Introduction to unbounded observable algebras

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M. Tomita defined the concepts of observables and observable algebras and studied them in order to bridge quantum mechanics and mathematics. Tomita's idea for this question is to consider the trio (A, x, y^*) consisting of an observable A, of two states x, y as an observable, which is called a trio observable, and to introduce an algebraic and topological structure in the set of trio observables. This means that an operator observable A having two different states can be regarded as two different observables. A. Inoue has introduced and developed this theory in [2]. The operator part of a Tomita's observable is always a bounded linear operator on $\mathcal{H}$, however an operator observable in quantum mechanics is unbounded, and the GNS-representation of a positive linear functional on a $*$-algebra is unbounded. This is our motivation for defining and studying unbounded observable algebras, which are an unbounded generalization of Tomita's observable algebras in [3]. Here we state this roughly. Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$ and write $\mathcal{D}^* = \{\xi^* \in \mathcal{H}^*;\ \xi \in \mathcal{D}\}$, where ξ^* is an element of the dual space $\mathcal{H}^*$ of $\mathcal{H}$ defined by $< \xi^*, x >:= (x|\xi)$ for all $x \in \mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D})$ the set of all linear operators X from $\mathcal{D}$ to $\mathcal{D}$ satisfying $D(X^*) \supset \mathcal{D}$ and $X^*\mathcal{D} \subset \mathcal{D}$, where X^* is the adjoint of X and $D(X^*)$ is the domain of X^*. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra consisting of closable operators in $\mathcal{H}$ equipped with the usual operations ($X+Y, \alpha X$ and XY) and the involution $X \mapsto X^\dagger := X^*\lceil_{\mathcal{D}}$ (the restriction of X^* to $\mathcal{D}$). A quadruplet $A = (A_0, \xi, \eta^*, \mu)$ of $A_0 \in \mathcal{L}^\dagger(\mathcal{D})$, $\xi, \eta \in \mathcal{D}$ and $\mu \in \mathbb{C}$ is called an (unbounded) quadruplet observable on $\mathcal{D}$ and denoted by $Q^\dagger(\mathcal{D})$ of all quadruplet observables on $\mathcal{D}$. Refering observables, states and expectations in the standard Hilbert space formulation of the quantum mechanics, we define the algebraic operations and involution $\dagger$ on $Q^\dagger(\mathcal{D})$ as follows:

$$\begin{aligned} A+B &= (A_0+B_0, \xi+\zeta, \eta^*+\chi^*, \gamma+\sigma), \\ \alpha A &= (\alpha A_0, \alpha\xi, \alpha\eta^*, \alpha\gamma), \\ AB &= (A_0B_0, A_0\zeta, (B_0^\dagger\eta)^*, (\zeta|\eta)), \\ A^\dagger &= (A_0^\dagger, \eta, \xi^*, \bar{\gamma}) \end{aligned}$$

for $A = (A_0, \xi, \eta^*, \gamma)$, $B = (B_0, \zeta, \chi^*, \sigma) \in Q^\dagger(\mathcal{D})$ and $\alpha \in \mathbb{C}$. A $*$-subalgebra of the $*$-algebra $Q^\dagger(\mathcal{D})$ is said to be a $Q^\dagger$-algebra on $\mathcal{D}$. Investigating $Q^\dagger$-algebras, we may deal with various physical phenomenons. For an element $A = (A_0, \xi, \eta^*, \gamma)$ of a $Q^\dagger$-algebra $\mathfrak{A}$ on $\mathcal{D}$ we write $\pi(A) = A_0$, $\lambda(A) = \xi$, $\lambda^*(A) = \eta^*$ and $\mu(A) = \gamma$. Then π is a (possibly unbounded) $*$-representation of $\mathfrak{A}$ on $\mathcal{D}$ (namely, a $*$-homomorphism of $\mathfrak{A}$ into $\mathcal{L}^\dagger(\mathcal{D})$), λ is a vector representation of $\mathfrak{A}$ into $\mathcal{D}$ (namely, a linear mapping of $\mathfrak{A}$ into $\mathcal{D}$ satisfying $\lambda(AB) = \pi(A)\lambda(B)$ for all $A, B \in \mathfrak{A}$), λ^* is a vector representation of $\mathfrak{A}$ into $\mathcal{D}^*$ and μ is a positive linear functional on $\mathfrak{A}$. A trio $(\pi(A), \lambda(A), \lambda^*(A))$ obtained by cutting a quadruplet observable A on $\mathcal{D}$ is called a trio observable on $\mathcal{D}$, and the set $T^\dagger(\mathcal{D})$ of all trio observables on $\mathcal{D}$ is also a $*$-algebra without identity under operations and the involution $\dagger$ as those in the case of $Q^\dagger(\mathcal{D})$. A $*$-subalgebra of $T^\dagger(\mathcal{D})$ is called a $T^\dagger$-algebra on $\mathcal{D}$. When $\mathcal{D} = \mathcal{H}$, a quadruplet (resp. trio) observable A is a Tomita's quadruplet (resp. trio) observable, namely, $\pi(A)$ is a bounded linear operator on $\mathcal{H}$, $\lambda(A) \in \mathcal{H}$ and $\lambda^*(A) \in \mathcal{H}^*$. The set $Q^*(\mathcal{H})$ (resp. $T^*(\mathcal{H})$) of all Tomita's quadruplet (resp. trio) observables on $\mathcal{H}$ is a Banach $*$-algebra without identity equipped with the above operations $A + B, \alpha A, AB$, the involution $A^\sharp := (\pi(A)^*, \lambda^*(A)^*, \lambda(A)^*, \overline{\mu(A)})$ (resp. $A^\sharp = (\pi(A)^*, \lambda^*(A)^*, \lambda(A)^*)$) and the norm $\|A\| := \max(\|\pi(A)\|, \|\lambda(A)\|, \|\lambda^*(A)\|, |\mu(A)|)$ (resp. $\|A\| := \max(\|\pi(A)\|, \|\lambda(A)\|, \|\lambda^*(A)\|)$).

A $*$-subalgebra of $Q^*(\mathcal{H})$ (resp. $T^*(\mathcal{H})$) is called a Q^*-algebra (resp. a T^*-algebra) on $\mathcal{H}$, and a closed $*$-subalgebra of the Banach $*$-algebra $Q^*(\mathcal{H})$ (resp. $T^*(\mathcal{H})$) is called a CQ^*-algebra (resp. CT^*-algebra) on $\mathcal{H}$. Here we remark that in [2] the involution on $Q^*(\mathcal{H})$ and $T^*(\mathcal{H})$ is denoted by $A \to A^\sharp$ as we did, however we denote the involution on $Q^\dagger(\mathcal{D})$ and $T^\dagger(\mathcal{D})$ by $A \to A^\dagger$.

In this talk, we tried to build the basic theory of unbounded Tomita's observable algebras called $T^\dagger$-algebras which are related to unbounded operator algebras, especially unbounded Tomita-Takesaki theory [1], operator algebras on Krein spaces [5], studies of positive linear functionals on $*$-algebras and so on. And we defined the notions of regularity, semisimplicity and singularity of $T^\dagger$-algebras and characterized them. We shall proceed further studies of $T^\dagger$-algebras and investigate whether a $T^\dagger$-algebra is decomposable into a regular part and a singular part [4].

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[2] A. Inoue: *Tomita's Lectures on Observable Algebras in Hilbert Space*, LNM 2285, Springer-Verlag, 2021.

[3] A. Inoue and H. Inoue: *An Unbounded Generalization of Tomita's Observable Algebras.* Rep. Math. Phys., **89** (2022), 153–184.

[4] H. Inoue: *An Unbounded Generalization of Tomita's Observable Algebras II.* Rep. Math. Phys., **91** (2023), 251–276.

[5] Y. Nakagami and M. Tomita, *Triangular matrix representation for self-adjoint operators in Krein spaces.* Japanese J. Math. **14** (1988), 165–202.