A quadratic estimation for the Kühnel conjecture on embeddings

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Heawood inequality and the Kühnel conjecture

The classical Heawood inequality states that if the complete graph K_n on n vertices is embeddable into the sphere with g handles, then

$$g \geqslant \frac{(n-3)(n-4)}{12}.$$

- Denote by Δ_n^k the union of k-faces of n-simplex
- Denote by S_g the connected sum of g copies of the Cartesian product $S^k \times S^k$ of two k-dimensional spheres

A higher-dimensional analogue of the Heawood inequality is the Kühnel conjecture. In a simplified form it states that *for every integer $k > 0$ there is $c_k > 0$ such that if Δ_n^k embeds into S_g, then*

$$g \geqslant c_k n^{k+1}.$$

We present the following quadratic estimate.

Main theorem for skeleton of a simplex

Theorem (Skeleton)

If Δ_n^k embeds into a $2k$-dimensional manifold M, then

$$\dim H_k(M;\mathbb{Z}_2) \gtrsim \frac{n^2}{2^k(k+1)^2}.$$

- You may assume that $M = S_g$, then $\dim H_k(M;\mathbb{Z}_2) = 2g$
- For $k = 1$ this follows from the Heawood inequality above
- Linear estimates were proved in [Goaoc, Mabillard, Paták, Patáková, Tancer, Wagner; Israel J. Math.; arXiv:1610.09063], [Paták, Tancer; arXiv:1904.02404]
- We tacitly work in the piecewise linear (PL) category

Netflix problem

'Matrix completion is the task of filling in the missing entries of a partially observed matrix... One example is the movie-ratings matrix, as appears in the Netflix problem (from machine learning): Given a ratings matrix in which each entry (i, j) represents the rating of movie j by customer i, if customer i has watched movie j and is otherwise missing, we would like to predict the remaining entries in order to make good recommendations to customers on what to watch next...'

The remaining entries are predicted so as to minimize the *rank* of the completed matrix. For a brief overview of the history of this and related problems, see [Wiki: Matrix completion: Low rank matrix completion], [Kogan, arXiv:2104.10668, Remark 4]

A version of Netflix problem

We consider matrices with entries in the set $\mathbb{Z}_2 = \{0, 1\}$ of all residues modulo 2 (with the sum and product operations).

There are algorithms estimating minimal rank for the particular case of unknown elements *on the diagonal* [Kogan, arXiv:2104.10668].
They are useful for 'weak realizability' of 'graphs with rotations' on non-orientable surfaces [Bikeev, arXiv:2012.12070]

We study a more general problem, in which instead of knowing specific matrix elements, we know linear relations on such elements. We estimate the minimal rank of matrices with such relations.
This is useful for embeddings of graphs to surfaces, and of k-dimensional simplicial complexes to $2k$-dimensional manifolds

Main theorem for joinpower

Let $[n]^{*k+1}$ be the k-complex with vertex set $[k+1] \times [n]$, in which every $k+1$ vertices from different lines span a k-simplex.
For $k = 1$ this is the complete bipartite graph $K_{n,n}$.

Theorem (Joinpower)

If $[n]^{*k+1}$ embeds into a $2k$-dimensional manifold M, then

$$\dim H_k(M; \mathbb{Z}_2) \gtrsim \frac{n^2}{2^k}.$$

For $k = 1$ this is covered by a result of Heawood

The estimate for Δ_n^k follows from the estimate for $[s]^{*k+1}$ for $n \geqslant (k+2)s - 2$ because then $\Delta_n^k \supset [s]^{*k+1}$

A general criterion

Our quadratic estimates (and earlier linear estimates) are non-trivial in spite of the existence of algebraic criterion for embeddability of k-complexes to $2k$-manifolds, due to Harris-Krushkal-Johnson-Paták-Tancer and [Kogan-Skopenkov, arXiv:2112.06636]

The structure of the proof

Our main achievement is to fit what we can prove in topology to what is sufficient for algebra. Thus our main idea is the notion of an (n, k)-**matrix.** Before we introduce it, we show how it works. The quadratic estimate for joinpower is implied by the following theorems. Thus the proof is split into two independent parts.

Theorem (Topological)

If $[n]^{*k+1}$ embeds into a $2k$-dimensional manifold M, then there is an (n, k)-matrix of rank at most $\dim H_k(M; \mathbb{Z}_2)$.

Theorem (Linear algebraic)

For $n \geqslant 4$ the rank of any (n, k)-matrix is at least $(n-3)^2/2^k$.

For $k = 1$ these results are implicit in [Fulek, Kynčl; arXiv:1903.08637]. Our proof is a higher-dimensional generalization of the case $k = 1$.

Cycles and rectangles

For $a \in [n]$ let a and a' be vertices of $K_{n,n}$ from different parts.
For any 4-cycle in $K_{n,n}$ there are 2-element subsets $P_1 = \{a, b\}$ and $P_2 = \{u, v\}$ of $[n]$ such that the cycle is

$$P = P(P_1, P_2) := au'bv'.$$

Idea of proof of the Topological Theorem ($k = 1$)

Let M be a 2-manifold, and $f : [n]^{*2} = K_{n,n} \to M$ a map.
For cycles $P, Q \subset K_{n,n}$ of length 4 denote

$$A(f)_{P,Q} := f(P) \cap_M f(Q) \in \mathbb{Z}_2.$$

Here $\cap_M$ the mod 2 algebraic intersection of closed curves; for a simple accessible to non-specialists definition see [Manifold Atlas: Intersection form]

The obtained square matrix $A(f)$ is symmetric and has $\binom{n}{2}^2$ rows

The matrix $A(f)$ is the Gram matrix (with respect to $\cap_M$) of some homology classes in $H_1(M; \mathbb{Z}_2)$. Hence $\operatorname{rk} A(f) \leqslant \dim H_1(M; \mathbb{Z}_2)$

Below we additionally assume that f is an embedding

Independence for $k = 1$

Let A be a symmetric matrix of $\binom{n}{2}^2$ rows (whose rows and whose columns correspond to all cycles of length 4 in $K_{n,n}$) with entries in $\mathbb{Z}_2$.

The matrix A is said to be **independent** if $A_{P,Q} = 0$ for any disjoint cycles P, Q of length 4 in $K_{n,n}$

It is obvious that $A(f)$ is independent

Additivity for $k = 1$

Denote by $\oplus$ the sum modulo 2 (i. e., symmetric difference)

The matrix A is said to be **additive** if $A_{P,Q} = A_{X,Q} + A_{Y,Q}$ for any cycles P, Q, X, Y of length 4 in $K_{n,n}$ such that $P = X \oplus Y$

The additivity property holds, e. g., for cycles $X = 1u'2v'$, $Y = 1u'2w'$ and $P = 1v'2w'$.

The matrix $A(f)$ is additive since $\cap_M$ distributes over $\oplus$

Non-triviality for $k = 1$

Two cycles of length 4 in $K_{3,3} \subset K_{n,n}$ are said to be **complementary** if they share a common edge $33'$ and their opposite edges are disjoint

The matrix A is said to be **non-trivial** if the sum of $A_{P,Q}$ over the two unordered pairs $\{P, Q\}$ of complementary cycles is equal to 1. The two pairs are $\{11'33', 22'33,\}$ and $\{12'33', 21'33'\}$

The non-triviality of $A(f)$ is known to be equivalent to *the van Kampen number* of $K_{3,3}$ being 1

$(n, 1)$-matrix

An $(n, 1)$-**matrix** is an independent additive non-trivial matrix.
So the Topological Theorem is implicitly known for $k = 1$.

Octahedra and parallelepipeds

An **octahedron** is a subcomplex of $[n]^{*3}$ isomorphic to $[2]^{*3} \cong S^2$.
Such a subcomplex is defined by the set $1 \times P_1 \sqcup 2 \times P_2 \sqcup 3 \times P_3$ of its vertices,
i. e., by a **parallelepiped** $P_1 \times P_2 \times P_3 \subset [n]^3$.
Then its faces are $\{(1, a_1), (2, a_2), (3, a_3)\}$ for $a_i \in P_i$.

Idea of proof of the Topological Theorem ($k=2$)

Let M be a 4-manifold, and $f\colon [n]^{*3} \to M$ a map. For octahedra P, Q in $[n]^{*3}$ denote

$$A(f)_{P,Q} := f(P) \cap_M f(Q) \in \mathbb{Z}_2.$$

The obtained square matrix $A(f)$ is symmetric and has $\binom{n}{2}^3$ rows

The matrix $A(f)$ is the Gram matrix (with respect to $\cap_M$) of some homology classes in $H_2(M;\mathbb{Z}_2)$. Hence $\operatorname{rk} A(f) \leqslant \dim H_2(M;\mathbb{Z}_2)$

Below we additionally assume that f is an embedding

Independence for $k = 2$

Let A be a symmetric matrix of size $\binom{n}{2}^3 \times \binom{n}{2}^3$ (whose rows and whose columns correspond to all octahedra in $[n]^{*3}$) with entries in $\mathbb{Z}_2$.

The matrix A is said to be **independent** if for any octahedra $P, Q \subset [n]^{*3}$

$$A_{P,Q} = 0 \text{ if } P \cap Q = \varnothing$$

It is obvious that $A(f)$ is independent

Additivity for $k = 2$

The matrix A is said to be **additive** if for any octahedra $P, Q, X, Y \subset [n]^3$ such that $P = X \oplus Y$

$$A_{P,Q} = A_{X,Q} + A_{Y,Q}$$

The matrix $A(f)$ is additive analogously to the case $k = 1$

The additivity holds, e. g., for octahedra
$X = [2]^{*2} * \{u, v\}$, $Y = [2]^{*2} * \{u, w\}$, and $P = [2]^{*2} * \{v, w\}$

Non-triviality for $k = 2$

Two octahedra in $[3]^{*3} \subset [n]^{*3}$ are said to be **complementary** if they share a common face 3^{*3} and their opposite faces are disjoint

The matrix A is said to be **non-trivial** if the sum of $A_{P,Q}$ over the four unordered pairs $\{P, Q\}$ of complementary octahedra is equal to 1

Example and $(n, 2)$-matrix

Denote $j^+ := \{j, 3\}$.
The four pairs are

$$\{1^+ * 1^+ * 1^+,\ 2^+ * 2^+ * 2^+\}$$
$$\{1^+ * 1^+ * 2^+,\ 2^+ * 2^+ * 1^+\}$$
$$\{1^+ * 2^+ * 1^+,\ 2^+ * 1^+ * 2^+\}$$
$$\{2^+ * 1^+ * 1^+,\ 1^+ * 2^+ * 2^+\}.$$

An $(n, 2)$-**matrix** is a symmetric independent additive non-trivial matrix.
So in order to prove the Topological Theorem for $k = 2$ it remains to prove the non-triviality of $A(f)$. This is one of the novel and non-trivial parts of the proof. This is the analogue of [Paták, Tancer] for $[n]^{*k+1}$ instead of Δ_n^k; the most novel part is purely combinatorial and is as follows.

Combinatorial proposition

The set of all ordered pairs of disjoint edges of $K_{3,3}$ (the *simplicial deleted product*) is the sum of products $P \times Q$ over all ordered pairs (P, Q) of complementary cycles of length 4 (this is possibly known)

The set of all ordered pairs of disjoint faces of $[3]^{*3}$ (the *simplicial deleted product*) is the sum of products $P \times Q$ over all ordered pairs (P, Q) of complementary octahedra

A 'converse' to the Topological Theorem

Let M be a closed $(k-1)$-connected $2k$-manifold.
A general position map $f\colon [n]^{*k+1} \to M$ is called a $\mathbb{Z}_2$-**embedding** if $|f\sigma \cap f\tau|$ is even for any non-adjacent k-faces σ, τ.
Let I_M be (the matrix of $\cap_M$ in some 'good' basis, i. e.)

the identity matrix if there is $x \in H_k(M;\mathbb{Z}_2)$ such that $x \cap_M x = 1$, and

the sum of $d(M)/2$ hyperbolic matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, otherwise, where

$d(M) := \dim H_k(M;\mathbb{Z}_2)$ is known to be even

Theorem

For $n \geqslant 4$ a $\mathbb{Z}_2$-embedding $[n]^{*k+1} \to M$ exists if and only if there is an $\binom{n}{2}^{k+1} \times d(M)$-matrix Y such that $Y^T I_M Y$ is an (n,k)-matrix.

The 'only if' part is proved here, the 'if' part is drafted by Styrt-S.

Definition of blocks

Let $A \in \mathbb{Z}_2^{\binom{n}{2}^3 \times \binom{n}{2}^3}$. For each 2-element subsets $U, V \subset [n]$ define the 2-*coordinate block*

$$A_{U,V} \in \mathbb{Z}_2^{\binom{n}{2}^2 \times \binom{n}{2}^2} \quad \text{by} \quad (A_{U,V})_{P,Q} = A_{U*P,\, V*Q}$$

for cycles $P, Q \subset [n]^2$.

Reduction of $(n,2)$ to $(n,1)$

Denote

$$B := A_{1^+,2^+} + A_{2^+,1^+}.$$

Then

$$\operatorname{rk} A \geqslant \frac{1}{2}\left(\operatorname{rk} A_{1^+,2^+} + \operatorname{rk} A_{2^+,1^+}\right) \geqslant \frac{1}{2}\operatorname{rk} B \geqslant \frac{(n-3)^2}{4}.$$

Here

- the first inequality follows since $\operatorname{rk} A \geqslant \operatorname{rk} A_{U,V}$ for any U, V
- the second inequality follows by subadditivity of rank
- the third inequality follows by the $(n,1)$ case applied to B, since below we check that B is an $(n,1)$-matrix

Independence, additivity and symmetry of B

The independence and the additivity hold for

$$B = A_{1^+,2^+} + A_{2^+,1^+}$$

since they hold for $A_{1^+,2^+}$ and $A_{2^+,1^+}$

Since A is symmetric, we have

$$A_{1^+*P,\,2^+*Q} = A_{2^+*Q,\,1^+*P}, \quad \text{i. e.,} \quad A_{2^+,1^+} = A^T_{1^+,2^+}.$$

Hence $B = A_{1^+,2^+} + A^T_{1^+,2^+}$ is symmetric

Non-triviality of B

For a matrix $X \in \mathbb{Z}_2^{\binom{n}{2}^{k+1} \times \binom{n}{2}^{k+1}}$ denote by SX the sum over 2^k unordered pairs of complementary rectangles/parallelepipeds.
The non-triviality holds for B since

$$SB = SA_{1^+,2^+} + SA_{2^+,1^+} =$$
$$= \left(A_{1^+,2^+}\right)_{1^+*1^+,\,2^+*2^+} + \left(A_{1^+,2^+}\right)_{1^+*2^+,\,2^+*1^+} +$$
$$+ \left(A_{2^+,1^+}\right)_{1^+*1^+,\,2^+*2^+} + \left(A_{2^+,1^+}\right)_{1^+*2^+,\,2^+*1^+} =$$
$$= A_{1^+*1^+*1^+,\,2^+*2^+*2^+} + A_{1^+*1^+*2^+,\,2^+*2^+*1^+} +$$
$$+ A_{2^+*1^+*1^+,\,1^+*2^+*2^+} + A_{2^+*1^+*2^+,\,1^+*2^+*1^+} = SA = 1,$$

where the last equality follows since A is non-trivial.

Proof of the Topological Theorem (beginning)

$$1 \stackrel{?}{=} \sum_{\{P,Q\}\in C} f(P) \cap_M f(Q) \stackrel{(LL)}{=} \sum_{\{P,Q\}\in C} \sum_{\{\sigma,\tau\}\in[P\times Q]} \left|g(\sigma) \cap g(\tau)\right|_2 = \dots$$

C is the set of unordered pairs of complementary octahedra

Localization Lemma: there is a general position map $g\colon [3]^{*3} \to \mathbb{R}^4$ such that (LL) holds

$[P \times Q]$ is the set of 2-element subsets $\{\sigma,\tau\} \subset [3]^{*3}$ such that either $\sigma \subset P$ and $\tau \subset Q$, or vice versa (this set is the image of $P \times Q$ under the projection to the $\mathbb{Z}_2$-quotient of $[3]^{*3} \times [3]^{*3}$)

Proof of the Topological Theorem (conclusion)

$$\dots = \sum_{\{P,Q\}\in C}\ \sum_{\{\sigma,\tau\}\in[P\times Q]} \left|g(\sigma)\cap g(\tau)\right|_2 \overset{(CP)}{=} \sum_{\{\sigma,\tau\}\in D} \left|g(\sigma)\cap g(\tau)\right|_2 \overset{(VK)}{=} 1.$$

D is the set of unordered pairs of disjoint faces of $[3]^{*3}$

equality (CP) is the Combinatorial Proposition for the sets over which the sum is

equality (VK) is a 1932 result of van Kampen

Proof of the Localization Lemma: choice of g

Recall that $I = [0,1] \subset \mathbb{R}$

Since $A(f)$ is independent of homotopy of f, we may assume that the image of f is in the interior of M

Recall that we shorten $\{3\}^{*3}$ to 3^{*3}

Since $3^{*3} \cong I^2$ is collapsible, there is an embedding $i\colon I^4 \to M$ in general position to f, and such that $iI^4 \supset f(3^{*3})$ and $iI^4 \cap f([2]^{*3}) = \varnothing$

Take any general position map $g\colon [3]^{*3} \to \mathbb{R}^4$ such that

$$f^{-1}(iI^4) = g^{-1}(I^4) =: Z \quad \text{and} \quad f|_Z = ig|_Z.$$

Here the property $f|_Z = ig|_Z$ means that $f(x) = ig(x)$ when $f(x) \in iI^4$ (or, equivalently, when $g(x) \in I^4$)

Proof of the Localization Lemma: choice of g'

It remains to prove that for any complementary octahedra P, Q

$$f(P) \cap_M f(Q) = \sum_{\{\sigma,\tau\} \in [P \times Q]} \left| g(\sigma) \cap g(\tau) \right|_2.$$

For the proof take any general position map $g' : [3]^{*3} \to \mathbb{R}^4$ such that

the restrictions of g and g' to $g^{-1}(\mathbb{R}^4 - I^4) = g'^{-1}(\mathbb{R}^4 - I^4)$ coincide, i. e., $g|_{g^{-1}(\mathbb{R}^4 - I^4)} = g'|_{g'^{-1}(\mathbb{R}^4 - I^4)}$,

g and g' are close and in general position inside $g^{-1}((0,1)^4) = g'^{-1}((0,1)^4)$

Proof of the Localization Lemma: beginning

Then we have

$$f(P) \cap_M f(Q) \overset{(1)}{=} \left|(ig(P) \cap iI^4) \cap (ig'(Q) \cap iI^4)\right|_2 \overset{(2)}{=}$$
$$\overset{(2)}{=} \left|(g(P) \cap I^4) \cap (g'(Q) \cap I^4)\right|_2 \overset{(3)}{=} \dots$$

equality (1) is proved as follows: $P \cap Q = 3^{*3}$, hence $f(P) \cap f(Q)$ is contained in iI^4; this, $f|_Z = ig|_Z$, and the property that g, g' are close and in general position over $(0,1)^4$, imply (1)

equality (2) follows since i is an embedding

Proof of the Localization Lemma: middle step

$$\dots \overset{(2)}{=} \left|(g(P)\cap I^4)\cap(g'(Q)\cap I^4)\right|_2 \overset{(3)}{=}$$

$$\overset{(3)}{=} \left|(g(P)-I^4)\cap(g'(Q)-I^4)\right|_2 \overset{(4)}{=}$$

$$\overset{(4)}{=} \left|(g(P)-I^4)\cap(g(Q)-I^4)\right|_2 \overset{(5)}{=} \dots$$

equality (3) follows since $|g(P)\cap g'(Q)|_2 = 0$ as the intersection of two general position 2-cycles in $\mathbb{R}^4$; these 2-cycles are in general position since

- g and g' are close and in general position over $(0,1)^4$
- $g|_{g^{-1}(\mathbb{R}^4-I^4)} = g'|_{g'^{-1}(\mathbb{R}^4-I^4)}$
- g is a general position map
- P, Q are complementary

equality (4) follows since $g|_{g^{-1}(\mathbb{R}^4-I^4)} = g'|_{g'^{-1}(\mathbb{R}^4-I^4)}$

Proof of the Localization Lemma: conclusion

$$\dots \overset{(4)}{=} \left|(g(P) - I^4) \cap (g(Q) - I^4)\right|_2 \overset{(5)}{=} \sum_{\{\sigma,\tau\}\in[P\times Q]} \left|g(\sigma) \cap g(\tau)\right|_2.$$

Equality (5) is proved as follows. Since $f^{-1}(iI^4) = g^{-1}(I^4)$, $f(3^{*3}) \subset iI^4$, and for any face $\delta \subset [2]^{*3} \subset [3]^{*3}$ (since it is disjoint with $3 * 3 * 3$) we have $f(\delta) \subset M - iI^4$, so

$$g(\delta) \cap g(3^{*3}) \subset (\mathbb{R}^4 - I^4) \cap I^4 = \varnothing.$$

Hence

$$(g(P) - I^4) \cap (g(Q) - I^4) = \bigsqcup_{\{\sigma,\tau\}\in[P\times Q]} g(\sigma) \cap g(\tau).$$

Proof of the Combinatorial Proposition

Denote by $\widetilde{C}$ the set of ordered pairs of complementary octahedra

Denote by $\widetilde{D}$ the set of ordered pairs of disjoint faces

The equality $\bigoplus\limits_{(P,Q)\in\widetilde{C}} P \times Q = \widetilde{D}$ follows from the two parts:

('Disjoint part')

For disjoint $\alpha, \beta \in [3]^3$ there is exactly one pair $(P, Q) \in \widetilde{C}$ such that $(\alpha, \beta) \in P \times Q$.

('Non-disjoint part')

For non-disjoint $\alpha, \beta \in [3]^3$ there is an even number of pairs $(P, Q) \in \widetilde{C}$ such that $(\alpha, \beta) \in P \times Q$.

Another description of complementary octahedra

For a face $\sigma \in [2]^3$ denote by

$$\Diamond\sigma := \sigma_1^+ * \sigma_2^+ * \sigma_3^+ = \{\sigma_1, 3\} * \{\sigma_2, 3\} * \{\sigma_3, 3\}$$

an octahedron in $[3]^{*3}$, whose opposite faces are ω and 3^{*3}

Obviously, if $\sigma, \tau \in [2]^{*3}$ are disjoint, then $\Diamond\sigma$ and $\Diamond\tau$ are complementary

Moreover, if P, Q are complementary then there is unique pair (σ, τ) of disjoint faces from $[2]^{*3}$ such that $\Diamond\sigma = P$ and $\Diamond\tau = Q$

Proof of the 'disjoint' part

Take any complementary octahedra P and Q such that $\alpha \in P$ and $\beta \in Q$. Take σ and τ such that $\Diamond\sigma = P$ and $\Diamond\tau = Q$. Take any $i \in [3]$

Suppose that $\alpha_i \neq 3$. Since $\alpha \subset \Diamond\sigma$ it follows that $\sigma_i = \alpha_i$. Since σ and τ are disjoint, $\tau_i = 3 - \sigma_i = 3 - \alpha_i$

Suppose that $\alpha_i = 3$. Then $\beta_i \neq 3$. Hence $\tau_i = \beta_i$ and $\sigma_i = 3 - \beta_i$ analogously to the previous point

Hence the i-th coordinates σ_i and τ_i are uniquely defined for each $i \in [3]$. Thus an ordered pair (σ, τ) of disjoint faces from $[2]^{*3}$ such that $(\alpha, \beta) \in \Diamond\sigma \times \Diamond\tau$ exists and is unique. Then an ordered pair (P, Q) of complementary octahedra such that $(\alpha, \beta) \in P \times Q$ exists and is unique

Proof of the 'non-disjoint' part: beginning

Since the faces α and β are not disjoint, we may assume that $\alpha_1 = \beta_1$ (the other cases are analogous)

Suppose that $\alpha_1 \neq 3$

Take any complementary octahedra P, Q

Take disjoint faces σ and τ such that $\Diamond\sigma = P$ and $\Diamond\tau = Q$

Since σ, τ are disjoint, we have $\sigma_1 \neq \tau_1$. Hence either $\sigma_1 \neq \alpha_1$ or $\tau_1 \neq \beta_1$. Then either $\alpha \not\subset \Diamond\sigma$ or $\beta \not\subset \Diamond\tau$. Then $(\alpha, \beta) \notin P \times Q$

Proof of the 'non-disjoint' part: conclusion

Suppose that $\alpha_1 = 3$

For every face $\delta = \delta_1 * \delta_2 * \delta_3 \subset [2]^{*3}$ denote $\delta' := (3 - \delta_1) * \delta_2 * \delta_3$. Clearly, $\delta'' = \delta$, and if two faces σ, τ are disjoint, then σ', τ' are also disjoint

Thus the summands in the sum from the statement split into couples corresponding to 'opposite' pairs $(\Diamond\sigma, \Diamond\tau)$ and $(\Diamond\sigma', \Diamond\tau')$. Since $\alpha_1 = 3$, the face α is contained either in both $\Diamond\sigma$ and $\Diamond\sigma'$, or in none. Analogously for β

Then for every couple $\{(\Diamond\sigma, \Diamond\tau), (\Diamond\sigma', \Diamond\tau')\}$ the pair (α, β) is contained either in both $\Diamond\sigma \times \Diamond\tau$ and $\Diamond\sigma' \times \Diamond\tau'$, or in none. This implies the result

Idea of the estimation for $(n, 1)$-matrices

The main idea of Fulek and Kynčl is to construct a 'good' matrix D (with a quadratic estimate on rank) from the given $(n, 1)$-matrix A by making row operations, adding low rank matrices, and taking submatrices.

Lemma on independence and additivity

Recall that a square matrix Y with $\mathbb{Z}_2$-entries is said to be *tournament* if $Y_{a,b} + Y_{b,a} = 1$ for all $a \neq b$

Recall that

$$[m_1] \sqcup [m_2] \sqcup \ldots \sqcup [m_\ell] = [m_1] \times \{1\} \sqcup [m_2] \times \{2\} \sqcup \ldots \sqcup [m_\ell] \times \{\ell\},$$

so that the disjoint union of ℓ copies of $[m]$ is $[m] \times [\ell]$

Lemma

Suppose that $n \geqslant 4$ and $A \in \mathbb{Z}_2^{s_{n,1} \times s_{n,1}}$ is independent and additive. Define the matrix

$$B \in \mathbb{Z}_2^{[n-1]^2 \times [n-1]^2} \quad \text{by} \quad B_{(a,i)(b,j)} := A_{\{a,n\}\times\{i,n\},\{b,n\}\times\{j,n\}}.$$

Then for any pairwise distinct $i, j, s \in [n-1]$ the residue $B_{(a,i)(b,j)} + B_{(a,s)(b,j)}$ is independent of distinct $a, b \in [n-1]$.

Proof of the lemma

Denote $P := \{a, n\} \times \{i, s\}$ and $Q(b) := \{b, n\} \times \{j, n\}$. By the additivity,

$$B_{(a,i)(b,j)} + B_{(a,s)(b,j)} = A_{\{a,n\}\times\{i,n\},\, Q(b)} + A_{\{a,n\}\times\{s,n\},Q(b)} = A_{P,Q(b)}.$$

The independence of b follows since for any $b' \in [n-1]$ distinct from a and b, by the additivity and the independence

$$A_{P,Q(b)} + A_{P,Q(b')} = A_{P,\,\{b,b'\}\times\{n,j\}} = 0.$$

The independence of a follows analogously.
Now the independence of both a and b follows since $n - 1 \geqslant 3$.

Useful notation

Take a matrix $X \in \mathbb{Z}_2^{([m_1]\sqcup\ldots\sqcup[m_\ell])^2}$

For $i, j \in [\ell]$ define the *block* $X_{i,j} \in \mathbb{Z}_2^{m_i \times m_j}$ by $(X_{i,j})_{a,b} := X_{(a,i)(b,j)}$

The matrix X is *tournament-like* if for every $i \in [\ell]$ the diagonal block $X_{i,i}$ is a tournament matrix

The matrix X *diagonal-like* if it is obtained by removing some rows and columns symmetric to the rows, from a matrix $\widetilde{X} \in \mathbb{Z}_2^{([m]\times[\ell])^2}$ with the following properties:

- $m \geqslant m_i$ for every $i \in [\ell]$,
- the under-diagonal block $\widetilde{X}_{i,j}$ is a diagonal matrix for every $i, j \in [\ell]$, $i > j$

A 'good' matrix to which the obtained matrix B is reduced

Lemma (Tournament Matrix)

The rank of a tournament matrix of size $m \times m$ is at least $\dfrac{m-1}{2}$.

This lemma is pretty simple.

Lemma (Good Matrix)

The rank of any tournament-like diagonal-like matrix in $\mathbb{Z}_2^{([m_1]\sqcup\ldots\sqcup[m_\ell])^2}$ is at least $\sum\limits_{i=1}^{\ell} \dfrac{m_i-1}{2}$.

Beginning of the proof of the Good Matrix Lemma

The simple case when there are no units in the under-diagonal blocks follows by Tournament Matrix Lemma because all under-diagonal blocks are zero matrices

The proof is by induction on $m_1 + \ldots + m_\ell$

The base follows by the above simple case

Let us prove the inductive step in the case when there is a unit in the union of under-diagonal blocks

Inductive step: beginning

Denote by D the given matrix

Arrange the rows and the columns of D lexicographically

Let (a, i) be the lexicographically maximal (i. e., the lowest) row whose intersection with the union of under-diagonal blocks of D is non-zero

Let (b, j) be the lexicographically minimal (i. e., the leftmost) column whose intersection with the row (a, i) is non-zero

Inductive step: construction of a 'good' matrix

Let D' be the matrix obtained from D by adding the row (a, i) to all rows whose intersection with the column (b, j) is non-zero

Let D'' be the matrix obtained from D' by adding the column (b, j) to all columns whose intersection with the row (a, i) is non-zero

Let D''' be the matrix obtained from D'' by removing the rows (a, i), (b, j), and the columns (a, i), (b, j)

Then $D''' \in \mathbb{Z}_2^{([n_1]\sqcup\ldots\sqcup[n_\ell])^2}$ for $n_i = m_i - 1$, $n_j = m_j - 1$, and $n_s = m_s$ for $s \neq i, j$, and D''' is tournament-like and diagonal-like

Then by induction hypothesis

$$\operatorname{rk} D \geqslant \operatorname{rk} D''' + 1 \geqslant \sum_{s=1}^{\ell} \frac{n_s - 1}{2} + 1 = \sum_{s=1}^{\ell} \frac{m_s - 1}{2}$$