegenerate zeros on the half-open interval $[0, 2\pi)$.

The purpose of the report is to investigate for any $n \in \mathbb{N}$ and "sign $\pm$" the functional of eigenvalues on the manifold of potentials.

$$\Lambda_n^{\pm} : P \to \mathbb{R}, \quad \Lambda_n^{\pm}(p) := \lambda_n^{\pm}(p).$$

By special parametrization we give an analytic and qualitative description of the functional of eigenvalues.

The manifold Y_n of eigenfunctions with exactly 2n zeros

$$Y_n := \{y \in H^2[0,2\pi] : \int_0^{2\pi} y^2 dx = 1,\ y \cong -y,$$
$$\exists p \in P : (1)\ \textit{is true with}\ \lambda = \lambda_n^{\pm}(p)\}.$$

The set Y_n $(n = 0, 1, ...)$ consists of all functions y such that:

1. $y(0) - y(2\pi) = y'(0) - y'(2\pi) = 0$;
2. there are exactly 2n points $x_i \in [0, 2\pi)$ at which $y(x_i) = 0$;
3. $y'(x_i) \neq 0 \quad (i = 1, ..., 2n)$;
4. $$\int_0^{2\pi} \left(\frac{y''}{\sin 2(x - x_1) \ldots \sin 2(x - x_{2n})} \right)^2 dx < +\infty.$$

Restoration of eigenvalue and potential by eigenfunction

There are mappings which recover the eigenvalue and potential:

$$\Lambda_n : Y_n \to \mathbb{R},\ \Lambda_n(y) = \lambda := -\frac{1}{2\pi}\int_0^{2\pi} \frac{y''}{y}dx;$$

$$\Phi_n : Y_n \to P,\ \Phi_n(y) = p := \frac{y''}{y} + \Lambda_n(y).$$

The reciprocal eigenfunctions and lacuna

For $n \in \mathbb{N}$ a pair $(y, y_r) \in Y_n \times Y_n$ is said to be **reciprocal** if these functions are generated by the same potential p and $\int\limits_0^{2\pi} y y_r dx = 0$.

$\forall\ y \in Y_n\ \exists\ !\ y_r = I(y)$ and $I^2(y) = y$.
If $\lambda_n^-(p) < \lambda_n^+(p)$ then $I(y^{\pm}(p)) = y^{\mp}(p)$.

By definition, **Lacuna**$(y) = \Delta\Lambda_n(y) := \Lambda_n(y) - \Lambda_n(I(y))$,

$$Y_n(\Delta\lambda) := \{y \in Y_n :\ \Delta\Lambda_n(y) = \Delta\lambda\}, \quad Y_n = \cup_{\Delta\lambda \in \mathbb{R}} Y_n(\Delta\lambda).$$

The subset $Y_n(0) \subset Y_n$ is called **degenerate**.

$$|\Delta\Lambda_n(p)| := \lambda_n^+(p) - \lambda_n^-(p) \geq 0,$$

$$P_n(|\Delta\lambda|) := \{p \in P :\ |\Delta\Lambda_n(p)| = |\Delta\lambda|\},\ P = \cup_{|\Delta\lambda| \geq 0} P_n(|\Delta\lambda|).$$

The subset $P_n(0) \subset P$ is called **degenerate**.

The manifold H_n

Let $\rho(x) := (y^2(x) + y_r^2(x))^{1/2}$. By polar coordinates (ρ, θ),

$$y(x) = \rho(x)\cos\theta(x), \;\; y_r(x) = \rho(x)\sin\theta(x), \;\; \theta(x) := \varphi + \int_0^x \eta(t)\,dt.$$

Then of the four pairs $(\pm y, \pm y_r)$, $(\pm y, \mp y_r)$ only one satisfies the conditions the Wronskian $W(y, y_r) > 0$ and $\varphi \in [0, \pi) \in \mathbb{R}P^1$.

Let H_n denote the subset of all functions η, then

$$H_n = \{\eta \in H^2[0, 2\pi] : \; \eta(x) > 0, \; \int_0^{2\pi} \eta(x)dx = 2\pi n,$$

$$\eta(0) - \eta(2\pi) = \eta'(0) - \eta'(2\pi) = 0$$

$$\int_0^{2\pi} \frac{\sin 2\int_0^x \eta(t)dt}{\eta(x)}dx = \int_0^{2\pi} \frac{\cos 2\int_0^x \eta(t)dt}{\eta(x)}dx = 0.\}$$

The subset $H_n \subset H^2[0, 2\pi]$ is a homotopically trivial, C^∞-smooth Banach submanifold of codimension 5.

The parametrization of manifold Y_n and P

The triple $(\eta, \varphi, \Delta\lambda) \in H_n \times \mathbb{R}P^1 \times \mathbb{R}$ determines the eigenfunction $y \in Y_n$ and corresponding potential $p \in P$:

$$\begin{tikzcd} Y_n \arrow[r, "\Phi_n"] \arrow[d, "G_n"'] & P \arrow[d, "F_n"] \\ H_n \times \mathbb{R}P^1 \times \mathbb{R} \arrow[r, "\pi_n"'] & H_n \times \mathbb{R}^2 \end{tikzcd}$$

where

1. G_n is bijection and C^∞-parametrization of Y_n,
2 F_n is C^∞-diffeomorphisms,
3. $\Phi_n, \; \pi_n \in C^\infty$,
4. the diagram is commutative.

The parametrization of manifold Y_n and the mapping π_n

By definition $\theta(x;\varphi,\eta) := \varphi + \int_0^x \eta(t)dt$, where $\varphi \in \mathbb{R}P^1$

$G_n: \ Y_n \to H_n \times \mathbb{R}P^1 \times \mathbb{R}$ is C^∞-diffeomorphism.

$$G_n^{-1} : H_n \times \mathbb{R}P^1 \times \mathbb{R} \to Y_n,$$

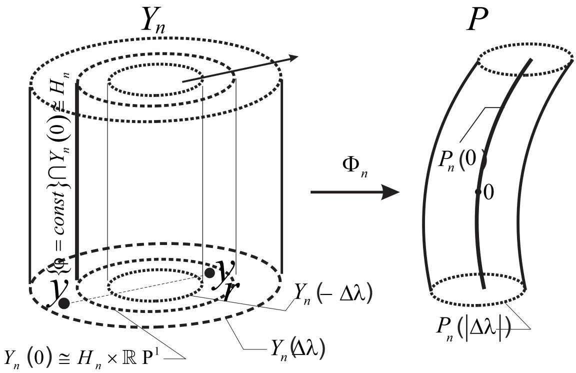
$$G_n^{-1}(\eta,\varphi,\Delta\lambda) :=$$

$$y^{sign(\Delta\lambda)} = \frac{const}{\eta^{1/2}(x)} \exp\left(\frac{\Delta\lambda}{4}\int_0^x \frac{\sin 2\theta(\tau;\varphi,\eta)}{\eta(\tau)}d\tau\right)\cdot\cos(\theta(x;\varphi,\eta))$$

$$y_r := I(y) = I(\eta,\varphi,\Delta\lambda) = (\eta, \varphi + \frac{\pi}{2}, -\Delta\lambda)$$

$$\pi_n(\eta,\varphi,\Delta\lambda) := (\eta, \Delta\lambda\cos 2\varphi, \Delta\lambda\sin 2\varphi)$$

The map Φ_n



1. The map Φ_n is C^∞-morphism.
2. For $\Delta\lambda \neq 0$, $\Phi_n|_{Y_n(\Delta\lambda)} : Y_n(\Delta\lambda) \to P(|\Delta\lambda|)$ is C^∞-diffeomorphism.
3. $\Phi_n|_{Y_n(0)} : Y_n(0) \to P(0)$ is C^∞-bundle with $\mathbb{R}P^1$ as fiber;
4. For any $\Delta\lambda$, $Y_n(\Delta\lambda) \cong P(0) \times \mathbb{R}P^1$.
5. $H_n \cong P(0) \subset P$ $codimP(0) = 2$; $P \setminus P(0) \cong P(0) \times \mathbb{R}P^1$.

The parametrization of manifold P and $\lambda(\eta, \varphi, \Delta\lambda)$

$$\Phi_n \cdot G_n^{-1} : \; H_n \times \mathbb{R}P^1 \times \mathbb{R} \to P,$$
$$\Phi_n \cdot G_n^{-1}(\eta, \varphi, \Delta\lambda) = p(x) = r(x) + \lambda,$$

where

$$r(x) := \frac{(y)''}{y} = -\frac{\eta''}{2\eta} + \frac{3(\eta')^2}{4\eta^2} - \eta^2 +$$
$$\frac{\Delta\lambda\eta' \sin 2\theta(x; \varphi, \eta)}{2\eta^2} + \frac{\Delta\lambda^2 \sin^2 2\theta(\dots)}{16\eta^2} + \Delta\lambda \cos 2\theta(\dots) - \frac{\Delta\lambda}{2},$$

$$\lambda = -\frac{1}{2\pi} \int_0^{2\pi} r(x) dx =$$
$$\frac{1}{2\pi} \int_0^{2\pi} \left(\eta^2 - \left(\frac{\eta'}{2\eta} \right)^2 - \left(\frac{\Delta\lambda \sin 2\theta}{4\eta} \right)^2 \right) dx + \frac{1}{2}\Delta\lambda.$$

Graph of eigenvalue, the function η is fixed

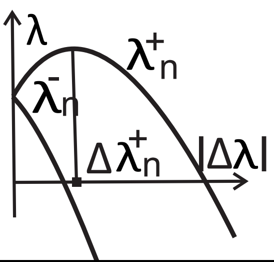
If cartesian coordinates are

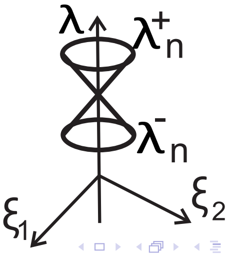
$$\xi_1 = |\Delta\lambda| \cos 2\varphi, \;\; \xi_2 = |\Delta\lambda| \sin 2\varphi,$$

then

$$\lambda_n^{\pm} = \frac{1}{2\pi} \int_0^{2\pi} \left(\eta^2 - \left(\frac{\eta'}{2\eta} \right)^2 \right) dx + \pm \frac{1}{2} \sqrt{\xi_1^2 + \xi_2^2} -$$

$$- \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{16\eta^2} \left(\xi_1 \sin(2 \int_0^x \eta(t)dt) + \xi_2 \cos(2 \int_0^x \eta(t)dt) \right)^2 \right) dx.$$

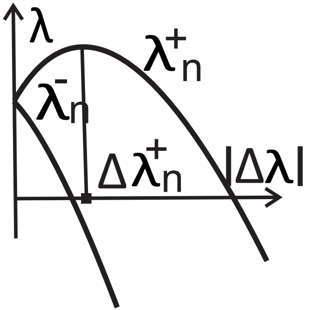


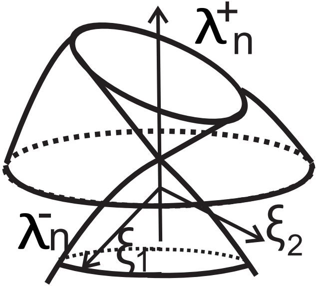


Graph of eigenvalue, the function η is fixed

$$\Delta\lambda_n^+(\eta,\varphi) = \frac{1}{2\pi}\left(\int_0^{2\pi}\left(\frac{\sin\left(2\varphi + 2\int_0^x \eta(t)dt\right)}{4\eta}\right)^2 dx\right)^{-1}, \ \varphi \in \mathbb{R}P^1$$

Above the plane $\{\xi_1, \xi_2\}$, the graph λ_n^+ looks like a volcanic crater with one maximum, one minimum, two saddle points on its ridge and conic minimum; and the graph λ_n^- looks like a pointed peak.





Graph of eigenvalue, a degenerate lacuna $\Delta\lambda = 0$

We study the problem for the extremum of the functional

$$\Lambda_n^0 : P_n(0) \cong \ H_n \to \mathbb{R}, \quad \lambda(\eta) = \frac{1}{2\pi} \int_0^{2\pi} \left(\eta^2 - \left(\frac{\eta'}{2\eta} \right)^2 \right) dx$$

There is a unique critical point of the functional Λ_n^0; this is the function $\eta_n(x) \equiv n$. Thus, the critical value is $\lambda_n^* = n^2$, and for any $n \in \mathbb{N}$ the critical point is the function

$$p(x) = -\frac{\eta''}{2\eta} + \frac{3(\eta')^2}{4\eta^2} - \eta^2 + \lambda = -n^2 + n^2 = 0.$$

Graph of eigenvalue, a degenerate lacuna $\Delta\lambda = 0$

At the critical point $\eta_n(x) \equiv n$ the tangent space is

$$T_{\eta_n} H_n = \{\delta \in H^2[0, 2\pi] : \int_0^{2\pi} \delta(x) dx = 0,$$

$$\delta(0) - \delta(2\pi) = \delta'(0) - \delta'(2\pi) = 0,$$

$$\int_0^{2\pi} (2n \cos 2nx \int_0^x \delta(t) dt - \sin 2nx \cdot \delta(x)) dx = 0,$$

$$\int_0^{2\pi} (-2n \sin 2nx \int_0^x \delta(t) dt - \cos 2nx \cdot \delta(x)) dx = 0\} =$$

$$= \{\cos x, sinx, ..., \cos(2n-1)x, \sin(2n-1)x, \cos(2n+1)x, \sin(2n+1)x ...\}.$$

Thus,

$$\delta(x) = \sum_{k \neq 2n}^{\infty} (a_k \cos kx + b_k \sin kx) \subset H^2[0, 2\pi].$$

At the critical point $\eta_n(x) \equiv n$ the second differential is

$$d^2\Lambda_n^0(n, \delta(x)) = \int_0^{2\pi} (2\delta^2(x) - \frac{(\delta'(x))^2}{2n^2})dx =$$

$$= \frac{\pi}{2n^2} \sum_{k \neq 2n}^{\infty} (2n - k)(2n + k)(a_k^2 + b_k^2).$$

Thus, on the manifold $P_n(0)$ of degenerate potentials, the functional Λ_n^0 of eigenvalues is Morse functional with unique critical point $p(x) \equiv 0$; Morse index of critical point is equal to $2(2n - 1)$.

Literature

1. *Ya.M. Dymarskii, A.A. Bondar'* An eigenfunction manifold generated by a family of periodic boundary value problems // Sbornik: Mathematics 212:9, 2021, P. 1208–1227.
2. *Ya. M. Dymarskii, Yu. A. Evtushenko* Foliation of the space of periodic boundary-value problems by hypersurfaces to fixed lengths of the n-th spectral lacuna // Sbornik: Mathematics 207:5, 2016, P. 678–701.

Thank you for your attention!