Andrews Skolemization May Shorten Resolution Proofs Non-Elementarily

Matthias Baaz[1] and Anela Lolić[2]

[1]Institute of Discrete Mathematics and Geometry, TU Wien, Austria
[2]Kurt Gödel Society, Institute of Logic and Computation, TU Wien, Austria

1 Introduction

The most prominent method of classical automated theorem proving in first-order logic is resolution, which is based on the negation of the target formula, elimination of strong quantifiers by the introduction of new Skolem function symbols, the deletion of weak quantifiers and the transformation of the remaining formula in the clause form. The most prominent method of Skolemization is structural Skolemization, where the positive existential (negative universal) quantifiers are replaced by Skolem functions depending on all weak quantifiers in whose scope the replaced quantifier occurs.

In this work structural Skolemization is compared to Andrews Skolemization [2, 3], where the positive existential quantifier is replaced by a Skolem function depending only on those weak quantifiers of the scope which bind in the sub-sequent formula. This sometimes reduces the dependencies of Skolem functions.

This contribution is based on [4], where a sequence of formulas $F_1, F_2, \dots$ with resolution proofs $\pi_1, \pi_2, \dots$ of these formulas after Andrews Skolemization is constructed s.t. there is no elementary bound in the complexity of $\pi_1, \pi_2, \dots$ of resolution proofs $\pi'_1, \pi'_2, \dots$ after structural Skolemization. The proofs are based on the elementary relation of resolution derivations with Andrews Skolemization to cut-free $\mathbf{LK}^+$-derivations and of resolution derivations with structural Skolemization to cut-free $\mathbf{LK}$-derivations. Therefore, this paper develops an application of the concept of globally sound but possibly locally unsound calculi to automated theorem proving.

2 Globally Sound Proofs and Andrews Skolemization

This work is based on the sequent calculi $\mathbf{LK}^+$ and $\mathbf{LK}^{++}$ introduced in [1]. They are obtained from $\mathbf{LK}$ by weakening the eigenvariable conditions. The resulting calculi are therefore globally sound but possibly locally unsound.

In Andrews' Skolemization method the introduced Skolem functions do not depend on the weak quantifiers $(Q_1x_1)\dots(Q_nx_n)$ dominating the strong quantifier (Qx), but on the subset of $\{x_i, \dots, x_j\}$ appearing (free) in the subformula dominated by (Qx). This method leads to smaller Skolem terms in general.

It can be shown that, like $\mathbf{LK}$-proofs can be Skolemized by substitution w.r.t. structural Skolemization, also $\mathbf{LK}^+$-proofs can be Skolemized w.r.t. Andrews Skolemization conserving the length and without introducing additional cuts.

Example 1. *Consider the following* $\mathbf{LK}^+$*-proof, which is not an* $\mathbf{LK}$*-proof:*

$$\dfrac{\dfrac{\dfrac{\dfrac{P(h(b),a) \vdash P(h(b),a)}{P(h(b),a) \vdash \forall y P(h(b),y)}}{P(h(b),a) \vdash \exists x \forall y P(x,y)}}{\forall y P(h(b),y) \vdash \exists x \forall y P(x,y)}}{\exists x \forall y P(h(x),y) \vdash \exists x \forall y P(x,y)}$$

with $a < b$*. Its Andrews Skolemization is*

$$\dfrac{\dfrac{P(h(c),f(h(c))) \vdash P(h(c),f(h(c)))}{P(h(c),f(h(c))) \vdash \exists x P(x,f(x))}}{\forall y P(h(c),y) \vdash \exists x P(x,f(x))}$$

Moreover, we show that there is an elementary transformation of an Andrews Skolemized cut-free $\mathbf{LK}$-proof into a cut-free $\mathbf{LK}^{++}$-proof with the original end-sequent depending on the complexity of the proof and the complexity of the original end-sequent. It follows that there is no elementary transformation of a cut-free $\mathbf{LK}^+$-proof in its structural Skolemized version. Of course there is an elementary transformation of a cut-free $\mathbf{LK}^+$-proof in its structured Skolemized version by adding cuts.

Example 2. *Consider the following* $\mathbf{LK}^+$*-derivation* φ*:*

$$\dfrac{\dfrac{\dfrac{\dfrac{A(g(a)) \vdash A(g(a)) \qquad A(g(a)) \vdash A(g(a))}{A(g(a)) \vee A(g(a)) \vdash A(g(a)), A(g(a))}}{\forall x(A(x) \vee A(x)) \vdash A(g(a)), \forall x A(g(x))}}{\forall x(A(x) \vee A(x)) \vdash A(g(a)) \vee \forall x A(g(x))}}{\forall x(A(x) \vee A(x)) \vdash \exists y(A(y) \vee \forall x A(g(x)))}$$

φ *has no structural Skolemization, as it would be necessary to substitute* $f(g(a))$ *for* a*. In contrast,* φ *can be Skolemized using Andrews Skolemization. The corresponding end-sequent is* $\forall x(A(x) \vee A(x)) \vdash \exists y(A(y) \vee A(g(c)))$*, and the Andrews Skolemized proof is obtained by substituting the Skolem constant* c *for* $g(a)$*:*

$$\dfrac{\dfrac{\dfrac{\dfrac{A(g(c)) \vdash A(g(c)) \qquad A(g(c)) \vdash A(g(c))}{A(g(c)) \vee A(g(c)) \vdash A(g(c)), A(g(c))}}{\forall x(A(x) \vee A(x)) \vdash A(g(c)), A(g(c))}}{\forall x(A(x) \vee A(x)) \vdash A(g(c)) \vee A(g(c))}}{\forall x(A(x) \vee A(x)) \vdash \exists y(A(y) \vee A(g(c)))}$$

Under assumption of any elementary clause form transformation we show that Andrews Skolemization might lead to a non-elementary speed-up compared to structural Skolemization.

Theorem 1. *There is a sequence of refutable formulas* $F_1, F_2, \ldots$ *with corresponding resolution refutations* $\pi_1, \pi_2, \ldots$ *with clause forms based on Andrews Skolemization s.t. no elementary bound in the complexity of* $\pi_1, \pi_2, \ldots$ *exists for the shortest sequence of corresponding resolution refutations based on structural Skolemization.*

References

[1] Juan P. Aguilera and Matthias Baaz. Unsound inferences make proofs shorter. *J. Symb. Log.*, 84(1):102–122, 2019.

[2] Peter B. Andrews. Resolution in type theory. *J. Symb. Log.*, 36(3):414–432, 1971.

[3] Peter B. Andrews. Theorem proving via general matings. *J. ACM*, 28(2):193–214, 1981.

[4] Matthias Baaz and Anela Lolic. Andrews Skolemization may shorten resolution proofs non-elementarily. In *LFCS*, volume 13137 of *Lecture Notes in Computer Science*, pages 9–24. Springer, 2022.