# Geometric approach to stable homotopy groups of spheres II; Arf-Kervaire Invariants

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**Аннотация**

The Kervaire Invariant 1 Problem until recently was an open problem in algebraic topology. Hill, Hopkins, and Ravenel solved this problem for all dimensions except $n = 126$, [H-H-R]. In dimension $n = 126$, the problem has not been solved and has the status of a hypothesis by V.P. Snaith (2009). We consider an alternative geometric to the Kervaire Invariant 1 Problem.

A notion of Abelian (for skew-framed immersions), bi-cyclic (for $\mathbb{Z}/2^{[3]}$-framed immersions) and quaternion-cyclic structure (for $\mathbb{Z}/2^{[4]}$-framed immersions) are introduced.

## 1 Self-intersections of generic immersions and the Kervaire invariant; the problem statement

The Kervaire Invariant 1 Problem until recently was an open problem in algebraic topology. In dimension $n = 126$, the problem has not been solved and has the status of a hypothesis by V.P. Snaith (2009) [S]. A geometrical approach toward the solution is proposed. The author assumes that the Arf-Kervaire invariant one Problem in all dimensions, including the dimension 126, will be solved, because additional assumptions concerning compression (see Main Result in the Introduction) could be omitted. Our approach develops an approach by L.S.Pontryagin [P], and by V.A.Rokhlin [Ro].

Let us consider a smooth generic immersion $f : M^{n-1} \looparrowright \mathbb{R}^n$, $n = 2^\ell - 2$, $\ell > 1$ of the codimension 1. Denote by $g : N^{n-2} \looparrowright \mathbb{R}^n$ the immersion of self-intersection manifold of $f$.

Let us recall a definition of the cobordism group $Imm^{sf}(n-k, k)$, a particular case $k = 1$ is better known: $Imm^{sf}(n-1, 1)$. The cobordism group is defined as equivalent classes of triples up to the standard cobordism relation, equipped with a disjoint union operation;

- $f : M^{n-k} \looparrowright \mathbb{R}^n$ is a codimension $k$ immersion;

• $\Xi : \nu(f) \cong k\varkappa_M^*(\gamma)$ is a bundle map, which is invertible, an invertible bundle map is a fibrewized isomorphism;

• $\varkappa_M \in H^1(M^{n-k};\mathbb{Z}/2)$ is a prescribed cohomology class, which is a mapping $M^{n-k} \to \mathbb{RP}^\infty = K(\mathbb{Z}/2,1)$.

By $\nu(f)$ is denoted the normal bundle of the immersion $f$, by $\varkappa_M^*(\gamma)$ is denoted the pull-back of the universal line bundle $\gamma$ over $\mathbb{RP}^\infty$ by the mapping $\varkappa_M$, by $k\varkappa_M^*(\gamma)$ is denoted the Whitney sum of the $k$ copies of the line bundles, below for short we will write $k\varkappa_M$. The isomorphism $\Xi$ is called a skew-framing of the immersion $f$.

The Kervaire invariant is an invariant of cobordism classes, which is homomorphism

$$\Theta^{sf} : Imm^{sf}(n-k,k) \to \mathbb{Z}/2. \qquad (1)$$

Let us recall the homomorphism (1) in the case $k=1$.

The normal bundle $\nu_g$ of the immersion $g : N^{n-2} \looparrowright \mathbb{R}^n$ is a 2-dimensional bundle over $N^{n-2}$, which is equipped by a $\mathbf{D}$–framing $\Xi$, where $\mathbf{D}$ is the dihedral group of the order 8. The classifying map of this bundle (and also the corresponding characteristic class) is denoted by $\eta_N : N^{n-2} \to K(\mathbf{D},1)$. Triples $(g,\eta_N,\Xi)$ up to a cobordism relation represents an element in the group $Imm^{\mathbf{D}}(n-2,2)$. The correspondence $(f,\varkappa,\Psi) \mapsto (g,\eta_N,\Xi)$ defines a homomorphism

$$\delta^{\mathbf{D}} : Imm^{sf}(n-1,1) \to Imm^{\mathbf{D}}(n-2,2). \qquad (2)$$

**Definition 1.** The Kervaire invariant of an immersion $f$ is defined by the formula:

$$\Theta_{sf}(f) = \langle \eta_N^{\frac{n-2}{2}}; [N^{n-2}] \rangle. \qquad (3)$$

It is not difficult to prove that the formula (3) determines a homomorphism

$$\Theta^{\mathbf{D}} : Imm^{\mathbf{D}}(n-2,2) \to \mathbb{Z}/2. \qquad (4)$$

The homomorphism (4) is called the Kervaire invariant of a $\mathbf{D}$–framed immersion. The composition of the homomorphisms (2), (4) determines the homomorphism (1).

## Main Result

Assume an element $x \in Imm^{sf}(n-1,1)$ admits a compression of the order 7 (see Definition 4). Then in the case $n = 2^\ell - 2$, $\ell \geq 7$, the homomorphism $\Theta^{sf}$ (3) on $x$ is trivial: $\Theta^{sf}(x) = 0$.

**Remark 2.** In the case $\ell = 7$ the statement of Main Result + a compression assumption imply the positive solution of the Snaith Conjecture [S]. The Snaith Conjecture is an open conjecture, because if an arbitrary element $x \in Imm^{sf}(125,1)$ admits a compression of the order 7 is unknown. A result, which is closed to imply a positive solution of the Conjecture, is proved [R-L]. The same time, a negative solution of Snath Conjecture is announced in [M2].

Main Result with an assumption $\ell \geq 8$ is a corollary of [H-H-R]. If an arbitrary element $x$, $\ell \geq 8$, admits a compression of the order 7 is unknown. Main Result and the following Compression Theorem gives an alternative proof of Hill-Hopkins-Ravenel theorem for all $n \geq 256$, probably, except a finite number of exceptional cases.

**A short proof of Main Result**

Main Result is proved in Theorem 30, where the Definition (17) of standardized $\mathbb{Z}/2^{[4]}$-framed immersion is used. The standardized immersion is a result of a Smale-Hirsh control principle [Hi] for iterated self-intersection points of a standardized $\mathbf{D}$-framed immersion (Subsection 2.2). The Arf-invariant for standardized $\mathbb{Z}/2^{[4]}$-framed immersion is determined by a homology class, described in Definition 18. $\square$

**Compression Theorem [Akh]**

For an arbitrary positive integer $d$ there exists $\ell = \ell(d)$, such that an arbitrary element in $Imm^{sf}(n-1,1)$ admits a compression of the order $d$.

Let us explain steps in the proof Main Theorem. Definition of the cobordism group $Imm^{\mathbf{D}}(n-2,2)$ of immersions with a dihedral framing is standard and analogous to the above definition of the skew-framed cobordism group $Imm^{sf}(n-k,k)$ for $k=1$. This group also admits a standard generalization as the cobordism group $Imm^{\mathbf{D}}(n-2k,2k)$ with a parameter $k \geq 1$. Each element in $Imm^{\mathbf{D}}(n-2k,2k)$ is represented by a triple $(g,\Psi,\eta_N)$, where $g : N^{n-2k} \looparrowright \mathbb{R}^n$ is an immersion, $\Psi$– is a dihedral framing in the codimension $2k$, i.e. an isomorphism $\Xi : \nu_g \cong k\eta_N^*(\tau)$, where $\eta_N : N \to K(\mathbf{D},1)$ is the classifying mapping, the universal 2-dimensional $\mathbf{D}$-bundle over $K(\mathbf{D},1)$ is denoted by $\tau$. The mapping $\eta_N$ determines a dihedral framing of the normal bundle $\nu_g$ by $\Xi$.

The Kervaire homomorphism is defined in a more general case using the following generalizations of the homomorphisms (1) and (4):

$$\Theta_k^{sf} : Imm^{sf}(n-k,k) \to \mathbb{Z}/2, \qquad \Theta_k^{sf} := \Theta_k^{\mathbf{D}} \circ \delta_k^{\mathbf{D}}. \tag{5}$$

$$\Theta_k^{\mathbf{D}} : Imm^{\mathbf{D}}(n-2k,2k) \to \mathbb{Z}/2, \qquad \Theta_k^{\mathbf{D}}[(g,\Psi,\eta_N)] = \langle \eta_N^{\frac{n-2k}{2}}; [N^{n-2k}] \rangle. \tag{6}$$

In the case $k=1$ the homomorphism (6) coincides with the homomorphism (4) above, the following commutative diagram, in which the homomorphisms $J^{sf}$, $J^{\mathbf{D}}$ are known, is well-defined.

$$\begin{array}{ccccc} Imm^{sf}(n-1,1) & \stackrel{\delta^{\mathbf{D}}}{\longrightarrow} & Imm^{\mathbf{D}}(n-2,2) & \stackrel{\Theta^{\mathbf{D}}}{\longrightarrow} & \mathbb{Z}/2 \\ \downarrow J_k^{sf} & & \downarrow J_k^{\mathbf{D}} & & || \\ Imm^{sf}(n-k,k) & \stackrel{\delta_k^{\mathbf{D}}}{\longrightarrow} & Imm^{\mathbf{D}}(n-2k,2k) & \stackrel{\Theta_k^{\mathbf{D}}}{\longrightarrow} & \mathbb{Z}/2. \end{array} \qquad (7)$$

Let us remind, that an idea to investigate intersection points as in Theorem 22 was communicated to me by A.V.Chernavskii (1996). S.A.Melikhov (2005) has noted that a quaternion analogue of the Chernavskii mapping as in Lemma 27 is required. An idea to make a $\mathbb{Z}/2$-control as in the formula (47) was realized after a talk by E.V.Scepin (2004). As a conclusion, results was collected-out at the A.S.Mishchenko Seminar in (2008)-(2023). Alexandr S. Mischenko and Theodor Yu. Popelenskii allow discussions, in particular, we will use Popelenskii's formulas in Section 7.

# 2 First step

## 2.1 The dihedral group $\mathbb{Z}/2^{[2]} = \mathbf{D}$

In the present and the next sections the cobordism groups $Imm^{sf}(n-k,k)$, $Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k)$ will be used. In the case the first argument in the bracket is strongly positive, the cobordism group is finite.

The dihedral group $\mathbb{Z}/2^{[2]} = \mathbf{D}$ is defined by its corepresentation

$$\{a, b \,|\, b^4 = a^2 = e, [a,b] = b^2\}.$$

This group is represented by rotations a subgroup in the group $O(2)$ of orthogonal transformation of the standard plane. Elements transforms the base $\{\mathbf{e}_1, \mathbf{e}_2\}$ on the plane $Lin(\mathbf{e}_1, \mathbf{e}_2)$ to itself, a non-ordered pair of coordinate lines on the plane are preserved by transformations. The element $b$ is represented by the rotation of the plane by the angle $\frac{\pi}{2}$. The element $a$ is represented by the reflection of the plane relative to the straight line $l_1 = Lin(\mathbf{e}_1 + \mathbf{e}_2)$ parallel to the vector $\mathbf{e}_1 + \mathbf{e}_2$.

Let us consider a subgroup $\mathbf{I}_{a \times \dot{a}} = \mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbb{Z}/2^{[2]}$ in the dihedral group, which is generated by the elements $\{a, b^2 a\}$. This is an elementary 2-group of the rank 2. Transformations of this group keep each line $l_1$, $l_2$ with the base vectors $\mathbf{f}_1 = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{f}_2 = \mathbf{e}_1 - \mathbf{e}_2$ correspondingly. The cohomology group $H^1(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1); \mathbb{Z}/2)$ contains two generators $\varkappa_a, \varkappa_{\dot{a}}$.

Let us define the cohomology classes

$$\varkappa_a \in H^1(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1); \mathbb{Z}/2)), \quad \varkappa_{\dot{a}} \in H^1(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1); \mathbb{Z}/2)). \qquad (8)$$

Denote by $p_a : \mathbf{I}_a \times \dot{\mathbf{I}}_a \to \mathbf{I}_a$ a projection, the kernel of $p_a$ consists the symmetry transformation with respect to the bisector of the second coordinate angle and the identity.

Denote $\varkappa_a = p_a^*(t_a)$, where $e \ne t_a \in H^1(K(\mathbf{I}_a,1);\mathbb{Z}/2) \simeq \mathbb{Z}/2$. Let us denote by $p_{\dot{a}} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \to \dot{\mathbf{I}}_a$ the projection, the kernel of $p_{\dot{a}}$ consists of the symmetry with respect to the bisector of the first coordinate angle and the identity.

Let us denote $\varkappa_{\dot{a}} = p_{\dot{a}}^*(t_{\dot{a}})$, where $e \ne t_{\dot{a}} \in H^1(K(\dot{\mathbf{I}}_a,1);\mathbb{Z}/2) \cong \mathbb{Z}/2$.

## 2.2 A standardized immersion with a dihedral framing

Let us consider a $\mathbf{D}$-framed immersion $(g,\Psi,\eta_N)$ of codimension $2k$. Let us assume that the image of $g$ contains in a regular neighborhood $U(\mathbb{R}\mathrm{P}^2)$ of the embedding $\mathbb{R}\mathrm{P}^2 \subset \mathbb{R}^n$. The following mapping $\pi \circ \eta_N : N^{n-2k} \to K(\mathbf{D},1) \to K(\mathbb{Z}/2,1)$ is well-defined, where $K(\mathbf{D},1) \to K(\mathbb{Z}/2,1)$ is the epimorphism with the kernel $\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbf{D}$. It is required that this mapping coincides to the composition $i \circ PROJ \circ \eta_N$, where $PROJ : U(\mathbb{R}\mathrm{P}^2) \to \mathbb{R}\mathrm{P}^2$ is the projection of the neighborhood onto its central line, $i : \mathbb{R}\mathrm{P}^2 \subset K(\mathbb{Z}/2,1)$ is the standard inclusion, which transforms the fundamental class into the generator.

### 2.2.1 Condition C1 for standardized immersions

Let us consider the submanifold $N_{sing}^{n-2k-2} \subset N^{n-2k}$, $N_{sing}^{n-2k-2} = PROJ^{-1}(\mathbb{R}\mathrm{P}^0), \mathbb{R}\mathrm{P}^0 \subset \mathbb{R}\mathrm{P}^2$. Require that the mapping $\eta_N$, restricted to $N_{sing}$, be skipped trough the skeleton of $K(\mathbf{I}_a \times \dot{\mathbf{I}}_a,1)$ of the dimension strictly less then $\frac{3n}{4}$.

Geometrically, this means that the submanifold $N_{sing}$ is a defect of a reduction of the structured mapping of the normal bundle with a control of the subgroup $\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbf{D}$ over $\mathbb{R}\mathrm{P}^2$ to a mapping with a control of the same type over $\mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2$.

In the case $k \to 1+$ the structuring mapping on the defect manifold is skipped through a polyhedron of the dimension $\frac{3n}{4}$.

The complement $N \setminus N_{sing}$ to the defect we get an open manifold, for which the classifying normal bundle mapping is given by

$$\eta_N : N \setminus N_{sing} \to K(\mathbf{I}_a \times \dot{\mathbf{I}}_a) \rtimes \mathbb{R}\mathrm{P}^1.$$

**Definition 3.** Let us say that a $\mathbf{D}$-framed immersion $(g,\Psi,\eta_N;\Upsilon)$ of the codimension $2k$ is standardized by $\Upsilon$ with a defect $\delta < \frac{3n}{4}$, if the above condition $C1$ is satisfied with a prescribed defect.

Quadruples $(g,\Psi,\eta_N;\Upsilon)$, up to cobordism relations generate an abelian group $Imm^{D;stand}(n-2k,2k)$; there is natural forgetful homomorphism:

$$FG : Imm^{\mathbf{D};stand}(n-2k,2k) \longrightarrow Imm^{sf}(n-2k,2k), \qquad (9)$$

after a control structure $\Upsilon$ is omitted.

**Definition 4.** Assume $[(f,\Xi,\varkappa_M)] \in Imm^{sf}(n-k,k)$, $f : M^{n-k} \looparrowright \mathbb{R}^n$, $\varkappa_M \in H^1(M^{n-k};\mathbb{Z}/2)$, $\Xi$ is a skew-framing. Let us say that the pair $(M^{n-k},\varkappa_M)$ admits a compression of an order $q$, if the mapping $\varkappa_M : M^{n-k} \to \mathbb{R}\mathrm{P}^\infty$ is represented (up to homotopy) by the following composition: $\varkappa = I \circ \varkappa'_M$ :

$M^{n-k} \to \mathbb{R}\mathrm{P}^{n-k-q-1} \subset \mathbb{R}\mathrm{P}^{\infty}$. Let us say that the element $[(f, \Xi, \varkappa_M)]$ admits a compression of an order $q$, if in its cobordism class exists a triple $(f', \Xi', \varkappa_{M'})$, which admits a compression of the order $q$.

Below we assume $n_\sigma = 14$, $k = \frac{n-n_\sigma}{16}$. In particular, in the case $n = 126$ we get $k = 7$ and in the case $n = 254$ we get $k = 15$.

**Theorem 5.** *Assume that* $\ell \geq 7$ *and that an element* $x \in Imm^{sf}(n - \frac{n-m_\sigma}{16}, \frac{n-m_\sigma}{16}) = Imm^{sf}(n-k,k)$ *admits a compression of the order* $q = 7$ *By this additional assumption one may get an element* $y = \delta_k^{\mathbf{D}} \in Imm^{\mathbf{D}}(n-2k, 2k)$, [1] *such that there exist an element* $z \in Imm^{sf;stand}(n-k;k)$, $z = [(g, \eta_N; \Xi; \Upsilon)]$, $FG(z) = y$ *by* (9), $\Upsilon$ *is the normalized structure with an additional condition:*

*a submanifold* $N^{n_\sigma}_{a \times \dot{a}} \subset N^{n-2k}$,

$$N^{n_\sigma}_{a \times \dot{a}} = \eta_N^{-1}(K(\infty - 2(q+1)) \subset K(\mathbf{D}, 1)), \tag{10}$$

*where* $K(\infty - 2(q+1)) \subset K(\mathbf{D}, 1)$ *is a skeleton of the codimension* $2(q+1)$ *in the universal space* $K(\mathbf{D}, 1)$, *determined by the Euler class of the bundle* $(q+1)\tau$ *over* $K(\mathbf{D}, 1)$: *the Whitney sum of* $(q+1)$ *copies of the universal* 2*-bundle* $\tau$, *admits a reduction of its structured subgroup* $\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbf{D}$ *of the normal bundle (equivalently, the mapping* $\pi \circ g : N^{n-2k} \looparrowright U(\mathbb{R}\mathrm{P}^2) \to \mathbb{R}\mathrm{P}^2$, *restricted on the submanifold* $N^{n_\sigma}_{a \times \dot{a}}$, *homotopic to the constant mapping).*

### 2.2.2 A preliminary construction for Theorem 5

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Let

$$d^{(2)} : \mathbb{R}\mathrm{P}^{n-k'} \times \mathbb{R}\mathrm{P}^{n-k'} \setminus \mathbb{R}\mathrm{P}^{n-k'}_{diag} \to \mathbb{R}^n \times \mathbb{R}^n \tag{11}$$

be an arbitrary $(T_{\mathbb{R}\mathrm{P}^{n-k'} \times \mathbb{R}\mathrm{P}^{n-k'}}, T_{\mathbb{R}^n \times \mathbb{R}^n})$–equivariant mapping, which is transversal along the diagonal $\mathbb{R}^n_{diag} \subset \mathbb{R}^n \times \mathbb{R}^n$. The diagonal in the preimage is mapped into the diagonal of the image, by this reason the equivariant mapping $d^{(2)}$ is defined on the open manifold outside of the diagonal (in the pre-image).

(In Theorem 5 $k'$ has to be defined by the formula: $k' = k+q+1$ (parameters $k$,$q$ correspond to denotations of Theorem 5).)

Let us re-denote $(d^{(2)})^{-1}(\mathbb{R}^n_{diag})/T_{\mathbb{R}\mathrm{P}^{n-k'} \times \mathbb{R}\mathrm{P}^{n-k'}}$ by $\mathbf{N}_\circ = \mathbf{N}(d^{(2)})_\circ$ for short, this polyhedron is called a polyhedron of (formal) self-intersection of the equivariant mapping $d^{(2)}$.

The polyhedron $\mathbf{N}_\circ$ is an open polyhedron, this polyhedron admits a compactification, which is denoted by $\mathbf{N}$ with a boundary $\partial\mathbf{N}$. The boundary consists of all critical (formal critical) points of the mapping $d^{(2)}$. One has: $\mathbf{N} \setminus \partial\mathbf{N} = \mathbf{N}_\circ$. Let us denote by $U(\partial\mathbf{N})_\circ$ a thin regular neighborhood of the boundary (= a diagonal) $\partial\mathbf{N}$.

The polyhedron $\mathbf{N}_\circ$ is equipped by the mapping $d^{(2)}$, which admits the following lift:

$$\eta_{Ab} : \mathbf{N}_\circ \to K(\mathbf{I}_{a \times \dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1). \tag{12}$$

[1] The homomorphism $Imm^{sf}(n-k,k) \to Imm^{sf;stand}(n-k,k)$ is not claimed

The relation between the mapping $d^{(2)}$, called the structured mapping, and its lift (12) is the following:

$$\eta_{\circ} : \mathbf{N}_{\circ} \stackrel{\eta_{Ab}}{\longrightarrow} K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}}), \mathbb{Z}, 1) \longrightarrow K(\mathbb{Z}/2^{[2]}, 1). \quad (13)$$

On the polyhedron $U(\partial \mathbf{N})_{\circ}$ the mapping $\eta_{circ;Ab}$ gets the values into the following subcomplex: $K(\mathbf{I}_{a\times\dot{a}} \times 2\mathbb{Z}, 1) \subset K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$ (the projection on $K(\mathbb{Z}, 1) = S^1$ is the analogue of the projection of the Moebius band onto its central line).

**Definition 6.** Let us call a formal (equivariant) mapping $d^{(2)}$, given by (141), is holonomic, if this mapping is the formal extension of a mapping

$$d : \mathbb{RP}^{n-k'} \to \mathbb{R}^n. \quad (14)$$

**Definition 7.** Assume a formal (equivariant) mapping (141) is holonomic. Let us say $d^{(2)}$ admits an Abelian structure, if the following two conditions are satisfied.

– 1. On the open polyhedron $\mathbf{N}_{\circ}$ the mapping (12) is well-defined, which is a lift of the structured mapping (13), i.e. the composition of the lift $\eta_{Ab}$ with a natural mapping $K(\mathbf{I}_{a\times\dot{a}}) \rtimes_{\chi^{[2]}} \mathbb{Z}, 1) \to K(\mathbb{Z}/2^{[2]}, 1)$ coincides with the structured mapping (13).

–2. Let us consider the Moebius band $M^2$ and represent the Eilenberg-MacLane space $K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$ as a skew-product $K(\mathbf{I}_{a\times\dot{a}}, 1) \tilde{\times} M^2 \to M^2$; the restriction of the bundle over the boundary circle $S^1 = \partial(M^2)$ is identified with the subspace $K(\mathbf{I}_{a\times\dot{a}} \times 2\mathbb{Z}, 1) \subset K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$. Let us include the space $K(\mathbf{I}_{a\times\dot{a}}, 1) \tilde{\times} M^2$ into $K(\mathbf{I}_{a\times\dot{a}}, 1) \tilde{\times} \mathbb{RP}^2$ by a gluing of the trivial bundle over the boundary $S^1$ by the trivial bundle over a small disk with a central point $\mathbf{x}_{\infty} \in \mathbb{RP}^2$.

The resulting space is denoted by $K(\mathbf{I}_{a\times\dot{a}}, 1) \rtimes \mathbb{RP}^2$. The following mapping:

$$\eta_{\circ;Ab} : (\mathbf{N}_{\circ}, U(\partial \mathbf{N}_{\circ})) \to (K(\mathbf{I}_{a\times\dot{a}}, 1) \rtimes \mathbb{RP}^2, K(\mathbf{I}_{a\times\dot{a}}, 1) \times \mathbf{x}_{\infty}), \quad (15)$$

is well-defined, where the inverse image by $\eta_{\circ;Ab}$ of a layer $K(\mathbf{I}_{a\times\dot{a}}, 1) \times \mathbf{y}_{\infty} \subset K(\mathbf{I}_{a\times\dot{a}}, 1) \rtimes \mathbb{RP}^2$, $(\mathbf{y}_{\infty} \approx \mathbf{x}_{\infty})$ is dominated by a subpolyhedron of the dimension $\frac{n-3k'}{2} = \frac{\dim(\mathbf{N}_{\circ})}{2} - k'$ (up to a small constant $d = 2$), which is equipped by the structured mapping.

**Remark 8.** The condition –2 in Definition 7 can be formulated, alternatively, as following: in the polyhedron $\mathbf{N}_{\circ}$ there exists a subpolyhedron $\mathbf{N}_{\circ;reg} \subset U(\partial \mathbf{N}_{\circ}) \subset \mathbf{N}_{\circ}$, which is inside its proper thin neighborhood, and there exists a polyhedron $P$ of the dimension (up to a small constant $d = 2$) $dim(P) = \frac{\dim(\mathbf{N}_{\circ})}{2} - k' = \frac{n-3k'}{2}$; there exists a control mapping $\mathbf{N}_{\circ;reg} \to P$ and the structured mapping $\eta_P : P \to K(\mathbf{I}_{a\times\dot{a}}, 1) \times S^1$, such that the composition $\mathbf{N}_{\circ;reg} \to P \to K(\mathbf{I}_{a\times\dot{a}}, 1) \times S^1 \to K(\mathbf{I}_{a\times\dot{a}}, 1) \rtimes \mathbb{RP}^1 \subset K(\mathbf{I}_{a\times\dot{a}}, 1) \rtimes \mathbb{RP}^2$ is the structured mapping $\eta_{\circ;Ab} : \mathbf{N}_{\circ;reg} \to K(\mathbf{I}_{a\times\dot{a}}, 1) \rtimes_{\chi^{[2]}} \mathbb{RP}^2$ up to homotopy. Consider the pair $(U(\partial \mathbf{N}_{\circ}), U(\partial \mathbf{N}_{\circ}) \setminus \mathbf{N}_{\circ;reg})$, and the restriction of $\eta_{\circ;Ab}$ on this pair. The following mapping of pairs exists:

$$\eta_{\circ;Ab} : (U(\partial \mathbf{N}_\circ), U(\partial \mathbf{N}_\circ) \setminus \mathbf{N}_{\circ;reg}) \to (K(\mathbf{I}_{a\times\dot{a}} \times 2\mathbb{Z}, 1), K(\mathbf{I}_{a\times\dot{a}}, 1)). \qquad (16)$$

In the formula above $K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} 2\mathbb{Z}, 1) = K(\mathbf{I}_{a\times\dot{a}} \times 2\mathbb{Z}, 1)$; $K(\mathbf{I}_{a\times\dot{a}}, 1) \subset K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} 2\mathbb{Z}, 1)$ is the standard the inclusion.

The following lemma is proved in Section 7, Subsection 7.7.

**Lemma 9.** *Small Lemma*
*For*

$$n - k' \equiv 1 \pmod 2, \quad n \equiv 0 \pmod 2 \qquad (17)$$

*there exist a holonomic formal mapping* $d^{(2)}$*, which admits an Abelian structure, Definition* 7*.*

## 2.3 Mapping $d^{(2)}$ in Lemma 9

Let us consider the standard 2-sheeted covering $p : S^1 \to S^1$ over the circle. One may write-down: $p : S^1 \to \mathbb{RP}^1$, where $\mathbb{Z}/2 \times S^1 \to \mathbb{RP}^1$ is the standard antipodal action. Let us consider the join of $\frac{n-k'+1}{2} = r$-copies of $\mathbb{RP}^1$ $\mathbb{RP}^1 * \dots * \mathbb{RP}^1 = S^{n-k'}$, which is homeomorphic to the standard $n-k'$-dimensional sphere. Let us define the join of $r$ copies of the mapping

$$\tilde{d} : S^1 * \dots * S^1 \to \mathbb{RP}^1 * \dots * \mathbb{RP}^1.$$

In the preimage the group $\mathbb{Z}/2$ by the antipodal action is represented, this action is commuted with $\tilde{d}$. The required mapping (14) is the composition of the projection onto the factor $\tilde{d}/_{\mathbb{Z}/2} : \mathbb{RP}^{n-k'} \to S^{n-k'}$ with the standard inclusion

$$d : S^{n-k'} \subset \mathbb{R}^n. \quad \square$$

**Proof of Theorem** 5

Recall, $k = \frac{n-n_\sigma}{16}$. Assume that an element $x \in Imm^{sf}(n-k, k)$ is represented by a skew-framed immersion $(f, \Xi, \varkappa_M)$, $f : M^{n-k} \looparrowright \mathbb{R}^n$. By the assumption, there exist a compression $\varkappa_M : M^{n-k} \to \mathbb{RP}^{n-k-q-1}$, $q = \frac{n_\sigma}{2} = 7$, for which the composition $M^{n-k} \to \mathbb{RP}^{n-k-q-1} \subset K(\mathbf{I}_d, 1)$ coincides with the mapping $\varkappa_M : M^{n-k} \to K(\mathbf{I}_d, 1)$.

Define $k'$ as the biggest positive integer, which is less then $k+q+1$, and such that $n - k' \equiv 1 \pmod 2$. Because $k = 1 \pmod 2$, $q = 7$, we may put $k' = k + q + 1 \ge 7$.

By Lemma 9 there exist an (equivariant) formal mapping $d^{(2)}$, which admits an Abelian structure, see Definition 7.

Let us construct a skew-framed immersion $(f_1, \Xi_1, \varkappa_1)$, for which the self-intersection manifold is as required in Theorem 5.

An immersion $f_1 : M^{n-k} \looparrowright \mathbb{R}^n$, which is equipped by a skew-framing $(\varkappa_1, \Xi_1)$, is defined as a result of a small deformation of the composition $d \circ \varkappa : M^{n-k} \to \mathbb{RP}^{n-k-q-1} \subset \mathbb{RP}^{n-k'} \to \mathbb{R}^n$ in the prescribed regular cobordism class of the immersion $f$. A caliber of the deformation $d \circ \varkappa \mapsto f_a$ define less then a radius of a regular neighborhood of the polyhedron of intersection points of the mapping $d$. Because $f_1$ is regular homotopic to $f$, the immersion $f_1$ is naturally equipped by a skew-framed.

Let us denote by $N^{n-2k}$ the self-intersection manifold of the immersion $f_1$. The following decomposition over a common boundary are well-defined:

$$N^{n-2k} = N^{n-2k}_{a \times \dot{a}, \mathbf{N}(d^{(2)})} \cup_\partial N^{n-2k}_{reg}. \tag{18}$$

In this formula $N^{n-2k}_{a \times \dot{a}, \mathbf{N}(d^{(2)})}$ is a manifold with boundary, which is immersed into a regular (immersed) neighborhood $U_{\mathbf{N}(d)}$ of the polyhedron $\mathbf{N}(d)$ (with a boundary) of self-intersection points of the mapping $d$. The manifold $N^{n-2k}_{reg}$ with boundary is immersed into an immersed regular neighborhood of self-intersection points, outside critical points; let us denote this neighborhood by $U_{reg}$. A common boundary of manifolds $N^{n-2k}_{a \times \dot{a}, \mathbf{N}(d^{(2)})}$, $N^{n-2k}_{reg}$ is a closed manifold of dimension $n-2k-1$, this manifold is immersed into the boundary $\partial(U_{reg})$ of the immersed neighborhood $U_{reg}$.

The parametrized $\mathbb{Z}/2^{[2]}$–framed immersion of the immersed submanifold (18) denote by $g_{a \times \dot{a}}$. Let us prove that a $\mathbb{Z}/2^{[2]}$–framing over the component (18) is reduced to a framing with the structured group $\mathbf{I}_{a \times \dot{a}} \rtimes \mathbb{Z}/2$ with a control over $\mathbb{RP}^2$.

Define on the manifold $N^{n-2k}_{a \times \dot{a}, \mathbf{N}(d^{(2)})}$ a mapping $N^{n-2k}_{a \times \dot{a}, \mathbf{N}(d^{(2)})} \stackrel{\eta_{a \times \dot{a}}}{\longrightarrow} K(\mathbf{I}_{a \times \dot{a}}, 1) \rtimes_{\chi^{[2]}} \rtimes \mathbb{RP}^2$, as the result of a gluing of the two mappings.

Define on the component $N^{n-2k}_{a \times \dot{a}, \mathbf{N}(d^{(2)})}$ in the formula (18) the following mapping:

$$\eta_{a \times \dot{a}, \mathbf{N}(d^{(2)})} : N^{n-2k}_{a \times \dot{a}, \mathbf{N}(d^{(2)})} \to K(\mathbf{I}_{a \times \dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}), 1), \tag{19}$$

as the composition of the projection $N^{n-2k}_{a \times \dot{a}, \mathbf{N}(d^{(2)})} \to N(d)$ and the mapping $\eta_{a \times \dot{a}} : \mathbf{N}(d^{(2)}) \to K((\mathbf{I}_{a \times \dot{a}}) \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$, constructed in Lemma 9.

The restrictions of the mappings $\eta_{a \times \dot{a}, N(d^{(2)})}$, $\eta_{a \times \dot{a}, N_{reg}}$ on a common boundary $\partial N^{n-2k}_{a \times \dot{a}, N(d^{(2)})}$, $\partial N^{n-2k}_{reg}$ are homotopy, because the mapping $\eta_{\mathbf{N}(d^{(2)})}$ : $\mathbf{N}(d^{(2)}) \to K(\mathbf{I}_a \times \dot{\mathbf{I}}_a \rtimes_{\chi^{[2]}} \mathbb{Z}), 1)$ satisfies on $\partial \mathbf{N}(d^{(2)})$ right boundary conditions. Therefore, the required mapping

$$\eta_{a \times \dot{a}} : N^{n-2k}_{a \times \dot{a}} \to K(\mathbf{I}_{a \times \dot{a}}, 1) \rtimes_{\chi^{[2]}} \mathbb{RP}^2 \tag{20}$$

is well-defined as a result of a gluing of the mappings $\eta_{a \times \dot{a}, \mathbf{N}(d^{(2)})}$ and $\eta_{a \times \dot{a}, N_{reg}}$. The mapping (20) determines a reduction of $\mathbb{Z}/2^{[2]}$–framing of the immersion $g_{a \times \dot{a}}$.

Let us prove that Condition $C1$ in Subsubsection , see Definition 3, is a corollary of Condition –2 in Definition 7. The structuring mapping on

$\partial N^{n-2k}_{a\times\dot{a},N(d^{(2)})}$ in compressed into the skeleton of $K(\mathbf{I}_{a\times\dot{a}},1)$ of the dimension $\frac{3(n-k-q-1)}{4}+d$. In the case $k=7$, $q=7$, $\frac{3(n-k-q-1)}{4}+d<\frac{3n}{4}$, if $d<10$. In Lemma 49 $d<3$, this proves Condition –2.

By the construction, the image of the mapping $f_2 : M^{n-8k} \looparrowright \mathbb{R}^n$ belongs to a regular neighborhood of the image of the submanifold $\mathbb{RP}^{n-8k-q-1} \subset \mathbb{RP}^{n-7k-k'}$ by the mapping $d$. Because $n-8k-q-1 = n-\frac{n}{2}+\frac{n_\sigma}{2}-\frac{n_\sigma}{2}-1=\frac{n}{2}-1$, the immersed submanifold $d(\mathbb{RP}^{n-8k-q-1})$ is an embedded submanifold.

Denote by $N_2^{n-16k}$ the manifold of self-intersection points of the immersion $f_2$, this is a manifold of dimension $n-16k=m_\sigma$. The following inclusion is well-defined:

$$N_2^{n-16k} \subset N_{reg}^{n-2k}, \tag{21}$$

where the manifold $N_{reg}^{n-2k}$ is defined by the formula (18).

In particular, the following reduction of the classified mapping $\eta_2$ : $N_2^{n-16k} \to K(\mathbb{Z}/2^{[2]},1)$ to a mapping

$$\eta_{Ab,2} : N_2^{n-16k} \to K(\mathbf{I}_{b\times\dot{b}},1) \tag{22}$$

with the image $K(\mathbf{I}_{b\times\dot{b}},1) \subset K(\mathbb{Z}/2^{[2]},1)$ is well-defined.

Therefore, without loss of a generality, for $\sigma \ge 5$ one may assume that

$$N_2^{n-16k} \cap N^{n-2k}_{a\times\dot{a},\mathbf{N}(d)} = \emptyset. \tag{23}$$

Theorem 5 is proved. □

# 3 Local coefficients and homology groups

Let us define the group $(\mathbf{I}_a \times \dot{\mathbf{I}}_a) \rtimes_{\chi^{[2]}} \mathbb{Z}$ and the epimorphism $(\mathbf{I}_a \times \dot{\mathbf{I}}_a) \rtimes_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z}/2^{[2]}$. Consider the automorphism

$$\chi^{[2]} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \to \mathbf{I}_a \times \dot{\mathbf{I}}_a \tag{24}$$

of the exterior conjugation of the subgroup $\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbf{D}$ by the element $ba \in \mathbf{D}$, this element is represented by the reflection of the plane with respect to the line $Lin(\mathbf{e}_2)$. Let us define the automorphism (denotations are not changed)

$$\chi^{[2]} : \mathbb{Z}/2^{[2]} \to \mathbb{Z}/2^{[2]}, \tag{25}$$

by permutations of the base vectors. It is not difficult to check, that the inclusion $\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbb{Z}/2^{[2]}$ commutes with automorphisms (24), (25) in the image and the preimage.

Define the group

$$\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}. \tag{26}$$

Let us consider the quotient of the group $\mathbf{I}_{a\times\dot{a}} * \mathbb{Z}$ (the free product of the group $\mathbf{I}_{a\times\dot{a}}$ and $\mathbb{Z}$) by the relation $zxz^{-1} = \chi^{[2]}(x)$, where $z \in \mathbb{Z}$ is the generator, $x \in \mathbf{I}_{a\times\dot{a}}$ is an arbitrary element.

This group is a particular example of a semi-direct product $A \rtimes_{\phi} B$, $A = \mathbf{I}_{a\times\dot{a}}$, $B = \mathbb{Z}$, by a homomorphism $\phi : B \to Aut(A)$; the set $A \times B$ is equipped with a binary operation $(a_1, b_1) * (a_2, b_2) \mapsto (a_1\phi_{b_1}(b_2), b_1 b_2)$. Let us define the group (26) by this construction for $A = \mathbf{I}_{a\times\dot{a}}$, $B = \mathbb{Z}$, $\phi = \chi^{[2]}$.

The classifying space $K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$ is a skew-product over the circle $S^1$ with $K(\mathbf{I}_{a\times\dot{a}}, 1)$, where the shift mapping in the cyclic covering $K(\mathbf{I}_{a\times\dot{a}}, 1) \to K(\mathbf{I}_{a\times\dot{a}}, 1)$ over $K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$ is induced by the automorphism $\chi^{[2]}$. The projection onto the circle is denoted by

$$p_{a\times\dot{a}} : K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1) \to S^1. \qquad (27)$$

Take a marked point $pt_{S^1} \in S^1$ and define the subspace

$$K(\mathbf{I}_{a\times\dot{a}}, 1) \subset K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1) \qquad (28)$$

as the inverse image of the marked point $pt_{S^1}$ by the mapping (27).

A description of the standard base of the group $H_i(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z})$ is sufficiently complicated and is not required. The group $H_i(K(\mathbf{I}_{a\times\dot{a}}, 1); \mathbb{Z})$ is described using the Kunneth formula:

$$\begin{array}{c} 0 \to \bigoplus_{i_1+i_2=i} H_{i_1}(K(\mathbf{I}_a, 1); \mathbb{Z}) \otimes H_{i_2}(K(\dot{\mathbf{I}}_a, 1); \mathbb{Z}) \longrightarrow \\ \longrightarrow H_i(K(\mathbf{I}_{a\times\dot{a}}, 1); \mathbb{Z}) \longrightarrow \\ \longrightarrow \bigoplus_{i_1+i_2=i-1} Tor^{\mathbb{Z}}(H_{i_1}(K(\mathbf{I}_a, 1); \mathbb{Z}), H_{i_2}(K(\dot{\mathbf{I}}_a, 1); \mathbb{Z}) \to 0. \end{array} \qquad (29)$$

The standard base of the group $H_i(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1))$ contains the following elements:

$$x \otimes y/(x \otimes y) - (y \otimes x),$$

where $x \in H_j(K(\mathbf{I}_a, 1))$, $y \in H_{i-j}(K(\dot{\mathbf{I}}_a, 1))$ ($\mathbb{Z}/2$–coefficients is the formulas are omitted).

In particular, for odd $i$ the group $H_i(K(\mathbf{I}_{a\times\dot{a}}, 1); \mathbb{Z})$ contains elements, which are defined by the fundamental classes of the following submanifolds: $\mathbb{RP}^i \times pt \subset \mathbb{RP}^i \times \mathbb{RP}^i \subset K(\mathbf{I}_a, 1) \times K(\dot{\mathbf{I}}_a, 1) = K(\mathbf{I}_{a\times\dot{a}}, 1)$, $pt \times \mathbb{RP}^i \subset \mathbb{RP}^i \times \mathbb{RP}^i \subset K(\mathbf{I}_a, 1) \times K(\dot{\mathbf{I}}_a, 1) = K(\mathbf{I}_{a\times\dot{a}}, 1)$. Let us denote the corresponding elements as following:

$$t_{a,i} \in H_i(K(\mathbf{I}_{a\times\dot{a}}, 1); \mathbb{Z}), t_{\dot{a},i} \in H_i(K(\mathbf{I}_{a\times\dot{a}}, 1); \mathbb{Z}). \qquad (30)$$

The following analogues of the homology groups $H_i(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1))$, $H_i(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z})$ with local coefficients over the generator $\pi_1(S^1)$ is defined, the groups are denoted by

$$H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}/2), \qquad (31)$$

$$H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}). \qquad (32)$$

The following epimorphism

$$p_{a\times\dot{a}} : \mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z} \tag{33}$$

is well-defined by the formula: $x * y \mapsto y$, $x \in \mathbf{I}_{a\times\dot{a}}$, $y \in \mathbb{Z}$. The following homomorphism

$$\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2, \tag{34}$$

is well-defined by the formula $p_{a\times\dot{a}} \pmod 2$.

Let us define the group (32). Let us consider the group ring $\mathbb{Z}[\mathbb{Z}/2] = \{a+bt\}$, $a, b \in \mathbb{Z}$, $t \in \mathbb{Z}/2$. The generator $t \in \mathbb{Z}[\mathbb{Z}/2]$ is represented by the involution

$$\chi^{[2]} : K(\mathbf{I}_{a\times\dot{a}} \times 2\mathbb{Z}, 1) \to K(\mathbf{I}_{a\times\dot{a}} \times 2\mathbb{Z}, 1),$$

the restriction of the involution on the subspace $K(\mathbf{I}_{a\times\dot{a}}, 1) \subset K(\mathbf{I}_{a\times\dot{a}} \times 2\mathbb{Z}, 1)$ is the reflection, which is induced by the automorphism $\mathbf{I}_a \times \dot{\mathbf{I}}_a \to \mathbf{I}_a \times \dot{\mathbf{I}}_a$, which permutes factors; the restriction of the involution on the group $2\mathbb{Z}$ is the antipodal involution in the double covering $S^1 \stackrel{\times 2}{\longrightarrow} S^1$. Because all non-trivial homology classes of the space $K(\mathbf{I}_{a\times\dot{a}}, 1)$ are of the order 2, transformation of signs is not required. Nevertheless, a local orientation of transformed simplex is preserved.

Let us consider the local system of the coefficient $\rho_t : \mathbb{Z}/2[\mathbb{Z}/2] \to Aut(K(\mathbf{I}_{a\times\dot{a}}, 1) \times 2\mathbb{Z})$, using this local system: a chain $(a+bt)\sigma$ with the support on a simplex $\sigma \subset K(\mathbf{I}_{a\times\dot{a}} \times 2\mathbb{Z}, 1)$ is transformed into a chain $(at+b)\chi^{[2]}(\sigma)$. The group (32) is well-defined by the tensor product of a chain $\mathbb{Z}[\mathbb{Z}/2]$-complex with the integers $\mathbb{Z}$ by the augmentation $\mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z}$: $a + bt \mapsto a + b$.

The group (31) is defined analogously. A complete calculation of the groups (31), (32) is not required.

Let us define the following subgroup:

$$D_i^{loc}(\mathbf{I}_{a\times\dot{a}}; \mathbb{Z}) \subset H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}) \tag{35}$$

by the formula: $D_i^{loc}(\mathbf{I}_{a\times\dot{a}}; \mathbb{Z}) = \mathrm{Im}(H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}}, 1); \mathbb{Z}) \to H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}))$, where the homomorphism

$$H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}}, 1); \mathbb{Z}) \to H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z})$$

is induced by the inclusion of the subgroup.

From the definition, there exist a natural homomorphism:

$$H_i(K(\mathbf{I}_{a\times\dot{a}}, 1); \mathbb{Z}[\mathbb{Z}/2]) \to D_i^{loc}(\mathbf{I}_{a\times\dot{a}}; \mathbb{Z}), \tag{36}$$

this homomorphism is an isomorphism.

A subgroup

$$D_i^{loc}(\mathbf{I}_{a\times\dot{a}}; \mathbb{Z}/2) \subset H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}/2) \tag{37}$$

is defined analogously as (35). A description of this subgroup is more easy, because this group is generated by elements $X+tY$, $X=x\otimes y, Y=x'\otimes y'$, wherein $\chi^{[2]}_*(x\otimes y)=y\otimes x$.

In the case $i=2s$ the basis elements $y\in D^{loc}_{i-1}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z})$ are the following:

1. $y=r$, $r=t_{a,2s-1}+t_{\dot{a},2s-1}$, where $t_{a,2s-1},t_{\dot{a},2s-1}\in H_{2s-1}(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z})$ are defined by the formula (30).

2. $y = z(i_1,i_2)$, where $z(i_1,i_2) = tor(r_{a,i_1},r_{\dot{a},i_2})+tor(r_{a,i_2},r_{\dot{a},i_1})$, $i_1,i_2 \equiv 1 \pmod 2$, $i_1+i_2 = 2s-2$, $tor(r_{a,i_1},r_{\dot{a},i_2}) \in Tor^{\mathbb{Z}}(H_{i_1}(K(\mathbf{I}_a,1);\mathbb{Z}),H_{i_2}(K(\dot{\mathbf{I}}_a,1);\mathbb{Z}))$. The elements $\{z(i_1,i_2)\}$ belong to the kernel of the homomorphism:

$$D^{loc}_{i-1}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z})\to D^{loc}_{i-1}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z}/2), \tag{38}$$

which is defined as the modulo 2 reduction of the coefficients: $\mathbb{Z}\to\mathbb{Z}/2$.

**Lemma 10.** *The group $H^{loc}_{2s}(K(\mathbf{I}_{a\times\dot{a}}\rtimes_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z})$ contains a direct factor by the subgroup $D^{loc}_{2s}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z})$. The elements from the subgroup*

$$\bigoplus_{i_1+i_2=2s} H_{i_1}(K(\mathbf{I}_a,1);\mathbb{Z})\otimes H_{i_2}(K(\dot{\mathbf{I}}_a,1);\mathbb{Z})\subset D^{loc}_{2s}(\mathbf{I}_a\times\dot{\mathbf{I}}_a;\mathbb{Z}),$$

*is mapped monomorphic into the image* $\mathrm{Im}(A)$ *by the following homomorphism:*

$$A: H^{loc}_{2s}(K(\mathbf{I}_{a\times\dot{b}}\rtimes_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z})\to \tag{39}$$
$$H^{loc}_{2s}(K(\mathbf{I}_{a\times\dot{b}}\rtimes_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}/2),$$

*which is the modulo 2 reduction of the coefficient system.*

**Proof of Lemma** 10

Assume an element $x\otimes y\in H_{i_1}(K(\mathbf{I}_a,1);\mathbb{Z})\otimes H_{i_2}(K(\dot{\mathbf{I}}_a,1);\mathbb{Z})$ is a cycle in $H^{loc}_{2s}(K(\mathbf{I}_{a\times\dot{a}}\rtimes_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z})$, which is represented by a singular manifold $f: V^{2s}\longrightarrow K(\mathbf{I}_a,1)\times K(\dot{\mathbf{I}}_a,1)$. Take the boundary $W^{2s+1}$, $\partial W^{2s+1}=V^{2s}$ and consider the cycle

$$F:(W^{2s+1},V^{2s})\longrightarrow(K(\mathbf{I}_{a\times\dot{a}}\rtimes_{\chi^{[2]}}\mathbb{Z},1),K(\mathbf{I}_a,1)\times K(\dot{\mathbf{I}}_a,1))$$

with a local coefficient system, which is given by the collection of a paths from centers of simplexes to the marked point in the subspace of the pair. We may assume, that the singular cycle $F$ is such that the manifold is catted by the hypermanifold $U^{2s}\subset W^{2s+1}$, the local path system at the two components of the boundary of the cutted submanifold $W^{2s-1}\setminus U^{2s}$ have no intersection points of paths with $U^{2s}$. Then, the two copies $U^{2s}_+$, $U^{2s}_-$ of the boundary are represented by $\chi^{(2)}$-conjugated cycles and after the homotopy of $U^{2s}\subset W^{2s+1}$ into a subspace $K(\mathbf{I}_a,1)\times K(\dot{\mathbf{I}}_a,1)\subset K(\mathbf{I}_{a\times\dot{a}}\rtimes_{\chi^{[2]}}\mathbb{Z},1)$ we get an ordinary singular cycle with 3 components of boundary: $V^{2s}$, $U^{2s}$ and $-U^{2s}$, where $-\dots$ is denoted the cycle with the opposite orientation. Lemma 10 is proved. $\square$

Let us consider the diagonal subgroup: $i_{\mathbf{I}_d,\mathbf{I}_{a\times\dot{a}}}: \mathbf{I}_d \subset \mathbf{I}_a \times \dot{\mathbf{I}}_a = \mathbf{I}_{a\times\dot{a}}$. This subgroup coincides to the kernel of the homomorphism

$$\omega^{[2]} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \to \mathbb{Z}/2, \tag{40}$$

which is defined by the formula $(x \times y) \mapsto xy$.

Let us define the homomorphism

$$\Phi^{[2]} : \mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z}/2^{[2]} \tag{41}$$

by the formula: $\Phi^{[2]}(z) = ab$, $z \in \mathbb{Z}$ is the generator (the element $ab$ is represented by the inversion of the first basis vector $\mathbb{Z}^{[2]} \subset O(2)$),

$$\Phi^{[2]}|_{\mathbf{I}_{a\times\dot{a}}\times\{0\}} : \mathbf{I}_{a\times\dot{a}} \subset \mathbf{D}$$

is the standard inclusion;

$$\Phi^{[2]}|_{z^{-1}\mathbf{I}_a\times\dot{\mathbf{I}}_a z} : \mathbf{I}_{a\times\dot{a}} \subset \mathbf{D}$$

is the conjugated inclusion, by the composition with the exterior automorphism in the subgroup $\mathbf{I}_{a\times\dot{a}} \subset \mathbb{Z}/2^{[2]}$.

Let us define

$$(\Phi^{[2]})^*(\tau_{[2]}) = \tau_{a\times\dot{a}},$$

where $\tau_{a\times\dot{a}} \in H^2(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1))$, $\tau_{[2]} \in H^2(K(\mathbb{Z}^{[2]}, 1))$.

# 4 The fundamental class of the canonical covering over the self-intersection $D$-framed standardized immersion without a defect

Let us consider a $\mathbb{Z}/2^{[2]}$–framed immersion $(g, \Psi, \eta_N)$, $g : N^{n-2k} \looparrowright \mathbb{R}^n$, and assume that this immersion is standardized by $\Upsilon$, see Subsubsection 2.2.

The mapping $\eta_N$ admits a reduction, the image of this mapping belongs to the total bundle space over $\mathbb{RP}^2$ with the layer $K(\mathbf{I}_{a\times\dot{a}}, 1)$. Assume that the manifold $N^{n-2k}$ is connected. Assume that a marked point on $\mathbb{RP}^2$ is fixed and denote:

$$PROJ^{-1}(\mathbf{x}_\infty) \in N^{n-2k}. \tag{42}$$

Assume that the immersion $g$ translates a marked point into a small neighborhood of the central point $\mathbf{x}_\infty \in \mathbb{RP}^2$, over this point the defect is well-defined.

The image of the fundamental class $\eta_{N,*}([N^{n-2k}])$ belongs to $H_{n-2k}(\mathbf{D};\mathbb{Z})$. The standardization condition, see Definition 3, implies an existence of a natural lift of this homology class into the group $H_{n-2k}(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z};\mathbb{Z})$, because the condition C1 implies that the submanifold $N_{sing} \subset N^{n-2k}$, which is a layer over the point $\mathbf{x}_\infty \in \mathbb{RP}^2$ determines zero-chain.

Let us define the lift of the characteristic class $\eta_{N,*}$ and define the class

$$\eta_{N,*}([N^{n-2k}]) \in H^{loc}_{n-2k}((\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1);\mathbb{Z}). \tag{43}$$

Then let us prove that

**Theorem 11.** *The characteristic class (43) satisfies the following equation:*

$$\eta_{N,*}([N^{n-2k}]) \in D^{loc}_{n-2k}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z}), \tag{44}$$

*see the inclusion (35).*

### 4.0.1 Construction of the class (43)

Let us consider a skeleton of the space $K(\mathbf{I}_{a\times\dot{a}}, 1)$, which is realized as the structured group $\mathbf{I}_{a\times\dot{a}} \subset O(1+1)$ of a bundle over the corresponding Grasmann manifold $Gr_{\mathbf{I}_{a\times\dot{a}}}(1+1,n)$ of non-oriented $1+1=2$-plans in $n$–space. Denote this Grasmann manifold by

$$KK(\mathbf{I}_{a\times\dot{a}}) \cong K(\mathbf{I}_{a\times\dot{a}}, 1). \tag{45}$$

On the space (45) a free involution

$$\chi^{[2]} : KK(\mathbf{I}_{a\times\dot{a}}) \to KK(\mathbf{I}_{a\times\dot{a}}), \tag{46}$$

acts, this involution corresponds to the automorphism (24) and represents by a permutation of line of layers (by the antidiagonal matrix).

Let us define a family of spaces (45) with the prescribed symmetry group, are parametrized over the base $S^1$, the generator of $S^1$ transforms layers by the involution (24). This space is a subspace (a skeleton): $K(\mathbf{I}_{a\times\dot{a}}, 1) \rtimes \mathbb{RP}^2 \subset K(\mathbf{D}, 1)$.

Define the required space

$$KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{RP}^2) \tag{47}$$

by a quotient of the direct product $KK(\mathbf{I}_{a\times\dot{a}}) \times S^2$ with respect to the involution $\chi^{[2]} \times (-1)$, where $-1 : S^2 \to S^2$–is the antipodal involution. The following inclusion is well-defined:

$$KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}) \subset KK(\mathbf{I}_{a\times\dot{a}}) \rtimes \mathbb{RP}^2. \tag{48}$$

The involution (78) induces the $S^1$-fibrewise involution

$$\chi^{[2]} : KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}) \to KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}), \tag{49}$$

which is extended to the involution

$$\chi^{[2]} : KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{R}\mathrm{P}^2) \to KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{R}\mathrm{P}^2) \qquad (50)$$

on $KK(\mathbf{I}_{a\times\dot{a}}) \rtimes \mathbb{R}\mathrm{P}^2$, all this extended involutions denote the same.

The universal 2-dimensional $\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}/2$–bundle over the target space in (48) $KK(\mathbf{I}_{a\times\dot{a}},1) \rtimes \mathbb{R}\mathrm{P}^2$, which is denoted by $\tau_{a\times\dot{a},\rtimes}$, is well-defined. We will need mostly the restriction of this bundle to the subspace (47). The restriction of this bundle on the origin subspace of (45) is the involutive pull-back of the universal 2-bundle $\tau_{a\times\dot{a}}$ over the target space of (45).

The first direct factor of the $\mathbf{I}_{a\times\dot{a}} \rtimes \mathbb{Z}$-framing of the immersion $(g,\Psi,\eta_N)$ by $\Upsilon$, determines the Gaussean mapping in (47). The restriction of this mapping on (42) is in the $\frac{3n}{4}$-skeleton of the subspace (45). The restriction of this mapping on the complement to this submanifold is inside the subspace (48). Local coefficients system, described in Section determines the class (43).

#### 4.0.2 Proof of Theorem 11

Let us prove that the class (43) satisfies the equation (44). Take the decomposition (18). The first manifold in the right hand-side of the formula is mapped into the skeleton of the dimension $n-2k-2q$, where $q=7$. Therefore the fundamental class of the second term gives a contribution to the characteristic class (43). By construction, the formula (44) is satisfied. Theorem 11 is proved. □

In the case $dim(N)=n-2k$, we get $dim(\bar{N}^{n_\sigma}_{a\times\dot{a}}) = n_\sigma = n-16k$, see (10), Theorem 5; recall, in the case $n=126$ one gets $k=7$ and $n_\sigma = 14$.

Let us consider the element

$$\eta_*(N^{n_\sigma}_{a\times\dot{a}}) \in D^{loc}_{n_\sigma}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z}). \qquad (51)$$

Let us prove that the element (51) satisfies an additional property, which is called pure standardized Property, Definition 18 below.

**Lemma 12.** *–1. The element* (51) *is decomposed with respect to the standard base of the group* $H_{n_\sigma}(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a,1))$ *contains not more then the only monomial* $t_{a,i}\otimes t_{\dot{a},i}$, *see.* (30), $i = \frac{n_\sigma}{2} = \frac{n-16k}{2}$. *The coefficient of this monomial coincides with the Kervaire invariant, which is calculated for a* $\mathbb{Z}/2^{[2]}$*–framed immersion* $(g,\Psi,\eta_N)$.

**Proof of Lemma** 12

Let us proof the statement for the case $n_\sigma = 14$. Let us consider the manifold $N^{n_\sigma}_{a\times\dot{a}}$, which we re-denote in the proof by $N^{14}$ for sort. This manifold is equipped with the mapping $\eta : N^{14} \to K(\mathbf{I}_{a\times\dot{a}},1)$. Let us consider all the collection of characteristic $\mathbb{Z}/2$-numbers for the mapping $\eta$, which is induced from the universal classes. The Hurewitcz image $\eta_*([N^{14}])$ is in the group $D^{loc}_{n_\sigma}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z}/2)$.

Let us estimate the fundamental class $[N^{14}]$ and prove that this class is pure (see definition 18) below. Because $N^{14}$ is oriented, among characteristic numbers: $\varkappa_a\varkappa_{\dot a}^{13}$, $\varkappa_a^3\varkappa_{\dot a}^{11}$, $\varkappa_a^5\varkappa_{\dot a}^9$, $\varkappa_a^7\varkappa_{\dot a}^7$, $\varkappa_a^9\varkappa_{\dot a}^5$, $\varkappa_a^3\varkappa_{\dot a}^{11}$, $\varkappa_a\varkappa_{\dot a}^{13}$. the only number $\varkappa_a^7\varkappa_{\dot a}^7$ could be nontrivial.

Let us prove that the characteristic number $\varkappa_a\varkappa_{\dot a}^{13}$ is trivial. Let us consider the manifold $K^3 \subset N^{14}$, which is dual to the characteristic class $\varkappa_a\varkappa_{\dot a}^{10}$. The normal bundle $\nu_N$ over the manifold $N^{14}$ is isomorphic to the Whitney sum $k\varkappa_a \oplus k\varkappa_{\dot a}$, where the positive integer $k$ satisfies the equation: $k \equiv 0 \pmod 8$. The restriction of the bundle $\nu_N$ on the submanifold $K^3$ is trivial. Therefore, the normal bundle $\nu_K$ of the manifold $K^3$ is stably isomorphic to the bundle $\varkappa_a \oplus 2\varkappa_{\dot a}$.

Because the characteristic class $w_2(K^3)$ is trivial we get: $\langle \varkappa_{\dot a}^3; [K^3]\rangle = 0$. But, $\langle \varkappa_{\dot a}^3; [K^3]\rangle = \langle \varkappa_a\varkappa_{\dot a}^{13}; [N^{14}]\rangle$. This proves that the characteristic number $\varkappa_a\varkappa_{\dot a}^{13}$ is trivial. Analogously, the characteristic numbers $\varkappa_a^5\varkappa_{\dot a}^9$, $\varkappa_a^9\varkappa_{\dot a}^5$, $\varkappa_a\varkappa_{\dot a}^{13}$ are trivial.

Let us prove that the characteristic number $\varkappa_a^3\varkappa_{\dot a}^{11}$ is trivial. Let us consider the submanifold $K^7 \subset N^{14}$, which is dual to the characteristic class $\varkappa_a^2\varkappa_{\dot a}^5$. The normal bundle $\nu_K$ of the manifold $K^6$ is stably isomorphic to the Whitney sum $2\varkappa_a \oplus 5\varkappa_{\dot a}$. Because $w_5(K^7) = 0$, the characteristic class $\varkappa_{\dot a}^5$ is trivial. In particular, $\langle \varkappa_a\varkappa_{\dot a}^6; [K^7]\rangle = 0$. The following equation is satisfied: $\langle \varkappa_a\varkappa_{\dot a}^6; [K^7]\rangle = \langle \varkappa_a^3\varkappa_{\dot a}^{11}; [N^{14}]\rangle$.

It is proved that the characteristic number $\varkappa_a^3\varkappa_{\dot a}^{11}$ is trivial. Analogously, the characteristic number $\varkappa_a^{11}\varkappa_{\dot a}^3$ is trivial.

It is sufficiently to note, that the characteristic number $\langle \varkappa_a^7\varkappa_{\dot a}^7; [N^{14}]\rangle$ coincides with the characteristic number (6) in the lemma. Lemma 12 is proved. $\square$

**Definition 13.** Let $(g, \eta_N, \Psi; \Upsilon)$ be the standardized **D**-framed immersion in the codimension $2k$. Let us say that this standardized immersion is negligible, if its fundamental class $\pmod 2$ is in the subgroup

$$\eta_*(N^{n-2k}_{a\times\dot a}) \in D^{loc}_{n-2k}(\mathbf{I}_{a\times\dot a}; \mathbb{Z}/2) \qquad (52)$$

and is trivial.

Let us say that this standardized immersion is pure, if its fundamental class $\pmod 2$ satisfies the condition of Lemma 12, i.e. the Hurewich image of the fundamental class is in the group $\in D^{loc}_{n-2k}(\mathbf{I}_{a\times\dot a}; \mathbb{Z}/2)$ and contains no monomials $t_{a,\frac{n-2k}{2}+i} \otimes t_{\dot a,\frac{n-2k}{2}-i}$, $i \in \{\pm1; \pm2; \pm3; \pm4; \pm5; \pm6; \pm7\}$, but, probably, contains the monomial $t_{a,\frac{n-2k}{2}} \otimes t_{\dot a,\frac{n-2k}{2}}$.

We may reformulate results by Lemma 12 using Definition 18.

**Proposition 14.** *Assume an element* $[(g, \eta_N, \Psi)] \in Imm^D(n-2k, 2k)$ *is in the image of the vertical homomorphism in the diagram* (7)*. Then this element is represented by a standardized immersion, which is pure in the sense of Definition 18.*

# 5 $\mathbf{H}_{a\times\dot{a}}$–structure on $\mathbb{Z}/2^{[3]}$–framed immersion $\mathbf{J}_b \times \dot{\mathbf{J}}_b$–structure on $\mathbb{Z}/2^{[4]}$–framed immersion

The group $\mathbf{I}_b$ is defined as the cyclic subgroup of the order 4 in the dihedral group: $\mathbf{I}_b \subset \mathbb{Z}/2^{[2]}$.

Let us define an analogous subgroup

$$i_{\mathbf{J}_b\times\dot{\mathbf{J}}_b} : \mathbf{J}_b \times \dot{\mathbf{J}}_b \subset \mathbb{Z}/2^{[4]}, \qquad (53)$$

which is isomorphic to the Cartesian product of the two cyclic groups of the order 4.

The group $\mathbb{Z}/2^{[4]}$ (the monodromy group for the 3-uple iterated self-intersection of a skew-framed immersion) is defined using the base $(\mathbf{e}_1, \dots, \mathbf{e}_8)$ in the Euclidean space $\mathbb{R}^8$. Let us denote the generators of the subgroup $\mathbf{J}_b \times \dot{\mathbf{J}}_b$ by $b$, $\dot{b}$ correspondingly. Let us describe transformations in $\mathbb{Z}/2^{[4]}$, which corresponds to each generator.

Let us consider an orthogonal base

$$\{\mathbf{f}_1 = \mathbf{x}_1, \dots, \mathbf{f}_8 = \mathbf{y}_4\}, \qquad (54)$$

which is determined by the formulas: $\mathbf{x}_i = \frac{\mathbf{e}_{2i-1}+\mathbf{e}_{2i}}{\sqrt{2}}$, $\mathbf{y}_i = \frac{\mathbf{e}_{2i-1}-\mathbf{e}_{2i}}{\sqrt{2}}$, $i = 1, \dots, 4$. The linear spaces $Lin_{\mathbf{x}} = Lin(\{\mathbf{x}_i\})$, $i = 1, \dots, 4$; $Lin_{\mathbf{y}} = Lin(\{\mathbf{y}_j\})$, $j = 1, \dots, 4$ are equipped with the $\mathbb{Z}/4 \times \mathbb{Z}/4$ action by the generators $(b_1, b_2)$:

$$b_1(\mathbf{x}_1) = \mathbf{x}_2;\ b_1(\mathbf{x}_2) = -\mathbf{x}_1,\ b_1(\mathbf{x}_3) = \mathbf{x}_4;\ b_1(\mathbf{x}_4) = -\mathbf{x}_3;$$

$$b_2(\mathbf{x}_1) = \mathbf{x}_3;\ b_1(\mathbf{x}_2) = \mathbf{x}_4,\ b_1(\mathbf{x}_3) = -\mathbf{x}_1;\ b_1(\mathbf{x}_4) = -\mathbf{x}_2;$$

$$b_1(\mathbf{y}_1) = \mathbf{y}_2;\ b_1(\mathbf{y}_2) = -\mathbf{y}_1,\ b_1(\mathbf{y}_3) = \mathbf{y}_4;\ b_1(\mathbf{y}_4) = -\mathbf{y}_3;$$

$$b_2(\mathbf{y}_1) = \mathbf{y}_3;\ b_1(\mathbf{y}_2) = \mathbf{y}_4,\ b_1(\mathbf{y}_3) = -\mathbf{y}_1;\ b_1(\mathbf{y}_4) = -\mathbf{y}_2.$$

Denote by $\chi^{[4]} : Lin_{\mathbf{x}} \to Lin_{\mathbf{y}}$ the involutive isomorphism by

$$\chi^{[4]}(\mathbf{x}_1) = (\mathbf{y}_1);\ \chi^{[4]}(\mathbf{x}_2) = (\mathbf{y}_3);\ \chi^{[4]}(\mathbf{x}_3) = (\mathbf{y}_2); chi^{[4]}(\mathbf{x}_4) = (\mathbf{y}_4).$$

The automorphism is defined the blocks involution:

$$\chi^{[4]} : Lin_{\mathbf{x}} \oplus Lin_{\mathbf{y}} \longrightarrow Lin_{\mathbf{x}} \oplus Lin_{\mathbf{y}}.$$

The involution $\chi^{[4]}$ satisfies the following property:

$$\chi^{[4]} \circ b_1 = b_2 \circ \chi^{[4]}.$$

Let us denote the subgroup $i_{\mathbf{H}_{a\times\dot{a}},\mathbf{J}_b\times\dot{\mathbf{J}}_b} : \mathbf{H}_{a\times\dot{a}} \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b$, which is the product of the diagonal subgroup, which we denote by $\mathbf{I}_b \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b$ with the elementary subgroup $\mathbb{Z}/2$ of the second factor, which is denoted by $\dot{\mathbf{J}}_d \subset \dot{\mathbf{J}}_b$.

The subgroup $\mathbf{H}_{a\times\dot{a}}$ coincides with the preimage of the subgroup $\mathbb{Z}/2 \subset \mathbb{Z}/4$ by the homomorphism

$$\omega^{[4]} : \mathbf{J}_b \times \dot{\mathbf{J}}_b \to \mathbb{Z}/4, \tag{55}$$

which is defined by the formula $(x \times y) \mapsto xy$.

Define the subgroup $i_{\mathbf{I}_{a\times\dot{a}},\mathbf{H}_{a\times\dot{a}}} : \mathbf{I}_{a\times\dot{a}} \subset \mathbf{H}_{a\times\dot{a}}$ as the kernel of the epimorphism

$$\omega^{[3]} : \mathbf{H}_{a\times\dot{a}} \to \mathbb{Z}/2, \tag{56}$$

which is defined by the formula: $(x \times y) \mapsto x$ using generators $x, y$ of the group $\mathbf{J}_b \times \dot{\mathbf{J}}_b$.

Let us consider the inclusion $\mathbb{Z}/2^{[3]} \subset \mathbb{Z}/2^{[4]}$, which is defined by the transformation of the space $Lin(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2)$. The inclusion $i_{\mathbf{H}_{a\times\dot{a}}} : \mathbf{H}_{a\times\dot{a}} \subset \mathbb{Z}/2^{[3]}$, which is corresponded to the inclusion (53) is well defined, such that the following diagram is commutative:

$$\begin{array}{ccc}
\mathbf{I}_{a\times\dot{a}} & \stackrel{i_{a\times\dot{a}}}{\longrightarrow} & \mathbb{Z}/2^{[2]} \\
i_{a\times\dot{a},\mathbf{H}_{a\times\dot{a}}} \downarrow & & i^{[3]} \downarrow \\
\mathbf{H}_{a\times\dot{a}} & \stackrel{i_{\mathbf{H}_{a\times\dot{a}}}}{\longrightarrow} & \mathbb{Z}/2^{[3]} \\
i_{\mathbf{H}_{a\times\dot{a}},\mathbf{J}_b\times\dot{\mathbf{J}}_b} \downarrow & & i^{[4]} \downarrow \\
\mathbf{J}_b \times \dot{\mathbf{J}}_b & \stackrel{i_{\mathbf{J}_b\times\dot{\mathbf{J}}_b}}{\longrightarrow} & \mathbb{Z}/2^{[4]}.
\end{array} \tag{57}$$

Let us define automorphisms of the order 2:

$$\chi^{[3]} : \mathbf{H}_{a\times\dot{a}} \to \mathbf{H}_{a\times\dot{a}}, \tag{58}$$

$$\chi^{[4]} : \mathbf{J}_b \times \dot{\mathbf{J}}_b \to \mathbf{J}_b \times \dot{\mathbf{J}}_b, \tag{59}$$

and

$$\chi^{[3]} : \mathbb{Z}/2^{[3]} \to \mathbb{Z}/2^{[3]}, \tag{60}$$

$$\chi^{[4]} : \mathbb{Z}/2^{[4]} \to \mathbb{Z}/2^{[4]}, \tag{61}$$

which are marked with a loss of the strictness.

The following triple of $\mathbb{Z}$-extensions of the group $\mathbf{J}_b \times \dot{\mathbf{J}}_b$, defined below, is required. The group $\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}$ was defined above by the formula (26). Analogously, the groups

$$\mathbf{H}_{a\times\dot{a}} \rtimes_{\chi^{[3]}} \mathbb{Z}, \tag{62}$$

$$(\mathbf{J}_b \times \dot{\mathbf{J}}_b) \rtimes_{\chi^{[4]}} \mathbb{Z}, \tag{63}$$

are defined as semi-direct products of the corresponding groups, equipped by automorphisms, with the group $\mathbb{Z}$.

The classifying space $K(\mathbf{H}_{a\times\dot a} \rtimes_{\chi^{[3]}} \mathbb{Z}, 1)$ is a skew-product of the circle $S^1$ with the space $K(\mathbf{H}_{a\times\dot a}, 1)$, moreover, the mapping $K(\mathbf{H}_{a\times\dot a}, 1) \to K(\mathbf{H}_{a\times\dot a}, 1)$, which corresponds to a shift of the cyclic covering over $K(\mathbf{H}_{a\times\dot a} \rtimes_{\chi^{[3]}} \mathbb{Z}, 1)$, is defined by the involution, which is induced by the automorphism $\chi^{[3]}$. The definition of the group (63) is totally analogous.

### 5.0.1 Laurent extensions

For the second and third lines of the diagram (57) let us define the primary and secondary Laurent extensions, which are self-conjugated by by the automorphism $\chi^{[3]}$ and $\chi^{[4]}$ correspondingly. Let us denote the primary extension on the group $\mathbf{H}_{a\times\dot a}$ by the $\mathbb{Z}/2\times\mathbb{Z}/2$ automorphism $\mu^{(3)}_b : \mathbf{H}_{a\times\dot a} \to \mathbf{H}_{a\times\dot a}$, which inverses the cyclic generator and is given by the complex conjugation over the coordinates.

$$\mathbf{H}_{a\times\dot a} \rtimes_{\mu^{(3)}} \mathbb{Z}\times\dot{\mathbb{Z}}. \tag{64}$$

Let us denote the secondary extension with the analogous property by

$$\rtimes_{\mu^{(4)}_b}, \quad \rtimes_{\mu^{(4)}_{\dot b}}.$$

The Laurent expansion on the factors $\mathbf{J}_b$, $\dot{\mathbf{J}}_b$ are determined by the automorphisms The Laurent extension on the factor $\mathbf{J}_b$ is determined by the automorphism $\mu^{(4)}_b : \mathbf{J}_b \to \mathbf{J}_b$, $\mu^{(4)}_{\dot b} : \dot{\mathbf{J}}_b \to \dot{\mathbf{J}}_b$, which inverse the generators. The secondary Laurent extension itself is defined as a secondary extension of the primary extension (64), and is denoted by

$$\mathbf{J}_b\times\dot{\mathbf{J}}_b \rtimes_{\mu^{(3)},\mu^{(4)}_{b\times\dot b}} ((\mathbb{Z}\times\dot{\mathbb{Z}})\times(\mathbb{Z}\times\dot{\mathbb{Z}})). \tag{65}$$

The primary extension is $\chi^{[3]}$-equivariant, we get the total primary extension:

$$\mathbf{H}_{a\times\dot a} \rtimes_{\mu^{(3)}} \mathbb{Z}\times\dot{\mathbb{Z}} \rtimes_{\chi^{[3]}} \mathbb{Z}. \tag{66}$$

Let us define the total secondary extension:

$$\mathbf{J}_b\times\dot{\mathbf{J}}_b \rtimes_{\mu^{(3)},\mu^{(4)}_{b\times\dot b}} ((\mathbb{Z}\times\mathbb{Z})\times(\dot{\mathbb{Z}}\times\dot{\mathbb{Z}})) \rtimes_{\chi^{[4]}} \mathbb{Z}. \tag{67}$$

The Laurent extensions (66), and (67) are naturally represented into $\mathbb{Z}/2^{[4]}$. The diagram (57) is a subdiagram in the following diagram:

$$\begin{array}{ccc}
\mathbf{I}_{a\times\dot a} \rtimes_{\chi^{[2]}} \mathbb{Z} & \stackrel{i_{a\times\dot a}}{\longrightarrow} & \mathbb{Z}/2^{[2]} \\
i_{a\times\dot a,\mathbf{H}_{a\times\dot a}} \downarrow & & i^{[3]}\downarrow \\
\mathbf{H}_{a\times\dot a} \rtimes_{\mu^{(3)};\chi^{[3]}} (\mathbb{Z}\times\mathbb{Z}) \rtimes \mathbb{Z} & \stackrel{i_{\mathbf{H}_{a\times\dot a}}}{\longrightarrow} & \mathbb{Z}/2^{[3]} \\
i_{\mathbf{H}_{a\times\dot a},\mathbf{J}_b\times\dot{\mathbf{J}}_b} \downarrow & & i^{[4]}\downarrow \\
\mathbf{J}_b\times\dot{\mathbf{J}}_b \rtimes_{\mu^{(3)},\mu^{(4)}_{b\times\dot b};\chi^{[4]}} ((\mathbb{Z}\times\mathbb{Z})\times(\dot{\mathbb{Z}}\times\dot{\mathbb{Z}})) \rtimes \mathbb{Z} & \stackrel{i_{\mathbf{J}_b\times\dot{\mathbf{J}}_b}}{\longrightarrow} & \mathbb{Z}/2^{[4]}.
\end{array} \tag{68}$$

The primary resolution representation in the middle row of the diagram (68) is following. The primary generators of $\mathbb{Z}$ (of $\dot{\mathbb{Z}}$) is represented into $Lin(\mathbf{x}_1,\mathbf{x}_2)$ (into $Lin(\mathbf{y}_1,\mathbf{y}_2)$) by the element $A$, the reflection of the vector $\mathbf{x}_1$ (of the vector $\mathbf{y}_1$). The secondary representation in the third row of the diagram (68) extends the primary representation. This representation coincides with the direct product of the two primary representations over the secondary covering in the pair of the planes $Lin(\mathbf{x}_1,\mathbf{x}_2)$, $(Lin(\mathbf{y}_1,\mathbf{y}_2))$. The secondary generator of $\mathbb{Z}$ (of $\dot{\mathbb{Z}}$) is represented into $Lin(\mathbf{x}_1,\mathbf{x}_3)$ (into $Lin(\mathbf{y}_1,\mathbf{y}_3)$) by the antipodal reflection of the base vectors $\mathbf{x}_1$, $\mathbf{x}_3$ ($\mathbf{y}_1$, $\mathbf{y}_3$) correspondingly. The representation corresponds to the description in Subsubsection 8.2.4 and the construction of extensions in Subsubsection 8.2.6.

We will need only short secondary extension, this extension is defined by the forgetful mapping

$$\begin{gathered} Forg : \mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu^{(3)},\mu^{(4)}_{b\times\dot{b}}} ((\mathbb{Z}\times\mathbb{Z})\times(\dot{\mathbb{Z}}\times\dot{\mathbb{Z}})) \rtimes_{\chi^{[4]}} \mathbb{Z} \longrightarrow \\ \mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu^{(4)}_{b\times\dot{b}}} (\mathbb{Z}\times\dot{\mathbb{Z}}) \rtimes_{\chi^{[4]}} \mathbb{Z}, \end{gathered} \tag{69}$$

when the generators $\mathbb{Z}\times\dot{\mathbb{Z}}$ of the primary extension is omitted, when the representation itself in (68) is kept. The $\mathbb{Z}/2$–reductions, which are associated with the representations $\chi^{[3]}$, $\chi^{[4]}$ are required. Recall, there is a defect of the integer extensions (62), (63) the $\chi$-parametrization is not over $S^1$ but also over the projective plane $\mathbb{R}\mathrm{P}^2$. At the final step of the construction the defect is eliminated and only the integer $\chi$-reduction is required. But, in the construction this (minimal dimensional) $\mathbb{Z}/2$-parametrization is assumed.

#### 5.0.2 Complex line bundle $\beta_{b\times\dot{b}}$

Let us define the following $\mathbb{C}$-fibred bundle over the classifying space of the image (69):

$$\beta_{b\times\dot{b}}, \tag{70}$$

this classifying space is denoted by (71), using the automorphisms $\mu^{(4)}_{b\times\dot{b}}$, $\chi^{[4]}$.

Describe explicitly finite-dimensional skeleton of the classifying space

$$B(\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu^{(4)}_{b\times\dot{b}}} \mathbb{Z}\times\mathbb{Z} \rtimes_{\chi^{[4]}} \mathbb{Z}). \tag{71}$$

Firstly, define this bundle over the subspace of the subgroup (53).

Take the product $\mathbb{R}\mathrm{P}^N/\mathbf{i} \times \mathbb{R}\mathrm{P}^N/\mathbf{i}$, $N$ is an arbitrary great odd number. The automorphism $\mu^{(4)}_{b\times\dot{b}}$ corresponds to the pairs of involutions (a $\mathbb{Z}/2\times\mathbb{Z}/2$-equivariant mapping) by the conjugation of the factor:

$$F\mu^{(4)}_{b\times\dot{b}} : \mathbb{R}\mathrm{P}^N/\mathbf{i} \times \mathbb{R}\mathrm{P}^N/\mathbf{i} \longrightarrow \mathbb{R}\mathrm{P}^N/\mathbf{i} \times \mathbb{R}\mathrm{P}^N/\mathbf{i}. \tag{72}$$

Take the torus of the mapping (72), which is skew-product over $T^2 = S^1\times S^1$ with the layer $\mathbb{R}\mathrm{P}^N/\mathbf{i}\times\mathbb{R}\mathrm{P}^N/\mathbf{i}$. Take an additional involution $I$ on $\mathbb{R}\mathrm{P}^N/\mathbf{i}\times\mathbb{R}\mathrm{P}^N/\mathbf{i}$, which permutes the factors. This involution extends the pair of involutions $(\mu^{(4)}_{b\times\dot{b}})$ into the triple of pairwise commuted involutions $(\mu^{(4)}_{b\times\dot{b}}, I)$. This extra involution $I$ determines the involution on the mapping torus and the following fibred space, which is fibred over 3-dimensional cylinder $T^2 \rtimes_I S^1$:

$$(\mathbb{RP}^N/\mathbf{i} \times \mathbb{RP}^N/\mathbf{i}) \rtimes_{\mu_{b\times\dot{b}}} (T^2 \rtimes_I S^1). \tag{73}$$

Below in Section 6 this fibred space will be used to determine the corresponding local coefficients system, with inversions of cyclic factors (TypeII).

By the generator $I$ the generator $b \in \mathbf{J}_b$ of the subgroup (53) is transformed into the generator $\dot{b} \in \dot{\mathbf{J}}_b$; the complex conjugation $\mu_b$ is transformed into the complex conjugation $\mu_{\dot{b}}$. The bundle (70) over $2N$-skeleton of the space (71) is defined as following. Over the subspace $\mathbb{RP}^N/\mathbf{i} \times \mathbb{RP}^N/\mathbf{i}$ this bundle is the standard $\mathbf{I}_b \times \mathbf{I_b}$-bundle over generators.

#### 5.0.3 Local coefficients and homology groups

Let us consider the homology group $H_i(K((\mathbf{J}_b \times \dot{\mathbf{J}}_b),1);\mathbb{Z})$. In particular, for an odd $i$ the group contains the fundamental classes of the manifolds

$$S^i/\mathbf{i} \times pt \subset S^i/\mathbf{i} \times S^i/\mathbf{i} \subset K(\mathbf{J}_b,1) \times K(\dot{\mathbf{J}}_b,1),$$

$$pt \times S^i/\mathbf{i} \subset S^i/\mathbf{i} \times S^i/\mathbf{i} \subset K(\mathbf{J}_b,1) \times K(\dot{\mathbf{J}}_b,1),$$

which are denoted by

$$t_{b,i}, \in H_i(K(\mathbf{J}_b,1)); \quad t_{\dot{b},i'} \in H_{i'}(K(\dot{\mathbf{J}}_b,1)). \tag{74}$$

A complete description of the group $H_i(K((\mathbf{J}_b \times \dot{\mathbf{J}}_b),1);\mathbb{Z})$ is possible using the Kunneth formula, as in the case (29). A description of the standard base of the group $H_i(K(\mathbf{J}_{b\times\dot{b}},1);\mathbb{Z})$ is described by:

$$\begin{array}{c} 0 \to \bigoplus_{i_1+i_2=i} H_{i_1}(K(\mathbf{J}_b,1);\mathbb{Z}) \otimes H_{i_2}(K(\dot{\mathbf{J}}_b,1);\mathbb{Z}) \longrightarrow \\ \longrightarrow H_i(K(\mathbf{J}_{b\times\dot{b}},1);\mathbb{Z}) \longrightarrow \\ \longrightarrow \bigoplus_{i_1+i_2=i-1} Tor^{\mathbb{Z}}(H_{i_1}(K(\mathbf{J}_b,1);\mathbb{Z}), H_{i_2}(K(\dot{\mathbf{J}}_b,1);\mathbb{Z}) \to 0. \end{array} \tag{75}$$

Let us define the following local integer homology groups

$$H_i^{loc}(K((\mathbf{J}_b \times \dot{\mathbf{J}}_b) \rtimes_{\mu^{(4)}_{b\times\dot{b}}} (\mathbb{Z}\times\mathbb{Z}),1);\mathbb{Z}). \tag{76}$$

which was claimed as homology with interior system coefficients. Local cycles in (76) are represented by oriented chains of the classifying space of the group (64) equipped with local paths to a marked point. An analogous more simple system of interior local coefficients for the single generator was investigated in [A-P].

Type I. The local $\mathbb{Z}\times\dot{\mathbb{Z}}\rtimes\mathbb{Z}$-coefficients system $\mu^{(4)}_{b\times\dot{b};I}, \chi^{(4)}$ acts along all the generators $\mu_b$, $\mu_{\dot{b}}, \chi^{(4)}$ by preserving orientation of local chains. In particular, the generators of the local system acts (along paths) to global cycle as following: the generator $t_{b,i}$ for $i \equiv 1 \pmod 4$ is changed into the opposite along the path $\mu_b$ and is preserved for $i \equiv 1 \pmod 4$, the generator $t_{b,i}$ is preserved along the path $\mu_{\dot{b}}$; the generators $t_{\dot{b},i'}$ for $i' \equiv 1 \pmod 4$ is changed into the opposite along the path $\mu_{\dot{b}}$ and is preserved for $i' \equiv 3 \pmod 4$; the generator $t_{\dot{b},i'}$ along the path $\mu_b$ is preserved. Because the homology group is described using generators $t_{b,i}$, $t_{\dot{b},i'}$ by (75), a global action of the local system is well-defined.

Type II. The local $\mathbb{Z} \times \dot{\mathbb{Z}} \rtimes \mathbb{Z}$-coefficients system $\mu^{(4)}_{b\times\dot{b};IIb}, \chi^{(4)}$ is determined by the same monodromy of cycles as Type I. The difference is the following: the orientation of chains is changed along the path $\mu_b$ and along the path $\mu_{\dot{b}}$ and is preserved along the simplest basic path of $\chi^{(4)}$. The monodromy of the system is described using generators as following: $t_{b,i}$, $i \equiv 1 \pmod 4$ keeps the sign along $\mu_b$ and changes the sign for $i \equiv 3 \pmod 4$ along $\mu_b$, the generator keeps the sign along $\mu_{\dot{b}}$; for the generators $t_{\dot{b},i'}$ the formula is analogous.

Let us determines classifying space for Type I local system. Consider a skeleton of the space $B(\mathbf{J}_b) \rtimes_{\mu_b} S^1 \times B(\dot{\mathbf{J}}_b) \rtimes_{\mu_{\dot{b}}} S^1 \rtimes_{\chi^{(4)}} S^1$, which is realized by means of the Laurent extension of the structured group $\mathbf{J}_{b\times\dot{b}} \subset O(4+4)$, using (57),(68) of a bundle over the corresponding Grasmann manifold $Gr_{\mathbf{J}_{b\times\dot{b}}}(4{+}4,n)$ of non-oriented $4+4=8$-plans in $n$–space. Denote this Grasmann manifold (analogously to (212) by

$$KK(\mathbf{J}_{b\times\dot{b}}) \subset K(\mathbf{J}_{b\times\dot{b}},1) \rtimes_{\mu_{b\times\dot{b}}} (S^1 \times S^1). \qquad (77)$$

On the space (77) a free involution

$$\chi^{[4]} : KK(\mathbf{J}_{b\times\dot{b}}) \to KK(\mathbf{J}_{b\times\dot{b}}), \qquad (78)$$

acts, this involution corresponds to the automorphism (59) and represents by a permutation of planes of layers (by a block-antidiagonal matrix). The space for Type I system is constructed.

Let us determines classifying space for Type II local system. This is an analogous construction, but, the extended Grasmann manifold $Gr_{\mathbf{J}_{b\times\dot{b}}}(4+4+2,n)$ is required. The 2-dimensional extension relates with the bundle (70). The space for Type II system is constructed.

Recall a general facts concerning constructed local coefficients system. Take a market point $pt$ in (77) and take the image $\chi^{[4]}(pt) = pt^t$ of this point by (78), take a path $\lambda$, which joins $pt$ and $\chi^{[4]}(pt)$. Using the path $\lambda$ a conjugated pair of local systems is defined:

$$\chi^{[4]} : \mu^{loc}_{b\times\dot{b}} \mapsto \mu^{loc}_{\dot{b}\times b}. \qquad (79)$$

The left system in the pair admits the corresponding homology group (76), the right system admits the conjugated homology group:

$$H_i^{loc}(K((\dot{\mathbf{J}}_b \times \mathbf{J}_b) \rtimes_{\mu^{(4)}_{\dot{b}\times b}} (\mathbb{Z} \times \dot{\mathbb{Z}}),1);\mathbb{Z}). \qquad (80)$$

A pair of conjugated local systems (80) is well defined up to an equivalent relation.

The following lemma is analogous to Lemma 10.

**Lemma 15.** *For the local system Type I, Type II the following inclusion is well defined:*

$$\bigoplus_{i_1+i_2=2s} H_{i_1}(K(\mathbf{J}_b,1);\mathbb{Z}) \otimes H_{i_2}(K(\dot{\mathbf{J}}_b,1);\mathbb{Z}) \subset$$
$$H_{2s}^{loc}(K((\mathbf{J}_b \times \dot{\mathbf{J}}_b) \rtimes_{\mu^{(4)}_{b\times\dot{b}},\chi^{(4)}} (\mathbb{Z}\times\dot{\mathbb{Z}}) \rtimes \mathbb{Z},1);\mathbb{Z}).$$

**Proof of Lemma 15**

The lemma is analogous to Lemma 10. □

# Standardized $\mathbf{J}_b \times \dot{\mathbf{J}}_b$-immersions

Let us formulate notions of standardized (and pre-standardized) $\mathbf{J}_b \times \dot{\mathbf{J}}_b$-immersion.

Let us consider a $\mathbb{Z}/2^{[4]}$-framed immersion $(g,\Psi,\eta_N)$ of the codimension $8k$. Assume that the image of the immersion $g$ belongs to a regular neighborhood of an embedding $I : S^1_b \times S^1_{\dot{b}} \rtimes_{\chi^{[4]}} S^1 \subset \mathbb{R}^n$.

In this formula the third factor $S^1$ is not a direct, the monodromy (an involution) of a layer $S^1_b \times S^1_{\dot{b}}$ along the exterior circle is given by the permutation of the factors. Denote by $U \subset \mathbb{R}^n$ a regular thin neighborhood of the embedding $I$.

Let us consider a $\mathbb{Z}/2^{[4]}$–framed immersion $g : N^{n-8k} \looparrowright U \subset \mathbb{R}^n$, for which the following condition (Y) of a control of the structured group of the normal bundle is satisfied.

## Condition (Y)

The immersion $g$ admits a reduction of a symmetric structured group to the group (69). Additionally, the projection $\pi \circ g : N^{n-8k} \to S^1_b \times S^1_{\dot{b}} \rtimes_{\chi^{[4]}} S^1$ of this immersion onto the central manifold of $U$ is agreed with the projection of the structured group $(\mathbb{Z}\times\dot{\mathbb{Z}}) \rtimes \mathbb{Z}$ onto the factors of the extension.

Formally, a weaker condition is following.

## Condition (Y1)

Assume an immersion $g$ admits a reduction of a symmetric structured group to the group (69).

**Definition 16.** Let us say that a $\mathbb{Z}/2^{[4]}$-framed immersion $(g,\Psi,\eta_N)$ of the codimension $8k$ is standardized if Condition Y is satisfied.

Let us say that $\mathbb{Z}/2^{[4]}$-framed immersion $(g,\Psi,\eta_N)$ of the codimension $8k$ is pre-standardized if the definition above week Condition (Y1) instead of Condition (Y) is formulated. Classification problems of standardized and of pre-standardized immersions are equivalent.

Standardized $\mathbb{Z}/2^{[4]}$-framed immersions generate a cobordism group, this group is naturally mapped into $Imm^{\mathbb{Z}/4^{[4]}}(n-8k, 8k)$ when a standardization is omitted.

The following definition is analogous to Definition 18. In this definition a projection of the Hurewich image of a fundamental class onto a corresponding subgroup is defined using Lemma 15.

**Definition 17.** Let $(g, \eta_N, \Psi)$ be a standardized (a pre-standardized) $\mathbb{Z}/2^{[4]}$-framed immersion in the codimension $8k$.

Let us say that this standardized immersion is pure, if the Hurewich image of a fundamental class in $\in D^{loc}_{n-8k}(\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu_{b\times\dot{b}}} (\mathbb{Z} \times \dot{\mathbb{Z}}); \mathbb{Z}/2)$ contains not monomial $t_{b,\frac{n-8k}{2}+i} \otimes t_{\dot{b},\frac{n-8k}{2}-i}$, $i \in \{\pm1; \pm2; \pm3; \pm4; \pm5; \pm6; \pm7\}$, but, probably, contains the only non-trivial monomial $t_{b,\frac{n-8k}{2}} \otimes t_{\dot{b},\frac{n-8k}{2}}$.

**Definition 18.** Let us say that a $\mathbb{Z}/2^{[4]}$-framed immersion $(g, \eta_N, \Psi)$ is negligible immersion, if its $\mathbb{Z}/2^{[2]}$-cover $(f, \Xi, \eta)$ is a negligible standardized $D$-framed immersion in the sense of Definition 18.

## 5.1 Dense Principle for formal self-intersected immersions

Let $f : M^{n-k} \looparrowright \mathbb{R}^n$ be an immersion, $k \geq 1$. Let us consider the associated equivariant mapping of Cartesian products: $f \times f : M \times M \to \mathbb{R}^n \times \mathbb{R}^n$, this mapping is also an immersion outside the diagonal in the origin. The inverse image of the diagonal $(f \times f)^{-1}(\mathrm{diag}_{\mathbb{R}^n})$ outside the diagonal $\mathrm{diag}_M$ is a submanifold denoted by $\overline{\Delta}(f)$ (previously the denotation $\overline{N}$ was used).

Let us consider a sufficiently small $\varepsilon_2$ neighbourhood $U_{\varepsilon_2}(\mathrm{diag}_{\mathbb{R}^n})$ of the diagonal $\mathrm{diag}_{\mathbb{R}^n}$, such that the inverse image $(f \times f)^{-1}(\mathrm{diag}_{\mathbb{R}^n})$ has no common points with $\overline{\Delta}(f)$. Then take a sufficiently small $\varepsilon_1$–neighbourhood $U_{\varepsilon_1}(\mathrm{diag}_M)$ of the diagonal $\mathrm{diag}_M$, such that $f \times f(U_{\varepsilon_1}(\mathrm{diag}_M)) \subset U_{\varepsilon_2}(\mathrm{diag}_{\mathbb{R}^n})$.

Let us introduce the following denotation:

$$M^{(2)} = (M \times M) \setminus U_{\varepsilon_1}(\mathrm{diag}_M),$$

where the constant $\varepsilon_1$ is defined explicitly by the construction.

Let us define a class of immersions, for which Dense Principle is formulated.

**Definition 19.** Let $M^{n-k}$ be a closed manifold. Let us fix positive $\varepsilon_1$ and $\varepsilon_2$. By $T : X \times X$ $(X = M^{n-m}, X = \mathbb{R}^n)$ the involution $T(x,y) = (y,x)$ on the Cartesian product is denoted. Let us consider $T$-equivariant immersion $F : M^{(2)} \to \mathbb{R}^n \times \mathbb{R}^n$. Let us call such an immersion *pure immersion*, if

(1) $F(\partial M^{(2)}) \subset U_{\varepsilon_2}(\mathrm{diag}_{\mathbb{R}^n})$;

(2) $F(\partial M^{(2)}) \cap \mathrm{diag}_{\mathbb{R}^n} = \emptyset$.

The both conditions can be refodmulated as a common condition: $F(\partial M^{(2)}) \subset U_{\varepsilon_2}(\mathrm{diag}_{\mathbb{R}^n}) \setminus \mathrm{diag}_{\mathbb{R}^n}$.

In the considered example the restriction $f \times f$ on $M^{(2)}$ is a pure immersion.

**Proposition 20.** *Let $f_0 : M \looparrowright R$ be a smooth immersion of a compact closed manifold, assume that $R$ is equipped with a metric* dist, $\dim(M) < \dim(R)$. *Let $g : M \to R$ be a continous mapping, which is homotopic to the immersion $f_0$ (in our case $R = \mathbb{R}^n$ and this extra assumption is unrequired). Then $\forall \varepsilon > 0$ there exists an immersion $f : M \looparrowright R$, which is regular homotopic to the immersion $f_0$, for which* $\mathrm{dist}(g; f)_{C^0} < \varepsilon$ *in the metric induced by the metric* dist.

*Доказательство.* This is a reformulation of $h$-principle by Hirsh [H]. □

**Proposition 21.** *Let $(M^{(2)}, \partial M^{(2)})$ be a smooth manifold with a boundary, let $T_{M^{(2)}} : (M^{(2)}, \partial M^{(2)}) \to (M^{(2)}, \partial M^{(2)})$ be the transposition involution. Let $R^{(2)}$ be a smooth manifold, which is equipped with the metric dist, and with the transposition involution $T_{R^{(2)}} : R^{(2)} \to R^{(2)}$, which admits a smooth submanifold of fixed points $\Delta_R \subset R^{(2)}$, and* $\dim(R^{(2)}) = 2n$, $\dim(\Delta_R) = n$, $\dim(M^{(2)}) < \dim(R^{(2)})$. *We need the case $R^{(2)} = \mathbb{R}^n \times \mathbb{R}^n$. Assume that there exists a pure immersion (in the sence of Definition 19), i.e. $T_M, T_R)$ is an equivariant immersion $F_0^{(2)} : M^{(2)} \looparrowright R^{(2)}$, and, additionally, the image of the boundary has no intersection with the fixed point manifold:*

$$Im(F_0^{(2)}(\partial M^{(2)})) \subset R^{(2)} \setminus \Delta_R. \tag{81}$$

*Let $G^{(2)} : M^{(2)} \to R^{(2)}$ be a smooth $(T_{M^{(2)}}, T_{R^{(2)}})$–equivariant mapping, for which the following condition is satisfied:*

$$Im(G^{(2)}(\partial M^{(2)})) \subset R^{(2)} \setminus \Delta_R \tag{82}$$

*and which is equivariant homotopic to an immersion $F_0^{(2)}$ is the class of equivariant immersions described above. Then $\forall \varepsilon > 0$ there exists $(T_{M^{(2)}}, T_{R^{(2)}})$–equivariant pure immersion (in the sence of definition described in Theorem 19) $F_1^{(2)} : M^{(2)} \looparrowright R^{(2)}$, which is regular homotopic to an immersion $F_0^{(2)}$, for which the following conditions* $\mathrm{dist}(F_1^{(2)}; G^{(2)})_{C^0} < \varepsilon$ *in the metric, indused by the metric* dist, $R^{(2)}$.

**Proof of Proposition** 21

Let us consider a triangulation of the manifold $(M^{(2)}, \partial M^{(2)})$ of the caliber, much more smaller then $\varepsilon$. The induction over skeletons, analogously to Hirsh induction proves the proposition. □

Let us apply the Proposition 21 in the following situation. Let $(f, \varkappa, \Psi)$ be a skew-framed immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$. Let us consider an open manifold $M^{n-k} \times M^{n-k} \setminus \Delta_M$, which is equipped with the involution $T_M$. Denote by $M^{(2)}$ a spherical blow-up of the manifold $M^{n-k} \times M^{n-k} \setminus \Delta_M$ along the diagonal, equipped with the free involution $T_M^{(2)}$. (A spherical blow-up is homeomorphic to $M \times M$ without a small regular delated $T_M^{(2)}$-equivariant neighborhood of the diagonal. The boundary of the manifold $M^{(2)}$ is a fibred space of the spherical $S^{n-k-1}$ -bundle over $M^{n-k}$. Let us denote by $\hat{M}^{(2)}$ a quatient $M^{(2)}/T^{(2)}_M$. The boundary $\partial \hat{M}^{(2)}$ coinsids with the projectivisation $TP(M^{n-k})$ of the tangent bundle $T(M^{n-k})$. Let us denote $\hat{M}^{(2)} \setminus \partial \hat{M}^{(2)}$ by $\hat{M}^{(2)}_\circ$.

Let $(\mathbb{R}^n)^{(2)}$ be a manifold $\mathbb{R}^n \times \mathbb{R}^n$, equipped with the involution $T^{(2)}_{\mathbb{R}^n}$. The following mapping of configuration spaces

$$f^{(2)} : M^{(2)} \to \mathbb{R}^n \times \mathbb{R}^n, \tag{83}$$

which is a $(T^{(2)}_M, T^{(2)}_R)$–equivariant immersion is well-defined. The equivariant immersion (83) satisfies an analogous condition, as for $F_0$ in Proposition 21.

Let us use a skew-framed framing $(\varkappa, \Psi)$ for the immersion $f$ to compute the normal bundle $\nu_{f^{(2)}}$ of the immersion $f^{(2)}$. Evidently, the immersion $\Psi$ induces the isomorphism:

$$\bar{\Psi}^{(2)} : \nu_{f^{(2)}} = k(\varkappa_1 \oplus \varkappa_2), \tag{84}$$

where $\varkappa_i$ is a linear bundle, assotiated with the immersion of the $i$-th factor $i = 1, 2$.

The involution $T^{(2)}_M$ is covered by an involutive authomorphism of the bundle $\nu_{f^{(2)}}$ (which is not the identity on the base), which permutes the corresponding lines in the right hand side of the formula 84). Therefore a corresponding vector byndle over $M^{(2)}$ is well-defined, let us denote this bundle by $\nu_{f^{(2)}}$. The isomorphism (84) induces the isomorphism

$$\Psi^{(2)} : \nu^{(2)}_{\hat{f}} = k(\eta), \tag{85}$$

where $\eta$ is the 2-dimensional vector bundle with the structuring group $\mathbf{D}$.

Let us assume that the mapping (83) is transversal along the diagonal $\Delta_{\mathbb{R}^n} \subset \mathbb{R}^n \times \mathbb{R}^n$ and consider the inverce image $(f^{(2)})^{-1}(\Delta_{\mathbb{R}^n})$ of this diagonal, which is denoted by $\bar{N} \subset \bar{M}^{(2)}$. It is easy to check that $\bar{N}$ is a closed manifold of the dimension $n - 2k$, which in the holonomic case coinsids with the manifold, defined by the formula (**??**). The manifold $\bar{N}$ is equipped with a free involution $T^{(2)}_M|_{\bar{N}} : \bar{N} \to \bar{N}$, the factormanifold is denoted by $N^{n-2k}$. The manifold $N^{n-2k}$ is closed and coinsids with the manifold, defined by the formula (**??**).

This new definition is a more general, because this makes sence without assumption that the equivariant mapping $\bar{f}^{(2)}$ is holonomic. This definition is possible for $(T^{(2)}_M, T^{(2)}_R)$–equivariant mappings, which satisfies the condition (81), from a regular equivariant homotopy class of the mapping $f^{(2)}$. The immersion $\bar{N}^{n-2k} \looparrowright M^{n-k}$ and its restriction $f|_{\bar{N}^{n-2k}} : \bar{N}^{n-2k} \looparrowright \mathbb{R}^n$ are well-defined. The normal bundle of the mapping $f|_{\bar{N}^{n-2k}}$ is isomorphic to the restriction of the bundle $\nu_{f^{(2)}}$ on the submanifold $\bar{N} \subset M^{(2)}$.

Let us consider the mapping $g : N^{n-2k} \looparrowright \mathbb{R}^n$, which is defined as a factormapping of the restriction of $f^{(2)}$ on $\bar{N}$. From the transversality condition for $f^{(2)}$ along the diagonal we get that $g$ is a local embedding, i.e. an immersion. The normal bundle $\nu_g$ of the immersion $g$ is naturally isomorphic to the restriction of the bundle $\nu_f^{(2)}$ on the submanifold $N^{n-2k} \subset M^{(2)}$. The dihedral framing, which is defined by the formula (85), coinsids with a dihedral framing $\Psi$.

Assume that an arbitrary $(T_M^{(2)}, T_{\mathbb{R}^n}^{(2)})$–equivariant immersion $G^{(2)} : M^{(2)} \looparrowright \mathbb{R}^n \times \mathbb{R}^n$, which satisfies the condition (82), and is equivariant homotopic to the immersion (83) with a prescribed condition is well-defined. In particular, the isomorphism (85) is preserved by a regular homotopy. Assume that a regular homotopy keeps the condition (82) and is transwersal along $\Delta_{\mathbb{R}^n} \subset \mathbb{R}^n \times \mathbb{R}^n$. Let us consider the manifold $N^{n-2k}(G)$ in the target. This manifold is closed, and an immersion $g(G) : N^{n-2k}(G) \looparrowright \Delta_{\mathbb{R}^n} = \mathbb{R}^n$ is well-defined as the restriction of $G$. The normal bundle of this immersion, which is denoted by $\nu_{g(G)}$, is isomorphic to the restriuction of the nornmal bundle of the immersion $G/T_M$. The normal bundle $\nu_{g(G)}$ is equipped with a dihedral framing, which is defined by the formula(85). Therefore the following **D**–framed immersuion $(g(G), \eta(G), \Psi(G))$, which is regular homotopic to the **D**–framed immersion $(g, \eta, \Psi)$ of the self-intersection manifold of the origine skew-framed immersion $(f, \varkappa, \Xi)$ is well-defined. The constructed **D**-framed immersion $(g, \eta, \Psi)$ satisfies additional properties, which are defined explicitely strating from the immersion (83).

## 5.2 Dense Principle for formal iterated self-intersected immersions

### Formal regular homotopy

Let $g : N \looparrowright \mathbb{R}^n$ be an arbitrary immersion. This immersion induces a $(T_N, T_{\mathbb{R}^n})$-equivariant mapping $g^{(2)} : N^{(2)} = N \times N \looparrowright \mathbb{R}^n \times \mathbb{R}^n$, where $T_N$ is a permutation of points in the origin, $T_{\mathbb{R}^n}$ is a permutation in the target. We will call that $F_t^{(2)}$ : $N \times N \looparrowright \mathbb{R}^n \times \mathbb{R}^n$ is a formal generic regular deformation of $g^{(2)}$ $F_0^{(2)} = g^{(2)}$ and the following conditions are satisfied:

–1. a support $supp(F^{(2)}) \subset N \times N$ of $F^{(2)}$ is outside the diagonal $\Delta_N \subset N \times N$;

–2. $F^{(2)}$ is a regular $(T_N, T_{\mathbb{R}^n})$-equivariant homotopy on $supp(F)$;

–3. $F^{(2)}$ is regular along $\Delta_{\mathbb{R}^n}$.

For an arbitrary formal regular homotopy of an immersion $g$ to a formal immersion $g_1^{(2)}$ the self-intersection manifold $L$, $h : L \looparrowright \mathbb{R}^n$, of $g$ is regular cobordant to the manifold $L_1$ of formal self-intersection of $g_1^{(2)}$, which is defined by $(g_1^{(2)})^{-1}(\Delta_{\mathbb{R}^n})/T_N$. For generic $F^{(2)}$ the manifold $L_1$ is equipped with a parametrized immersion $h_1 : L_1 \looparrowright \mathbb{R}^n$ and the immersions $h$ and $h_1$ of manifolds $L$ and $L_1$ are regular cobordant. In the case $g$ is a $\mathbb{Z}/2^{[s]}$-framed immersion, the immersion $h$ is $\mathbb{Z}/2^{[s+1]}$-framed and $L_1$ and the cobordism between $L$ and $L_1$ are also $\mathbb{Z}/2^{[s+1]}$-framed. This statement is a straghtforward generalisation of the construction for skew-framed immersion, described in 5.1 to the case of $\mathbb{Z}/2^{[s]}$-framed immersions.

We need a generalisation of the construction described in Subsection 5.1

and we will consider a formal self-intersection immersed manifold of a formal self-intersection immersed manifold of an original immersion $g$ in a common construction.

The immersion $g$ induces a $(T^{(2)}_N, T^{(2)}_{\mathbb{R}^n})$- equivariant mapping $g^{(2+2)} : (N \times N) \times (N \times N) \looparrowright (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)$, where $T^{(2)}_N$ is the involution of involutions by permutation of pairs (in a common bracket) of points in the origin, $T^{(2)}_{\mathbb{R}^n}$ is a permutation of pairs of points in the target. The restriction $g^{(2+2)}$ on a formal self-intersection manofold of $g^{(2)}$ is an immersion and we may apply the iteration of the definition to consider a formal self-intersection points of this restriction.

We will call that

$$F^{(2+2)}_t : (N \times N) \times (N \times N) \looparrowright (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)$$

is a formal generic regular deformation of $g^{(2+2)}$, $F^{(2+2)}_0 = g^{(2+2)}$ if the following conditions are satisfied:

–1. a support $supp(F^{(2+2)}) \subset (N \times N) \times (N \times N)$ of $F^{(2+2)}$ is outside the thick diagonal $\Delta^{(2)}_N \subset (N \times N) \times (N \times N)$;

–2. $F^{(2+2)}$ is a regular $(T^{(2)}_N, T^{(2)}_{\mathbb{R}^n})$-equivariant homotopy on $supp(F)$;

–3. $F^{(2+2)}$ is regular along $\Delta^{(2)}_{\mathbb{R}^n} \subset (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)$.

An arbitrary generic immersion $g : N \looparrowright \mathbb{R}^n$ with a self-intersection manifold $h : L \looparrowright \mathbb{R}^n$ admits an iterated self-intersection manifold $f : M \looparrowright \mathbb{R}^n$ which is a self-intersection manifold of $h$. For an arbitrary formal regular homotopy $F^{(2+2)}$ of the extension $g^{(2+2)}$ $g$ to a formal immersion $g^{(2+2)}_1$ the iterated self-intersection manifold $M$, $f : M \looparrowright \mathbb{R}^n$, of $g$ is regular cobordant to the manifold $M_1$ of formal iterated self-intersection of $g^{(2+2)}_1$, which is defined by $(g^{(2+2)}_1)^{-1}(\Delta^{(2)}_{\mathbb{R}^n})/T^{(2)}_N$. For generic $F^{(2+2)}$ the manifold $M_1$ is equipped with a parametrized immersion $f_1 : M_1 \looparrowright \mathbb{R}^n$ and the immersions $f$ and $f_1$ of manifolds $M$ and $M_1$ are regular cobordant. In the case $g$ is a $\mathbb{Z}/2^{[s]}$-framed immersion, the immersion $f$ is $\mathbb{Z}/2^{[s+2]}$-framed and $M_1$ and the cobordism between $M$ and $M_1$ are also $\mathbb{Z}/2^{[s+2]}$-framed.

The definition of formal homotopies (with no regular condition) should be applied for generic singular mappings.

**Theorem 22.** *Let $(g, \Psi, \eta_N)$ be a standardized pure $\mathbf{D}$-framed immersion in the codimension $2k$ (see Definitions 3, 18), $k \geq 7$. Let $(h, \Psi_L, \eta_L)$ be the $\mathbb{Z}/2^{[3]}$-framed immersion of self-intersection points of $g$; $(f, \Psi_M, \zeta_M)$ be the $\mathbb{Z}/2^{[4]}$-framed immersion of iterated self-intersection points of $g$.*

*Then in each regular homotopy class of $\mathbf{D}$-framed standardized immersions there exist $(g, \Psi, \eta)$ and a formal regular deformation $g^{(2+2)} \mapsto \bar{g}^{(2+2)}$, the iterated self-intersection manifold of $\bar{g}^{(2+2)}$ is a disjoin union of a pre-standardized pure $\mathbb{Z}/2^{[4]}$-immersion $(f, \Xi_M, \eta_M)$ (see Condition (Y1)) and a negligible $\mathbb{Z}/2^{[4]}$-immersion.*

*Moreover, the cobordism class of the of $\mathbb{Z}/2^{[4]}$-standardized immersion $(f, \Xi_M, \eta_M)$ is well-defined for $(g, \Psi, \eta_N)$ in its regular cobordism class.*

**A sketch of a proof of Theorem 22**

In Subsubsection 5.3.1 a spesial (non-generic) $PL$-mapping $d^{(2)}$, which is a ramified $\mathbb{Z}/4\times\mathbb{Z}/4$ covering, is constructed. A formal self-intersections of $d^{(2)}$ is a polyhedron, which is explisitely described in Section 8. This polyhedron admits a subpolyhedron (98), which becomes a closed component after a vertical formal deformation of the mapping $d^{(2)}$, described in Lemma 25. This closed component is a support of a characteristic class, the cohomology class, which is detected Arf-invariant, as is proved in Lemma 48. The proof itself of Theorem 22 is presented at the last part of the section, estimations of the regular codimension for $n=126$ are restricted, in the case $n=254,\dots$ calculations are not restricted. Also for $n=254,\dots$ an alternative approach using a straghforvard iteration of the Dence Principle, described in Subsection 5.1 gives simplifications, because the most Section 8 is unnesesary.

## 5.3 Proof of Theorem 22

### 5.3.1 Construction of the mapping $d^{(2)}$

Consider the standard covering $p: S^1 \to S^1$ of the degree 4. It is convenient to write-down: $p: S^1 \to S^1/\{\mathbf{i}\}$, by the quotient: $\mathbb{Z}/4 \times S^1 \to S^1/\{\mathbf{i}\}$.

Consider the join of $\frac{n'-k+1}{2} = r$-copies of the circle $S^1/\{\mathbf{i}\}$ $S^1/\{\mathbf{i}\} * \dots * S^1/\{\mathbf{i}\} = S^{n'-k}$, which is PL-homeomorphic to the standard $n'-k$-dimensional sphere. Let us define the join of $r$ copies of the mapping $p$:

$$\tilde{d} : S^1 * \dots * S^1 \to S^1/\{\mathbf{i}\} * \dots * S^1/\{\mathbf{i}\}.$$

On the pre-image acts the group $\mathbb{Z}/4$ by the diagonal action, this action is commuted with $\tilde{d}$. The mapping $\bar{d}$ is defined by the composition of the quotient $\tilde{d}/\{\mathbf{i}\} : S^{n'-k}/\{\mathbf{i}\} \to S^{n'-k}$ with the standard inclusion

$$S^{n'-k} \subset \mathbb{R}^{n'}.$$

A formal (non-holonomic, vertical) small deformation of the formal (holonomic) extension of the mapping $\bar{d}$ is the required mapping $d^{(2)}$.

### 5.3.2 Construction of the mapping $d^{(2+2)}$

The mapping $\bar{d} : \mathbb{RP}^{n'-k} \to S^{n'-k}$ has to be generalized, using two-stages tower (89) of ramified coverings.

Let us recall, that a positive integer $m_\sigma = 14$. Denote by $ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$ the Cartesian product o standard lens space $\pmod 4$, namely,

$$ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} = S^{n-\frac{n-m_\sigma}{8}+1}/\mathbf{i} \times S^{n-\frac{n-m_\sigma}{8}+1}/\mathbf{i}. \qquad (86)$$

Evidently, $\dim(ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}) = \frac{7}{4}(n+m_\sigma)+2 > n$.

On the space $ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$ a free involution $\chi_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} : ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} \to ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$ acts by the formula: $\chi_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}(x \times y) = (y \times x)$.

Let us define a subpolyhedron (a manifold with singularities) $X_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} \subset ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$. Let us consider the following family $\{X_j, \quad j = 0, 1, \dots, j_{max}\}$, $j_{max} \equiv 0 \pmod 2$, of submanifolds $ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$:

$$X_0 = S^{n - \frac{n-m_\sigma}{8} + 1}/\mathbf{i} \times S^1/\mathbf{i},$$

$$X_1 = S^{n - \frac{n-m_\sigma}{8} - 1}/\mathbf{i} \times S^3/\mathbf{i}, \quad \dots$$

$$X_j = S^{n - \frac{n-m_\sigma}{8} + 1 - 2j}/\mathbf{i} \times S^{2j+1}/\mathbf{i},$$

$$X_{j_{max}} = S^1/\mathbf{i} \times S^{n - \frac{n-m_\sigma}{8} + 1}/\mathbf{i},$$

where

$$j_{max} = \frac{7n + m_\sigma}{16} = 2^{n-1}, \quad m_\sigma = 14. \tag{87}$$

The dimension of each manifold in this family equals to $n - \frac{n-m_\sigma}{8} + 2$ and the codimension in $ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$ equals to $n - \frac{n-m_\sigma}{8}$. Let us define an embedding

$$X_j \subset ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$$

by a Cartesian product of the two standard inclusions.

Let us denote by $\chi_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} : ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} \to ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$ the involution, which permutes coordinates. Evidently, we get: $\chi_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}(X_j) = X_{j_{max}-j}$.

A polyhedron $X_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} = \bigcup_{j=0}^{j_{max}} X_j \subset ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$ is well-defined. This polyhedron is invariant with respect to the involution $\chi_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$. The polyhedron $X_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$ can be considered as a stratified manifolds with strata of the codimension 2.

The restriction of the involution $\chi_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$ on the polyhedron $X_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$ denote by $\chi_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$.

Write-down the sequence of the subgroups of the index 2 of the diagram (68):

$$\mathbf{I}_a \times \dot{\mathbf{I}}_a \longrightarrow \mathbf{H}_{a \times \dot{a}} \longrightarrow \mathbf{J}_b \times \dot{\mathbf{J}}_b. \tag{88}$$

Define the following tower of 2-sheeted coverings, which is associated with the sequence (88):

$$ZZ_{a \times \dot{a}} \longrightarrow ZZ_{\mathbf{H}_a \times \dot{a}} \longrightarrow ZZ_{\mathbf{J}_b \times \dot{\mathbf{J}}_b}. \tag{89}$$

The bottom space of the tower (90) coincides to a skeleton of the Eilenberg-MacLane space: $ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} \subset K(\mathbf{J}_a, 1) \times K(\dot{\mathbf{J}}_a, 1)$. This tower (90) determines the tower (89) by means of the inclusion $ZZ_{\mathbf{J}_b \times \dot{\mathbf{J}}_b} \subset K(\mathbf{J}_b, 1) \times K(\dot{\mathbf{J}}_b, 1)$ of the bottom.

Let us define the following tower of double coverings:

$$X_{a \times \dot{a}} \longrightarrow X_{\mathbf{H}_{b \times \dot{b}}} \longrightarrow X_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}. \tag{90}$$

The bottom space of the tower (90) is a subspace of the bottom space of the tower (89) by means of an inclusion $X_{\mathbf{J}_b \times \dot{\mathbf{J}}_b} \subset ZZ_{\mathbf{J}_b \times \dot{\mathbf{J}}_b}$. The tower (90) determines as the restriction of the tower (89) on this subspace.

Let us describe a polyhedron $X_{a\times\dot{a}} \subset ZZ_{a\times\dot{a}}$ explicitly. Let us define a family $\{X'_0, X'_1, \dots, X'_{j_{max}}\}$ of standard submanifolds in the manifold $ZZ_{a\times\dot{a}} = \mathbb{RP}^{n-\frac{n-m_\sigma}{8}+1} \times \mathbb{RP}^{n-\frac{n-m_\sigma}{8}+1}$ by the following formulas:

$$X'_0 = \mathbb{RP}^{n-\frac{n-m_\sigma}{8}+1} \times \mathbb{RP}^1 \dots \tag{91}$$

$$X'_j = \mathbb{RP}^{n-\frac{n-m_\sigma}{8}+1-2j} \times \mathbb{RP}^{2j+1} \dots$$

$$X'_{j_{max}} = \mathbb{RP}^1 \times \mathbb{RP}^{n-\frac{n-m_\sigma}{8}+1}.$$

In this formulas the integer index $j_{max}$ is defined by the formula (87). The polyhedron $X_{a\times\dot{a}} \subset ZZ_{a\times\dot{a}}$ is defined as the union of standard submanifolds in this family. The polyhedron $X_{\mathbf{H}_{a\times\dot{a}}} \subset ZZ_{\mathbf{H}_{a\times\dot{a}}}$ a quotient of the double covering, which corresponds to the tower of the groups.

The spaces $X_{\mathbf{H}_{a\times\dot{a}}}$, $X_{a\times\dot{a}}$ admit free involutions, which are pullbacks of the involution $\chi_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$ by the projection on the bottom space of the tower.

The cylinder of the involution $\chi_{a\times\dot{a}}$ is well-defined, (correspondingly, of the involution $\chi_{\mathbf{H}_{a\times\dot{a}}}$), which is denoted by $X_{a\times\dot{a}} \rtimes_\chi S^1$ (correspondingly, by $X_{\mathbf{H}_{a\times\dot{a}}} \rtimes_\chi S^1$). The each space is embedded into the corresponding fibred space over $\mathbb{RP}^2$:

$$X_{a\times\dot{a}} \rtimes_\chi S^1 \subset X_{a\times\dot{a}} \rtimes_\chi \mathbb{RP}^2,$$

$$X_{\mathbf{H}_{a\times\dot{a}}} \rtimes_\chi S^1 \subset X_{\mathbf{H}_{a\times\dot{a}}} \rtimes_\chi \mathbb{RP}^2.$$

Then let us define a polyhedron $J_{b\times\dot{b}}$, which is a base of a ramified covering $X_{a\times\dot{a}} \to J_{b\times\dot{b}}$.

Then let us extend the ramified covering over the bottom space of the tower to the ramified covering: $X_{a\times\dot{a}} \rtimes_\chi \mathbb{RP}^2 \to J_{b\times\dot{b}} \rtimes_\chi \mathbb{RP}^2$, and the ramified covering over the middle space of the tower of the ramified covering $X_{\mathbf{H}_{a\times\dot{a}}} \rtimes_\chi \mathbb{RP}^2 \to J_{b\times\dot{b}} \rtimes_\chi \mathbb{RP}^2$.

Let us define a polyhedron (a manifold with singularities) $J_{b\times\dot{b}}$. For an arbitrary $j = 0, 1, \dots, j_{max}$, where $j_{max}$ is defined by the formula (87), let us define the polyhedron $J_j = S^{n-\frac{n-m_\sigma}{8}-2j+1} \times S^{2j+1}$ (the Cartesian product). Spheres (components of this Cartesian product) $S^{n-\frac{n-m_\sigma}{8}-2j+1}$, $S^{2j+1}$ are re-denoted by $J_{j,1}$, $J_{j,2}$ correspondingly. Using this denotations, we get:

$$J_j = J_{j,1} \times J_{j,2}.$$

The standard inclusion $i_j : J_{j,1} \times J_{j,2} \subset S^{\frac{n-m_\sigma}{8}+1} \times S^{\frac{n-m_\sigma}{8}+1}$ is well-defined, each factor is included into the target sphere as the standard subsphere.

The union $\bigcup_{j=0}^{j_{max}} Im(i_j)$ of images of this embeddings is denoted by

$$J_{b\times\dot{b}} \subset S^{\frac{n-m_\sigma}{8}+1} \times S^{\frac{n-m_\sigma}{8}+1}. \tag{92}$$

The polyhedron $J_{b\times\dot{b}}$ is constructed.

Let us define a ramified covering

$$\varphi_{a\times\dot{a}} : X_{a\times\dot{a}} \to J_{b\times\dot{b}}. \tag{93}$$

The covering (93) is defined as the union of the Cartesian products of the ramified coverings, which was constructed in Subsubsection 5.3.1.

The covering (93) is factorized into the following ramified covering:

$$\varphi_{\mathbf{H}_{a\times\dot{a}}} : X_{\mathbf{H}_{a\times\dot{a}}} \to J_{b\times\dot{b}}. \tag{94}$$

Because $X_{a\times\dot{a}} \to X_{\mathbf{H}_{a\times\dot{a}}} \to J_{b\times\dot{b}}$ is a double covering, the number of sheets of the covering (94) is greater by the factor $2^r$, where $r$ is the denominator of the ramification.

The polyhedron $J_{b\times\dot{b}}$ is equipped by the involution $\chi$, which is defined analogously to the involutions $\chi_{a\times\dot{a}}$, $\chi_{\mathbf{H}_{a\times\dot{a}}}$.

The cylinder of the involution is well-defined, let us denote this cylinder by $J_{b\times\dot{b}}\rtimes_{\chi}S^1$. The inclusion $J_{b\times\dot{b}}\rtimes_{\chi}S^1 \subset J_{b\times\dot{b}}\rtimes_{\chi}\mathbb{R}\mathrm{P}^2$. is well-defined.

The ramified covering (93) commutes with the involutions $\chi_{a\times\dot{a}}$, $\chi_{\mathbf{H}_{a\times\dot{a}}}$ in the origin and the target.

Therefore the ramified covering

$$c_X : X_{a\times\dot{a}}\rtimes_{\chi}\mathbb{R}\mathrm{P}^2 \to J_{b\times\dot{b}}\rtimes_{\chi}\mathbb{R}\mathrm{P}^2, \tag{95}$$

which is factorized into the ramified covering

$$c_Y : X_{\mathbf{H}_{a\times\dot{a}}}\rtimes_{\chi}\mathbb{R}\mathrm{P}^2 \to J_{b\times\dot{b}}\rtimes_{\chi}\mathbb{R}\mathrm{P}^2. \tag{96}$$

is well-defined.

**Lemma 23.** *There exist an inclusion*

$$i : J_{b\times\dot{b}}\rtimes_{\chi}\mathbb{R}\mathrm{P}^2 \times D^5 \rtimes D^3 \subset \mathbb{R}^n, \tag{97}$$

*where $D^5$ is the standard disks (of a small radius), the disk $D^5$ is a layer of the trivial 9-bundle over $\mathbb{R}\mathrm{P}^2$; $D^3$ is the standard disks (of a small radius), the disk $D^3$ is a layer of the non-trivial twisted 3-bundle $3\varkappa_{\chi}$ over $\mathbb{R}\mathrm{P}^2$, where $\varkappa_{\chi}$ is the line bundle with the generic class in $H^1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}/2)$.*

**Proof of Lemma** 23

Put $m_\sigma = 14$, $k = 7$, $n = 126$ and prove the statement with this restriction (this case is the most complicated). The cases $n = 2^l - 2, l \geq 8$, $k = \frac{n-m_\sigma}{16}, m_\sigma = 14$ are also possible.

Let us prove that the polyhedron $J_{b\times\dot{b}}$ is embeddable into the sphere $S^{n-11}$.

The sphere $S^{n-11}$ is the unite sphere (the target of the standard projection) of an $n-10$-dimensional layer. Take a collection of embedding: $S^1 \times S^{n-13} \subset S^{n-11}$, $S^3 \times S^{n-15} \subset S^{n-11}$, $\dots$, $S^1 \times S^{n-13} \subset S^{n-11}$. Take the Euclidean coordinates $x_1, \dots, x_{n-10}$ in $\mathbb{R}^{n-10}$, where the sphere $S^{n-11} \subset \mathbb{R}^{n-10}$ is the unite sphere with the center at the origin. Each embedding $S^{2k_1+1} \times S^{2k_2+1} \subset S^{n-11}$, $k_1 + k_2 = \frac{n-14}{2}$ is the hypersurface in $S^{n-11}$ given by the equation $x_1^2 + \dots + x_{2k_1+2}^2 - x_{2k_1+3}^2 - \dots - x_{n-10}^2 = 0$. The polyhedron $J_{b\times\dot{b}}$ is $PL$-homeomorphic to the union of the tori.

The normal bundle of the embedding $emb : \mathbb{RP}^2 \subset \mathbb{R}^n$ is the Whitney sum:

$$\nu(emb) = (\frac{n-12}{2} + 1)\varepsilon \oplus (\frac{n-12}{2} + 1)\varkappa_\chi \oplus 3\varkappa_\chi \oplus 5\varepsilon.$$

because $n - 12 = 2 \pmod{16}$. The polyhedron $J_{b\times\dot{b}} \rtimes \mathbb{RP}^2$ is embedded into $(\frac{n-12}{2} + 1)\varepsilon \oplus (\frac{n-12}{2} + 1)$.

Lemma 23 is proved. □

There exist a formal deformation of the ramified covering (95), which satisfies special properties, which are formulated in Definition 24.

Let $\bar{d}_X = i \circ c_{\bar{X}} : \bar{X}_{a\times\dot{a}} \rtimes_\chi \mathbb{RP}^2 \to \mathbb{R}^n$, $\bar{d}_X^{(2+2)}$ be a formal $\mathbb{Z}/2 \times \mathbb{Z}/2$-equivariant holonomic extension of the mapping $\bar{d}_X$ on the space of $(2+2)$-configurations: the space of non-ordered pairs of non-ordered two-points. A general facts on such extensions are not required, because we will considered the only following case.

Define a formal mapping $\bar{d}_X^{(2+2))}$ as a small formal deformation of the holonomic extension $\bar{d}_X^{(2+2)}$ of the mapping $\bar{d}_X$. The deformation is a codimension 4 vertical with respect to the line bundles described in Lemma 23 and is $\chi^{(4)}$-equivariant.

On the first stage of the construction we consider the mapping $\bar{d}_Y$ $\bar{d}_Y = i \circ c_Y : Y_{\mathbf{H}_{a\times\dot{a}}} \rtimes_\chi \mathbb{RP}^2 \to \mathbb{R}^n$ and its formal $\mathbb{Z}/2$-extension $(\bar{d}_Y)^{(2)}$. On the second stage we consider the double formal extension $((\bar{d}_X)^{(2)})^{(2)}$, this extension determines an extra self-intersections over self-intersection points of the mapping $(\bar{d}_Y)^{(2)}$. This resulting polyhedron is called a polyhedron of iterated self-intersections. To define this polyhedron two steps are required and one may changes an order of the steps, if required. By this, on the first step self-intersection points of the upper polyhedron in the tower is organized. On the second step self-intersection points of of the polyhedron by the mapping $\bar{d}_Y$ is organized. Recall the towers of coverings are parametrized over $\mathbb{RP}^2$.

Then we define a special deformation (formal) $((\bar{d}_X)^{(2)})^{(2)} \mapsto d_X^{(2+2)}$. Properties of iterated self-intersections are described in Definition 24 and then proved in Lemma 25.

**Definition 24.** Let us say that a formal mapping $d_X^{(2+2)}$ (by a formal deformation of a holonomic extension) admits a bi-cyclic structure, if the following condition is satisfied: the polyhedron of iterated self-intersections is divided into two subpolyhedra: a closed subpolyhedron $\mathbf{NN}_{b\times\dot{b}}$ and a subpolyhedron $\mathbf{NN}_\circ$ with a boundary. Additionally, the following conditions, concerning a reduction of the structuring mapping are satisfied:

– 1. $\mathbf{NN}_\circ$ is neglected (the Hurewicz image of the fundamental class (which exists) has the trivial characteristic numbers).

–2. On a closed polyhedron (a component of self-intersection polyhedron is closed and does not contain boundaries singular points) $\mathbf{NN}_{b\times\dot{b}}$, outside the critical subpolyhedron $Q \subset \mathbf{NN}_{b\times\dot{b}}$ of dimension less then 32 the structuring mapping admits the following reduction:

$$\eta_{b\times\dot{b}} : \mathbf{NN}_{b\times\dot{b}} \setminus Q \to K(\mathbf{J}_{b\times\dot{b}}, 1)\rtimes_{\mu_{b\times\dot{b}},\chi^{[2]}} S^1 \times S^1 \times \mathbb{RP}^2, \qquad (98)$$

where the structuring mapping corresponds to the group (69).

Additionally, the closed subpolyhedron $\mathbf{NN}_{b\times\dot{b}}$ contains a fundamental class $[\mathbf{NN}_{b\times\dot{b}}]$. The image of the fundamental by the structuring mapping (98) is estimated (is non-trivial) as following. Take an arbitrary submanifold $\mathbb{RP}^{\frac{n-2k}{2}+t} \times \mathbb{RP}^{\frac{n-2k}{2}-t} \subset \bar{X}_{a\times\dot{a}}$. Take a small generic (formal) alteration of the mapping $d_X$, restricted on this submanifold; consider the polyhedron $\mathbf{N}(t)$ of iterated self-intersections of this mapping, which are near $\mathbf{NN}_{b\times\dot{b}}$. The image of the fundamental class $[\mathbf{NN}(t)]$ by the mapping (98) (after the 4-ordered transfer) is the following element in $D^{loc}_{n-8k}(\mathbf{I}_{a\times\dot{a}}; \mathbb{Z}/2)$, which is calculate as $\varkappa_a^{\frac{n-8k}{2}+2t} \otimes \varkappa_{\dot{a}}^{\frac{n-8k}{2}-2t}$.

**Lemma 25.** *There exists a formal mapping, which satisfies Definition 24.*

### Proof of Lemma 25

For a simplicity of denotations, let us consider the case $n = 136$, $k = 7$. The case $n = 254$, $k = 15$ is more simple. A required deformation (formal and vertical) with bi-cyclic structure (see Definition 24). This deformation is the result of a gluing a collection of elementary deformations over each elementary product in the decomposition (87).

Take the normal bundle structure. Let us consider the factor $3\varepsilon$ with the standard torus $\gamma : T^2 \subset \mathbb{R}^3$ inside, the factor $2\varkappa_\chi$, which is called the elimination space of the secondary deformation, the factor $2\varepsilon$, which is called the elimination space for the primary deformation. Using Lemma 45, a vertical deformation along elimination spaces to kill the polyhedra $KK_{\mathbf{II}_d}$, $KK_{\mathbf{II}_b,\mathbf{I}_d}$ is well-defined. After this deformation iterated self-intersection is localized inside a closed polyhedron $KK_{\mathbf{II}_b;\mathbf{I}_b}$, all the last component give no contribution to the characteristic number by Lemma 44. Let us consider the deformation onto the torus with the image in $3\varepsilon$ constructed in Lemma 46. The self-intersection points of $KK_{\mathbf{II}_b,\mathbf{I}_b}$ will be regularized outside a singular polyhedron denoted by $S_1$ of the mapping (250) and belong to the subpolyhedron (244). By Theorem 46 the required mapping (98) is well-defined outside a singular subpolyhedron denoted by $S_2$. The dimension of singular polyhedron $S_1$, $S_2$ equal $58+6-2\alpha = 64-2\alpha$, where $\alpha$ is the codimension of the vertical deformation over all transversal directions, $\alpha = 3(14-8) = 18$. Because the polyhedron have dimension $28-2j+30+2j$ and this singular polyhedron are parametrized over 6-dimension: $(S^1)^4 \times \mathbb{RP}^2$. We get a singular polyhedron equals to 28, which is strictly less then 32. Lemma 25 is proved. □

## Proof of Theorem 22

Consider a standardized $\mathbf{D}$-framed immersion $(g, \Psi, \eta_N)$, where $g$ is closed to the composition of the structuring mapping $\eta_N : N^{n-2k} \to X_{\mathbf{H}_{a\times\dot{a}}} \rtimes_\chi \mathbb{R}\mathrm{P}^2$ with the mapping (137). Then a formal deformation of the immersion $g \mapsto g_0$, which is induced by the formal deformation $\bar{d} \mapsto d$ is considered. This formal deformation is fixed near diagonals. The bi-cyclic component of the iterated self-intersection is defined as the component in a neighborhood of a regular domain of the component $\mathbf{NN}_{b\times\dot{b}}$. By the hirsh principle, which can be explicitly applied for formal immersions, the formal immersion $g_0$ has a component $N^{n-8k}$, which is immersed into a regular neighborhood of $\mathbf{NN}_{b\times\dot{b}}$.

The codimension of the mapping $\eta_N$ equals to 8, including the extended torus $\gamma$. Then the codimension of the iterated self-intersection points of $g_0$ equals to 32. Because the singular polyhedron of the mapping (98) in $\mathbf{NN}_{b\times\dot{b}}$ has the dimension strictly less then 32, the induced structure on $N^{n-8k}$, described in Definition 16, is regular.

The image of the structuring mapping on $N^{n-8k}$, restricted to the canonical covering, belongs to the subspace (48), because the defect on the iterated self-intersection is empty, see Definition 3.

The structuring group of the characteristic mapping has the required reduction, as it is followed from Statement 2 in Definition 24. The characteristic number in Statement 2 Definition 24 is restricted using the collection of characteristic number for the polyhedra $\mathbf{NN}(t)$ (see Lemma 48). The immersion $g_0$ is pure in the sense of the Definition 17. The only characteristic number in the described collection could be non-trivial. Theorem 22 is proved. $\square$

# 6 $\mathbf{Q} \times \mathbb{Z}/4$–structure (quaternionic-cyclic structure) on self-intersection manifold of a standardized $\mathbb{Z}/2^{[4]}$–framed immersion

In this Section a mistake by the author, was pointed-out in [L] is corrected. We use arguments related with the Herbert immersion theorem [He]. Let us recall the definition of the quaternionic subgroup $\mathbf{Q} \subset \mathbb{Z}/2^{[3]}$, which contains the subgroup $\mathbf{J}_b \subset \mathbf{Q}$.

Let us define the following subgroups:

$$i_{\mathbf{J}_a\times\dot{\mathbf{J}}_a,\mathbf{Q}\times\mathbb{Z}/4} : \mathbf{J}_a \times \dot{\mathbf{J}}_a \subset \mathbf{Q} \times \mathbb{Z}/4, \tag{99}$$

$$i_{\mathbf{Q}\times\mathbb{Z}/4} : \mathbf{Q} \times \mathbb{Z}/4 \subset \mathbb{Z}/2^{[5]}, \tag{100}$$

$$i_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} : \mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2 \subset \mathbb{Z}/2^{[5]}. \tag{101}$$

Define the subgroup (99). Define the epimorphism $\mathbf{J}_b \times \dot{\mathbf{J}}_b \to \mathbb{Z}/4$ by the formula $(x \times y) \mapsto xy$. The kernel of this epimorphism coincides with the antidiagonal subgroup $\dot{\mathbf{I}}_b = antidiag(\mathbf{J}_b \times \dot{\mathbf{J}}_b) \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b$, and this epimorphism admits a section, the kernel is a direct factor (the subgroup $\mathbf{J}_b \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b$). This kernel is mapped onto the group $\mathbb{Z}/4$ by the formula: $(x \times x^{-1}) \mapsto x$. The subgroup (99) is well-defined.

Let us define subgroups (100), (101). Consider the basis (54) in the space $\mathbb{R}^8$. Let us define an analogous basis of the space $\mathbb{R}^{16}$. This basis contains 16 vectors, the basis vectors are divided into two subset $16 = 8 + 8$.

$$\mathfrak{h}_{1,*,**}, \mathfrak{h}_{2,*,**}; \tag{102}$$

$$\dot{\mathfrak{h}}_{1,*,**}, \dot{\mathfrak{h}}_{2,*,**}. \tag{103}$$

where the symbols $*$, $**$ takes values $+, -$ independently.

Let us define the subgroup (100). The representation $i_{\mathbf{Q}\times\mathbb{Z}/4}$ is given such that the generator $\mathbf{j}$ of the quaternionic factor $\mathbf{Q} \subset \mathbf{Q} \times \mathbb{Z}/4$ acts in each 4-dimensional subspace from the following list:

$$diag(Lin(\mathfrak{h}_{1,*,**}, \mathfrak{h}_{2,*,**}, \mathfrak{h}_{1,*,-**}, \mathfrak{h}_{2,*,-**}), \quad Lin(\dot{\mathfrak{h}}_{1,*,**}, \dot{\mathfrak{h}}_{2,*,**}, \dot{\mathfrak{h}}_{1,*,-**}, \dot{\mathfrak{h}}_{2,*,-**})), \tag{104}$$

$$diag(Lin(\mathfrak{h}_{1,-*,**}, \mathfrak{h}_{2,-*,**}, \mathfrak{h}_{1,-*,-**}, \mathfrak{h}_{2,-*,-**}), \quad Lin(\dot{\mathfrak{h}}_{1,-*,**}, \dot{\mathfrak{h}}_{2,-*,**}, \dot{\mathfrak{h}}_{1,-*,-**}, \dot{\mathfrak{h}}_{2,-*,-**})), \tag{105}$$

$$antidiag(Lin(\mathfrak{h}_{1,*,**}, \mathfrak{h}_{2,*,**}, \mathfrak{h}_{1,*,-**}, \mathfrak{h}_{2,*,-**}), \quad Lin(\dot{\mathfrak{h}}_{1,*,**}, \dot{\mathfrak{h}}_{2,*,**}, \dot{\mathfrak{h}}_{1,*,-**}, \dot{\mathfrak{h}}_{2,*,-**})), \tag{106}$$

$$antidiag(Lin(\mathfrak{h}_{1,-*,**}, \mathfrak{h}_{2,-*,**}, \mathfrak{h}_{1,-*,-**}, \mathfrak{h}_{2,-*,-**}), \quad Lin(\dot{\mathfrak{h}}_{1,-*,**}, \dot{\mathfrak{h}}_{2,-*,**}, \dot{\mathfrak{h}}_{1,-*,-**}, \dot{\mathfrak{h}}_{2,-*,-**})) \tag{107}$$

by the standard matrix, which is defined in the standard basis of the corresponding space. Each of 4-space, described above, corresponds to one of the pair $Lin_{\mathbf{x}}$, $Lin_{\mathbf{y}}$ of spaces, using (54).

The generator $\mathbf{i} \in \mathbf{Q}$ acts in the direct sum of the two exemplars of the corresponding space as the generator of the group $\mathbf{J}_b$ by the corresponding matrix. The generator of the factor $\mathbb{Z}/4 \subset \mathbf{Q} \times \mathbb{Z}/4$ acts of the direct sum of the two exemplars of the corresponding space as the generator of the group $antidiag(\mathbf{J}_b \times \dot{\mathbf{J}}_b) \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b$. The representation (100) is well-defined.

Let us define the representation (101) as following. The factor $\mathbf{J}_b \times \dot{\mathbf{J}}_b \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2$ is represented in each 4-dimensional subspace (104)-(107) by the formula (53), which is applied separately to standard basis of each spaces.

The factor $\mathbb{Z}/2 \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2$ is represented

–in 8-dimensional subspace, which is defined as the direct sum of the subspaces (104), (106) by the identity transformation.

–in 8-dimensional subspace, which is defined as the direct sum of the subspaces (105), (107) by the central symmetry.

The representation (101) is well-defined.

On the group $\mathbf{Q} \times \mathbb{Z}/4$ define the automorphism $\chi^{[5]}$ of the order 4. This automorphism on the subgroup (99) is defined as the restriction of the automorphism $\chi^{[4]}$. The extension of $\chi^{[4]}$ from the subgroup $\chi^{[5]}$ to the group is defined by the simplest way: the automorphism $\chi^{[5]}$ keeps the generator$\mathbf{j}$. It is easy to see that the automorphism with such property exists and uniquely.

Consider the projection

$$p_{\mathbf{Q}} : \mathbf{Q} \times \mathbb{Z}/4 \to \mathbf{Q} \tag{108}$$

on the first factor. The kernel of the homomorphism $p_{\mathbf{Q}}$ coincides with the antidiagonal subgroup $\dot{\mathbf{I}}_b \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b \subset \mathbf{Q} \times \mathbb{Z}/4$. Evidently, the following equality is satisfied on the first factor. The kernel of the homomorphism $p_{\mathbf{Q}}$ coincides with the antidiagonal subgroup $\dot{\mathbf{I}}_b \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b \subset \mathbf{Q} \times \mathbb{Z}/4$.

Evidently, the following equality is satisfied:

$$p_{\mathbf{Q}} \circ \chi^{[5]} = p_{\mathbf{Q}}. \tag{109}$$

Analogously, on the group $\mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2$ define the automorphism $\chi^{[5]}$ of the order 2 (this new automorphism denote the same). Define the projection

$$p_{\mathbb{Z}/4\times\mathbb{Z}/2} : \mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2 \to \mathbb{Z}/4 \times \mathbb{Z}/2, \tag{110}$$

the kernel of this projection coincides with the diagonal subgroup

Obviously, the following formula is satisfied the kernel of this projection coincides with the diagonal subgroup $\dot{\mathbf{I}}_b \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2$. Obviously, the following formula is satisfied: $\chi^{[5]} \circ p_{\mathbb{Z}/4\times\mathbb{Z}/2} = p_{\mathbb{Z}/4\times\mathbb{Z}/2}$.

This allows to define analogously with (26), (62), (63) the groups

$$(\mathbf{Q} \times \mathbb{Z}/4) \rtimes_{\chi^{[5]}} \mathbb{Z}, \tag{111}$$

$$(\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2) \rtimes_{\chi^{[5]}} \mathbb{Z}, \tag{112}$$

as semi-direct products of the corresponding groups with automorphisms with the group $\mathbb{Z}$.

Let us define the epimorphism:

$$\omega^{[5]} : (\mathbf{Q} \times \mathbb{Z}/4) \rtimes_{\chi^{[5]}} \mathbb{Z} \to \mathbf{Q}, \tag{113}$$

the restriction of this epimorphism on the subgroup (99) coincides with the epimorphism (108). For this definition use the formula (109) and define $z \in Ker(p_{\mathbf{Q}})$, where $z \in \mathbb{Z}$ is the generator.

Evidently, the epimorphism

$$\omega^{[5]} : (\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2) \rtimes_{\chi^{[5]}} \mathbb{Z} \to \mathbb{Z}/4 \times \mathbb{Z}/2, \tag{114}$$

analogously is well-defined, denote this automorphism as the automorphism (113).

In the group $\mathbb{Z}/2^{[5]}$ let us define the involution, which is denoted by $\chi^{[5]}$ as on the resolution group. In the standard basis of the spaces (104)-(107) the automorphism $\chi^{[5]}$ is defined by the same formulas as $\chi^{[4]}$, the each considered space is a proper space for $\chi^{[5]}$. This definition implies that $\chi^{[5]}$ is commuted with the representations (100), (101).

Moreover, the following homomorphisms

$$\Phi^{[5]} : (\mathbf{Q} \times \mathbb{Z}/4) \rtimes_{\chi^{[5]}} \mathbb{Z} \to \mathbb{Z}/2^{[5]}, \tag{115}$$

$$\Phi^{[5]} : (\mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2) \rtimes_{\chi^{[5]}} \mathbb{Z} \to \mathbb{Z}/2^{[5]}, \tag{116}$$

which extend the diagram (68), are well-defined, they are included into the following commutative diagrams (117), (118).

$$\begin{array}{ccc} (\mathbf{J}_a \times \dot{\mathbf{J}}_a) \rtimes_{\chi^{[4]}} \mathbb{Z} & \stackrel{\Phi^{[4]} \times \Phi^{[4]})}{\longrightarrow} & \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \\ i_{\mathbf{J}_a \times \dot{\mathbf{J}}_a, \mathbf{Q} \times \mathbb{Z}/4} \downarrow & & i_{[5]} \downarrow \\ (\mathbf{Q} \times \mathbb{Z}/4) \rtimes_{\chi^{[5]}} \mathbb{Z} & \stackrel{\Phi^{[5]}}{\longrightarrow} & \mathbb{Z}/2^{[5]}, \end{array} \tag{117}$$

In this diagram the left vertical homomorphism

$$i_{\mathbf{J}_b \times \dot{\mathbf{J}}_b, \mathbf{Q} \times \mathbb{Z}/4} : (\mathbf{J}_b \times \dot{\mathbf{J}}_b) \rtimes_{\chi^{[4]}} \mathbb{Z} \to (\mathbf{Q} \times \mathbb{Z}/4) \rtimes_{\chi^{[5]}} \mathbb{Z}$$

is induced by the homomorphism (99), the right vertical homomorphism

$$i_{[5]} : \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \subset \mathbb{Z}/2^{[5]}.$$

is the inclusion of the subgroup of the index 2.

The following diagram

$$\begin{array}{ccc} (\mathbf{J}_a \times \dot{\mathbf{J}}_a) \rtimes_{\chi^{[4]}} \mathbb{Z} & \stackrel{\Phi^{[4]} \times \Phi^{[4]}}{\longrightarrow} & \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \\ i_{\mathbf{J}_a \times \dot{\mathbf{J}}_a, \mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2} \downarrow & & i_{[5]} \downarrow \\ (\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2) \rtimes_{\chi^{[5]}} \mathbb{Z} & \stackrel{\Phi^{[5]}}{\longrightarrow} & \mathbb{Z}/2^{[5]}, \end{array} \tag{118}$$

the left vertical homomorphism

$$i_{\mathbf{J}_a \times \dot{\mathbf{J}}_a, \mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2} : (\mathbf{J}_a \times \dot{\mathbf{J}}_a) \rtimes_{\chi^{[4]}} \mathbb{Z} \to (\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2) \rtimes_{\chi^{[5]}} \mathbb{Z}$$

is an inclusion.

**Lemma 26.** *The homomorphism (115) is extended from the subgroup* $\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu^{(4)}_{b\times \dot{b}}} \mathbb{Z} \times \mathbb{Z} \rtimes_{\chi^{[4]}} \mathbb{Z}$ *of the index 2 on the group*

$$(\mathbf{Q} \times \mathbb{Z}/4) \rtimes_{\mu^{(5)}_{b\times \dot{b}}} \mathbb{Z} \times \mathbb{Z} \rtimes_{\chi^{[5]}} \mathbb{Z}.$$

## Proof of Lemma 26

Let us construct the extension

$$(\mathbf{Q} \times \mathbf{Q}) \rtimes_{\mu^{(5)}_{b\times \dot{b}}} \mathbb{Z} \times \mathbb{Z} \rtimes_{\chi^{[5]}} \mathbb{Z},$$

then let us pass to the required subgroup of the index 2.

The automorphism $\mu^{(5)}_{b\times \dot{b}}$ is induced from the automorphism $\mu_{\mathbf{Q}} : \mathbf{Q} \to \mathbf{Q}$ of the factors: $\mu_{\mathbf{Q}}(\mathbf{i}) = -\mathbf{i}$, $\mu_{\mathbf{Q}}(\mathbf{j}) = -\mathbf{j}$, $\mu_{\mathbf{Q}}(\mathfrak{k}) = \mathfrak{k}$. $\square$

Additionally, one has to consider, as this automorphism is defined on 3-dimensional quaternionic lens $S^3/\mathbf{Q}$ (the quaternionic action on $S^3$ is on the right).

The automorphism is given by the transformation of $S^3$, which is the right multiplication of the quaternion unite. The automorphism $\mu_{\mathbf{Q}}$ is commuted with the transformation of the quaternion units (the only non-obvious calculation concerns the unite ) $\mathfrak{k}$). Therefore on the quotient $S^3/\mathbf{Q}$ the automorphism is well-defined.

The trivialization of the tangent bundle on the quaternionic lens, which is defined by the left action of the units $(\mathbf{i}, \mathbf{j}, \mathfrak{k})$, is changed by corresponding automorphisms. Therefore the image of the quaternion framing by the differential

$$d(\mu_{\mathbf{Q}}) : T(S^3/\mathbf{Q}) \to T(S^3/\mathbf{Q}) \qquad (119)$$

is fibred isotopic to the identity. This isotopy consists of a family of rotations trough the angle $180^0$ in oriented planes, which are orthogonal to the vectors $\mathfrak{k}$.

The homomorphism (116) is extend to the homomorphism of the Laurent extension from the subgroup $\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu^{(4)}_{b\times \dot{b}}} \mathbb{Z} \times \mathbb{Z} \rtimes_{\chi^{[4]}} \mathbb{Z}$ of the index 2 to the all group:

$$(\mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2) \rtimes_{\mu^{(5)}_{b\times \dot{b}}} \mathbb{Z} \times \mathbb{Z} \rtimes_{\chi^{[5]}} \mathbb{Z} \mapsto \mathbb{Z}/2^{[5]}. \qquad (120)$$

**Definition 27.** Let us say that a $\mathbb{Z}/2^{[4]}$-framed immersion $(g, \Psi, \eta_N)$ of the codimension $8k$, which is standardized in the sense of Definition 16, is an immersion with $\mathbf{Q} \times \mathbb{Z}/4$-structure (with a quaternion-cyclic structure), if $g$ is self-intersects along a $\mathbb{Z}/2^{[5]}$-framed immersion $(h, \Xi, \zeta_L)$ with the self-intersection manifold $L$ of the dimension $n - 16k$ (for $k = 7$, $n = 126$ we have $n - 16k = 14$).

Additionally, let us assume that $(g, \Psi, \eta_N)$ is a pre-standardized immersion, see Definition 16. Then the fundamental class $[\bar{L}]$ of the canonical covering over the manifold $L$ of self-intersection, $dim(L) = m_\sigma$, is mapped by the characteristic mapping into the element of the dimension $m_\sigma$ in the subgroup described in Lemma (15).

**Lemma 28.** *Let $(g, \eta_N, \Psi)$ a a pre-standardized $\mathbb{Z}/2^{[4]}$-framed immersion in the codimension $8k$, see Definition 16. Then in a regular homotopy class of this immersion there exists a standardized immersion with quaternionic-cyclic structure, see Definition 27.*

**Proof of Lemma 28**

Let us define the space

$$ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} = S^{\frac{n+6}{2}+1}/\mathbf{i} \times S^{\frac{n+6}{2}+1}/\mathbf{i} \tag{121}$$

Evidently, $\dim(ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}) = n + 8 > n$.

Let us define a family $\{Z_0, \dots, Z_{j_{max}}\}$ of standard submanifolds in the manifold $ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$, where

$$j_{max} = \frac{n+6}{8}, \tag{122}$$

by the following formula:

$$Z_0 = S^{\frac{n+12}{2}} \times S^3/\mathbf{i} \subset S^{\frac{n+12}{2}}/\mathbf{i} \times S^{\frac{n+12}{2}}/\mathbf{i}, \tag{123}$$

$$Z_1 = S^{\frac{n+4}{2}} \times S^7/\mathbf{i} \subset S^{\frac{n+12}{2}}/\mathbf{i} \times S^{\frac{n+12}{2}}/\mathbf{i}, \dots$$

$$Z_j = S^{\frac{n+12-8j}{2}}/\mathbf{i} \times S^{4j+3}/\mathbf{i} \subset S^{\frac{n+12}{2}}/\mathbf{i} \times S^{\frac{n+12}{2}}/\mathbf{i}, \dots$$

$$Z_{j_{max}} = S^3/\mathbf{i} \times S^{\frac{n+12}{2}}/\mathbf{i} \subset S^{\frac{n+12}{2}}/\mathbf{i} \times S^{\frac{n+6}{2}}/\mathbf{i}.$$

In the formula $j_{max}$ is defined by the formula (122). A subpolyhedron

$$Z_{a \times \dot{a}} \subset ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} \tag{124}$$

is defined as the union of submanifolds of the family (123). Evidently, $\dim(Z_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}) = \frac{n+18}{2}$.

The standard involution, which is permuted the factors, is well-defined:

$$\chi^{[4]} : ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} \to ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}, \tag{125}$$

this involution is invariant on the subspace $Z_{a \times \dot{a}}$ (124).

Let us consider the standard cell-decomposition of the space $K(\mathbf{J}_a \times \dot{\mathbf{J}}_a, 1) = K(\mathbf{J}_a, 1) \times K(\dot{\mathbf{J}}_a, 1)$, this cell-decomposition is defined as the direct product of the cell-decomposition of the factors. The standard inclusion of the skeleton is well-defined:

$$Z_{a \times \dot{a}} \subset K(\mathbf{J}_a \times \dot{\mathbf{J}}_a, 1). \tag{126}$$

The skeleton (126) is invariant with respect to the involution (125). Therefore the inclusion

$$Z_{a\times\dot{a}}\rtimes_{\chi^{[4]}}S^1 \subset Z_{a\times\dot{a}}\rtimes_{\chi^{[4]}}S^1 \subset K((\mathbf{J}_a \times \dot{\mathbf{J}}_a)\rtimes_{\chi^{[4]}}\mathbb{Z},1) \qquad (127)$$

is well-defined.

Let us define a polyhedron $J_Z$, a subpolyhedron of the polyhedron $J_X$, see the formula (92). Denote by $JJ_{\mathbf{J}_a}$ the join of $j_{max}$ copies of the lens space $S^3/\mathbf{i}$. Denote by $JJ_{\mathbf{J}_a}$ the join of $j_{max}$ copies of the lens space $S^3/\mathbf{i}$.

Denote by $JJ_{\mathbf{Q}}$ the join of $j_{max}$ copies of the quaternionic lens space $S^3/\mathbf{Q}$, the second exemplar of this join. Recall that the positive integer $j_{max}$ is defined by the formula (122).

Let us define the subpolyhedron $J_{\bar{Z},j} \subset JJ_{\mathbf{J}_a} \times JJ_{\dot{\mathbf{J}}_a}$ by the formula

$$J_{\bar{Z},j} = JJ_{\mathbf{J}_a,j} \times JJ_{\dot{\mathbf{J}}_a,j}, 1 \le i \le j_{max},$$

where $JJ_{\mathbf{J}_a,j} \subset JJ_{\mathbf{J}_a}$ is the subjoin with the coordinates $1 \le i \le j_{max} - j$, $JJ_{\dot{\mathbf{J}}_a,j} \subset JJ_{\dot{\mathbf{J}}_a}$ is the subjoin with the coordinates $1 \le i \le j$. Let us define the subpolyhedron $J_{Z,j} \subset JJ_{\mathbf{Q}} \times JJ_{\dot{\mathbf{Q}}}$ by the formula:

$$JJ_{Z,j} = JJ_{\mathbf{Q},j} \times JJ_{\dot{\mathbf{Q}},j}, 1 \le i \le j_{max},$$

where $JJ_{\mathbf{Q},j} \subset JJ_{\mathbf{Q}}$ is the subjoin with coordinates $1 \le i \le j_{max} - j$, $JJ_{\dot{\mathbf{Q}},j} \subset JJ_{\dot{\mathbf{Q}}}$ is the subjoin with the coordinates $1 \le i \le j$.

Define $J_Z$ by the formula:

$$J_Z = \cup_{j=1}^{j_{max}} J_{Z,j} \subset JJ_{\mathbf{Q}} \times JJ_{\dot{\mathbf{Q}}}. \qquad (128)$$

Define $J_{\bar{Z}}$ by the formula:

$$J_{\bar{Z}} = \cup_{j=1}^{j_{max}} J_{\bar{Z},j} \subset JJ_{\mathbf{J}_a} \times JJ_{\dot{\mathbf{J}}_a}. \qquad (129)$$

On the polyhedron $Z_{a\times\dot{a}}$, $J_Z$ a free involution, which is denoted by $T_{\mathbf{Q}}$, and which is corresponded to the quadratic extension (99) is well-defined. The polyhedron (128) is invariant with respect to the involution $T_{\mathbf{Q}}$, the inclusion of the factor-polyhedron is defined by

$$(J_Z)/T_{\mathbf{Q}} = \cup_{j=1}^{j_{max}} J_{Z,j}/T_{\mathbf{Q}} \subset JJ_{\mathbf{J}_a}/T_{\mathbf{Q}} \times JJ_{\dot{\mathbf{J}}_a}/T_{\mathbf{Q}}. \qquad (130)$$

The standard 4-sheeted covering (with ramifications)

$$c_Z : Z_{a\times\dot{a}} \to J_Z, \qquad (131)$$

is well-defined. This covering is factorized to the ramified covering $\hat{c}_Z$. This covering is defined by the composition of the standard 2-sheeted covering

$$Z_{a\times\dot{a}}/T_{\mathbf{Q}} \to (J_Z)/T_{\mathbf{Q}} \qquad (132)$$

and the standard 2-sheeted covering: $Z_{a\times\dot{a}} \to Z_{a\times\dot{a}}/T_{\mathbf{Q}}$. The ramified covering (131) is equivariant with respect to the involution (125), which acts in the target and the source.

Let us consider the embedding $S^3/\mathbf{Q} \subset \mathbb{R}^4$, which is constructed in [Hi]. The Cartesian product of joins of corresponding copies of the embeddings gives the embedding

$$J_{Z,j} \subset \mathbb{R}^{5j_{max}+3}. \tag{133}$$

The following lemma is analogous to Lemma 23.

**Lemma 29.** *There exists an embedding*

$$\mathrm{i_{J_Z}} : \mathrm{J_Z} \rtimes_{\chi} \mathrm{S}^1 \subset \mathbf{D}^{\mathrm{n}-5} \times \mathrm{S}^1 \subset \mathbb{R}^{\mathrm{n}}. \tag{134}$$

The following extension of the polyhedron (127), which corresponds to the Laurent extension of the structuring groups, as in Lemma 26, is well-defined. Let us define a ramified covering:

$$c_Z : Z_{a \times \dot{a}} \rtimes_{\mu_{b \times \dot{b}}} S^1 \times S^1 \to J_Z \rtimes_{\mu_{b \times \dot{b}}} S^1 \times S^1, \tag{135}$$

which is an extension of the covering (131), as a parametrized covering over 2-dimensional torus. The factors of the torus acts on factors of the coverings (131), correspondingly with the extension of the formula (119) on the quaternion lens space.

The inclusion of the base of the covering (29) is extended to the following immersion (which is denoted by the same):

$$\mathrm{i_{J_Z}} : \mathrm{J_Z} \rtimes_{\mu_{\mathrm{b}\times\dot{\mathrm{b}}},\chi} \mathrm{S}^1 \looparrowright \mathbf{D}^{\mathrm{n}-5} \times \mathrm{S}^1 \times \mathrm{S}^1 \rtimes_{\chi} \mathrm{S}^1 \subset \mathbb{R}^{\mathrm{n}}. \tag{136}$$

An arbitrary immersion $g$ with a $\mathbf{Q} \times \mathbb{Z}/4$-structure is defined as a small approximation of a mapping, which is the composition of the mapping to the total space of the 2-covering (135), the ramified covering itself, and the immersion (239).

The decomposition to maximal strata of self-intersection points of the ramified covering (135) determines a control of the image of the fundamental class of self-intersection points of the immersion $g$ by the structuring mapping. Cycles of the types 1,2 correspond to structuring groups of maximal strata, cycles of type 3 can be exist, because (239) is an immersion and could have additional closed component of a self-intersection. Lemma 28 is proved. □

In the following Lemma the main result is proved.

**Theorem 30.** *Let $(g, \Psi, \eta_N)$ $\mathbb{Z}/2^{[4]}$ be a standardized (see Definition 16) immersion, with $\mathbf{Q} \times \mathbb{Z}/4$-structure (see Definition 27). Let us assume that $(g, \Psi, \eta_N)$ is pure (see Definition 17). Then $(g, \Psi, \eta_N)$ is negligible.*

**Proof of Theorem 30**

We use a non-standard modification of the Herbert formula with local $\mu_{b\times \dot{b}}, \chi^{(4)}$ coefficients valued in the group described in Lemma 15, and for immersions with additional marked component, which gives an extra contribution on the both sides of the Herbert formula. The Euler class of the bundle (70) is an immersed submanifold in the codimension 2, possibly, with double self-intersections. Denote this submanifold by $N_1^{n-8k-2} \subset N^{n-8k}$.

Let us restrict the immersion $g$ on the submanifold $N_1^{n-8k-2} \looparrowright N$ and let us consider the manifold $L_1^{n_\sigma-4} \looparrowright L^{n_\sigma}$ of self-intersection points of the immersed manifold $g(N_1^{n-8k-2})$.

The characteristic number, the image of the fundamental class $[\bar{L}_1]$ by the characteristic mapping $D^{loc}_{n_\sigma-4}(\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu_{b\times \dot{b}}} (\mathbb{Z}\times\dot{\mathbb{Z}});\mathbb{Z})$, is calculated using the Euler classes of the structured 4-bundle over $L$ and the image $[\bar{L}^{n_\sigma}]$, which is an element in $D^{loc}_{n_\sigma}(\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu_{b\times \dot{b}}} (\mathbb{Z}\times\dot{\mathbb{Z}});\mathbb{Z})$. This calculation of also possible by the Herbert theorem, which is applied to $[\bar{L}]$. The self-intersection points of the immersion $N_1^{n-8k-2} \looparrowright N^{n-8k}$ give an extra contribution to self-intersection points of $g|_{N_1}$. In the Herbert formula for $[\bar{L}_1^{n_\sigma-4}]$ this extra classes are cancelled in the both sides in the formula.

The characteristic classes takes values modulo 4. By arguments as in [A-P], this characteristic class in $D^{loc}_{n_\sigma-4}(\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu_{b\times \dot{b}}} (\mathbb{Z}\times\dot{\mathbb{Z}});\mathbb{Z})$ is opposite to itself. Because $g$ is pure, this proves that the characteristic number is even. Theorem 30 is proved. $\square$

# 7 Classifying space for self-intersections of skew-framed immersions

## 7.1 Preliminary constructions and definitions

In this section we assume that $n$, $k$ are even positive integers, $n > k$.

### 7.1.1 Axillary mappings $c : \mathbb{RP}^{n-k+1} \to \mathbb{R}^n$, $c_0 : \mathbb{RP}^{n-k+1} \to \mathbb{R}^n$, which are used in Lemma 9, in Theorem 22

Let us denote by $J_0 \subset \mathbb{R}^n$ the standard sphere of dimension $(n-k)$ in Euclidean space. This sphere is $PL$ - homeomorphic to the join of $\frac{n-k+2}{2} = r_0$ copies of the circles $S^1$. Let us denote by $i_{J_0} : J_0 \subset \mathbb{R}^n$ the standard embedding.

Let us denote by

$$i_J : J \subset \mathbb{R}^n \qquad (137)$$

the standard $(n-k+1)$-dimensional sphere in Euclidean space. This sphere is $PL$ - homeomorphic to the join of $\frac{n-k+2}{2} = r$ copies of the circles $S^1$.

Define the following mapping $p_0 : S^{n-k+1} \to J_0$ as the join of $r$ copies of the standard 4-sheeted coverings $S^1 \to S^1/\mathbf{i}$. The standard antipodal action $-1 \in \mathbf{I}_d \times S^{n-k+1} \to S^{n-k+1}$ (for the group $\mathbf{I}_d$ and analogous denotations below see Section 2) commutes with the mapping $p_0$. Therefore the ramified covering $p_0 : \mathbb{RP}^{n-k+1} \to J$ is well-defined. The mapping

$$c_0 : \mathbb{RP}^{n-k+1} \to \mathbb{R}^n \tag{138}$$

is defined as the composition $i_J \circ p_0$.

Replace the mapping $p_0$ defined above by the mapping $p$, the join of $r$ copies of 2-sheeted coverings $S^1 \to \mathbb{RP}^1$. The ramified covering $p : \mathbb{RP}^{n-k+1} \to J$ is analogously well-defined. The mapping

$$c : \mathbb{RP}^{n-k+1} \to \mathbb{R}^n \tag{139}$$

is the composition $i_J \circ p$.

### 7.1.2 Deleted square

In this and next sections we use denotations for subgroups in the dihedral group $\mathbf{D}$, denoted its generators by superscripts: $\mathbf{I_b}$, $\mathbf{I_{a,\dot{a}}}$ and analogous.

Let us consider the classifying space

$$(\mathbb{RP}^{n-k+1} \times \mathbb{RP}^{n-k+1} \setminus \Delta)/T' = \Gamma_\circ, \tag{140}$$

which is also called the "deleted product or, the "deleted square"of the space $\mathbb{RP}^{n-k+1}$. The subspace $\Delta \subset \mathbb{RP}^{n-k+1} \times \mathbb{RP}^{n-k+1}$ is the diagonal subspace of the Cartesian product. This space (140) is defined by the factor on the Cartesian product by the involution $T' : \mathbb{RP}^{n-k+1} \times \mathbb{RP}^{n-k+1} \to \mathbb{RP}^{n-k+1} \times \mathbb{RP}^{n-k+1}$, which permutes the coordinates, outside the diagonal. The deleted product is an open manifold.

### 7.1.3 The mapping $c_0$ (138), $c$ (139) as formal mappings with a formal self-intersection

Denote by $T_{\mathbb{RP}^{n-k+1} \times \mathbb{RP}^{n-k+1}}$, $T_{\mathbb{R}^n \times \mathbb{R}^n}$ the standard involutions in the spaces $\mathbb{RP}^{n-k+1} \times \mathbb{RP}^{n-k+1}$, $\mathbb{R}^n \times \mathbb{R}^n$, which permute the coordinates. Let

$$d^{(2)} : \mathbb{RP}^{n-k+1} \times \mathbb{RP}^{n-k+1} \to \mathbb{R}^n \times \mathbb{R}^n \tag{141}$$

be an arbitrary $T_{\mathbb{RP}^{n-k+1} \times \mathbb{RP}^{n-k+1}}$, $T_{\mathbb{R}^n \times \mathbb{R}^n}$–equivariant mapping, which is transversal along the diagonal in the target. Denote the polyhedron $(d^{(2)})^{-1}(\mathbb{R}^n_{diag})/T_{\mathbb{RP}^{n-k+1} \times \mathbb{RP}^{n-k+1}}$ by $\mathbf{N} = \mathbf{N}(d^{(2)})$, let us call this polyhedron a formal self-intersection polyhedron of the mapping $d^{(2)}$. In the case when the formal mapping $d^{(2)}$ is a holonomic mapping and is the formal extension of a mapping $d$ (the cases $d = c_0$, $d = c$ are required) the polyhedron $\mathbf{N}_\circ(d^{(2)})$ coincides with the polyhedron of self-intersection points of $d$, which is defined by the formula:

$$\mathbf{N}_\circ(d) = \{([\mathbf{x}, \mathbf{y}]) \in \Gamma_\circ : \mathbf{y} \neq \mathbf{x}, d(\mathbf{y}) = d(\mathbf{x})\}, \tag{142}$$

where $\Gamma_\circ$ is the deleted square (140).

### 7.1.4 The structuring mappings $\eta_{\mathbf{N}_\circ} : \mathbf{N}_\circ \to K(\mathbf{D},1)$ for the mappings $c_0$, $c$

Let us define the mapping

$$\eta_{\Gamma_\circ} : \Gamma_\circ \to K(\mathbf{D},1), \tag{143}$$

which is called the structuring mapping.

Denote, that the inclusion $\bar{\Gamma}_\circ \subset \mathbb{R}\mathrm{P}^{n-k+1} \times \mathbb{R}\mathrm{P}^{n-k+1}$ induces an isomorphism of the fundamental groups, because the codimension of the diagonal $\Delta \subset \mathbb{R}\mathrm{P}^{n-k+1} \times \mathbb{R}\mathrm{P}^{n-k+1}$ equals $n-k+1$ and satisfies the following inequality: $n-k+1 \geq 3$. Therefore, the following equation is satisfied:

$$\pi_1(\bar{\Gamma}_\circ) = H_1(\bar{\Gamma}_\circ;\mathbb{Z}/2) = \mathbb{Z}/2 \times \mathbb{Z}/2. \tag{144}$$

Let us consider the induced automorphism $T'_* : H_1(\bar{\Gamma}_\circ;\mathbb{Z}/2) \to H_1(\bar{\Gamma}_\circ;\mathbb{Z}/2)$. This automorphism is not the identity. Let us fix the standard isomorphism $H_1(\bar{\Gamma}_\circ;\mathbb{Z}/2)$ with $\mathbf{I}_{A\times\dot{A}}$, this isomorphism maps the generator of the first factor (correspondingly, the generator of the second factor) of the group $H_1(\bar{\Gamma}_\circ;\mathbb{Z}/2)$ (see (144)) to the element $A = ab \in \mathbf{I}_c \subset \mathbf{D}$ (correspondingly, to the element $\dot{A} = ba \in \mathbf{I}_c \subset \mathbf{D}$), which is represented in the representation of $\mathbf{D}$ by the symmetry with respect to the second (correspondingly, with respect to the first) coordinate axis.

It is easy to check that the automorphism of external conjugation of the subgroup $\mathbf{I}_{A\times\dot{A}} \subset \mathbf{D}$ on the element $b \in \mathbf{D} \setminus \mathbf{I}_{A\times\dot{A}}$ (in this formula the element $b \in \mathbf{I}_b$ is a generator), which is defined by the formula: $x \mapsto bxb^{-1}$, is transformed by the considered isomorphism into the automorphism $T'_*$. The fundamental group $\pi_1(\Gamma_\circ)$ is a quadratic extension of the group $\pi_1(\bar{\Gamma})$ by the element $b$, this extension is uniquely determined (up to an equivalence) by the automorphism $T'_*$. Therefore $\pi_1(\Gamma_\circ) \simeq \mathbf{D}$ and the mapping $\eta_{\Gamma_\circ} : \Gamma_\circ \to K(\mathbf{D},1)$ is well-defined.

The structuring mapping $\eta_{\mathbf{N}_\circ} : \mathbf{N}_\circ \to K(\mathbf{D},1)$ is defined by the restriction of $\eta_{\Gamma_\circ}$ on the open subpolyhedron (142). The construction depends of $c_0$ (138), $c$ (139).

## 7.2 Stratifications

In this construction the polyhedrons (142) for mappings (138), (139) are different. Let us start with the more complicated case $d = c_0$.

### 7.2.1 The stratification of the sphere $J$

Let us ordered 1-dimensional lens spaces (circles), which are generators of the join, by positive integers $1,\dots,r$, and denote by $J(k_1,\dots,k_s) \subset J$ the subjoin, which is defined by the join of the subcollection of circles $S^1/\mathbf{i}$ with the numbers $1 \leq k_1 < \dots < k_s \leq r$, $0 \geq s \geq r$. The stratification is induced by the stratification of the standard $r$-dimensional simplex $\delta^r$ by the projection $J \to lta^r$. The inverse images of the vertexes of the simplex are circles $J(j) \subset J$,

$J(j) \approx S^1$, $1 \le j \le r$, which are irreducible components of the join. The circle looks like the base $S^1/-1$ of 2-sheeted covering for the mapping (139), and like the base $S^1/\mathbf{i}$ for 4-sheeted covering for the mapping (138).

Let us define the space $J_1^{[s]}$ as a subspace in $J$ as the union of all subspaces $J(k_1, \dots, k_s) \subset J$.

Therefore, the following stratification

$$J^{(r)} \subset \dots \subset J^{(1)} \subset J^{(0)}, \tag{145}$$

of the space $J$ is well-defined. For a given stratum the number $r-s$ of omitted coordinates is called the deep of the stratum.

Let us introduce the following denotations:

$$J^{[i]} = J^{(i)} \setminus J^{(i+1)}. \tag{146}$$

#### 7.2.2 The stratification of $\mathbb{RP}^{n-k}$

Let us define the stratification of the projective space $\mathbb{RP}^{n-k+1}$.Denote the maximal open cell $p^{-1}(J(k_1, \dots, k_s))$ by $U(k_1, \dots, k_s) \subset S^{n-k+1}/-1$. This cell is called an elementary stratum of the deep $r-s$. A point on an elementary stratum $U(k_1, \dots, k_s) \subset S^{n-k+1}/-1$ is defined by the collection of coordinates $(\check{x}_{k_1}, \dots, \check{x}_{k_s}, l)$, where $\check{x}_{k_i}$ is the coordinate on the circle, which is covered the circle of the join with the number $k_i$, $l$ is a coordinate on $(s-1)$-dimensional simplex of the join. Points on an elementary stratum $U(k_1, \dots, k_s)$ belong to the union of subcomplexes of the join with corresponding coordinates.

#### 7.2.3 The stratification of the polyhedron $\mathbf{N}_\circ$ (142) for the mapping $c_0$ (138)

The polyhedron $\mathbf{N}_\circ$, (142) is the disjoin union of elementary strata, which are defined as inverse images of the strata (146). Let us denote tis strata by

$$K^{[r-s]}(k_1, \dots, k_s), \qquad 1 \le s \le r. \tag{147}$$

Let us consider exceptional antidiagonal strata on the anti-diagonal

$$\Delta_{anti} = \{([x,y]) \in \Gamma_0 : x = \mathbf{i}y\} \tag{148}$$

of the deleted square $\Gamma_\circ$, this strata also are included into $\mathbf{N}_\circ$. An open polyhedron, which is defined by the cutting of anti-diagonal strata from $\mathbf{N}_\circ$ is denoted by $K_\circ \subset \mathbf{N}_\circ$.

Let us describe an elementary stratum $K^{[r-s]}(k_1, \dots, k_s)$ by means of coordinate system. For the simplicity of denotations, let us consider the case $s = r$, this is the case of maximal elementary stratum. Assume a pair of points $(\mathbf{x}_1, \mathbf{x}_2)$ determines a point on $K^{[0]}(1, \dots, r)$, let us fixes for the pair a pair of points $(\check{\mathbf{x}}_1, \check{\mathbf{x}}_2)$ on the covering sphere $S^{n-k}$, which is mapped into the pair $(\mathbf{x}_1, \mathbf{x}_2)$ by the projection $S^{n-k+1} \to \mathbb{RP}^{n-k+1}$. With respect to constructions above, denote by $(\check{x}_{1,i}, \check{x}_{2,i})$, $i = 1, \dots, r$ the collection of spherical coordinates of each point. A spherical coordinate determines a point on the circle with the same number $i$, the circle is a covering over the corresponding coordinate lens $J(i) \subset J$ in the join. Recall, that the pair of coordinates with a common index determines a par in a layer of the standard cyclic $\mathbf{i}$-covering $S^1 \to S^1/\mathbf{i}$.

Collections of coordinates $\{(\check{x}_{1,i}, \check{x}_{2,i})\}$ are considered up to antipodal transformations of the full collection into the antipodal collection. Moreover, pairs of points $(\mathbf{x}_1, \mathbf{x}_2)$ are non-ordered and lifts of a point in $K$ to a pair of points $(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2)$ on the sphere $S^{n-k}$ are not uniquely defined. Therefore collection of coordinates are defined up to 8 different transformations. (Up to transformation of the group $\mathbf{D}$ of the order 8.)

Analogous statement is well-formulated for points in deeper elementary strata $K^{[r-s]}(k_1, \dots, k_s)$, $1 \le s \le r$.

### 7.2.4 The stratification of the polyhedron $\mathbf{N}_\circ$ (142) for the mapping $c$ (139)

The polyhedron $\mathbf{N}_\circ$, (142) is the disjoin union of elementary strata, which are defined as inverse images of the strata (146) analogously with the case in the subsection 7.2.3. Let us denote this strata by (216) as in the construction for the mapping $c_0$. In the considered case exceptional antidiagonal strata on $\mathbf{N}_\circ$ is absent.

Let us describe an elementary stratum $K^{[r-s]}(k_1, \dots, k_s)$ by means of coordinate system. Assume a pair of points $(\mathbf{x}_1,\ \mathbf{x}_2)$ determines a point on $K^{[0]}(1, \dots, r)$, let us fixes for the pair a pair of points $(\check{\mathbf{x}}_1, \check{\mathbf{x}}_2)$ on the covering sphere $S^{n-k}$, which is mapped into the pair $(\mathbf{x}_1,\ \mathbf{x}_2)$ by the projection $S^{n-k+1} \to \mathbb{RP}^{n-k+1}$. Denote by $(\check{\mathbf{x}}_{1,i}, \check{\mathbf{x}}_{2,i})$, $i = 1, \dots, r$ the collection of spherical coordinates of each point. Recall, that the pair of coordinates with a common index determines a pair in a layer of the standard 2-fold covering $S^1 \to S^1/-1$.

Collections of coordinates $(\check{x}_{1,i}, \check{x}_{2,i})$ are considered up to antipodal transformations of the full collection into the antipodal collection. Moreover, pairs of points $(\mathbf{x}_1, \mathbf{x}_2)$ are non-ordered and lifts of a point in $K$ to a pair of points $(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2)$ on the sphere $S^{n-k+1}$ are not uniquely defined. Therefore collection of coordinates are defined up to 8 different transformations. (Up to transformation of the group $\mathbf{D}$ of the order 8.)

Analogous statement is well-formulated for points in deeper elementary strata $K^{[r-s]}(k_1, \dots, k_s)$, $1 \le s \le r$.

### 7.2.5 Coordinates on $K_\circ \subset \mathbf{N}_\circ$, the mapping $c_0$ (138)

Let $x \in K^{[r-s]}(k_1, \dots, k_s)$ be a point on an elementary stratum. Let us consider collections of spherical coordinates $\check{x}_{1,i}$, $\check{x}_{2,i}$, $k_1 \le i \le k_s$, $1 \le i \le r_0$ of the point $x$. For each $i$ the following cases are possible: the pairs of $i$-th coordinates coincides; antipodal; the second coordinate is the result of the transformation of the first coordinate by the generator $\mathbf{i}$ of the cyclic cover $S^1 \mapsto S^1/\mathbf{i}$. Let us define for an ordered pair of coordinates $\check{x}_{1,i}$, $\check{x}_{2,i}$, $1 \le i \le r$ a residue $v_i$ with values $+1$, $-1$, $+\mathbf{i}$, $-\mathbf{i}$: $v_{k_i} = \check{x}_{1,k_i}(\check{x}_{2,k_i})^{-1}$.

By transformation of the collection of coordinates $\check{x}_2$ into the antipodal collection, the collection of residues are transformed by the antipodal opposition:

$$\{(\check{x}_{1,k_i}, \check{x}_{2,k_i})\} \mapsto \{(-\check{x}_{1,k_i}, \check{x}_{2,k_i})\}, \quad \{v_{k_i}\} \mapsto \{-v_{k_i}\},$$

$$\{(\check{x}_{1,k_i}, \check{x}_{2,k_i})\} \mapsto \{(\check{x}_{1,k_i}, -\check{x}_{2,k_i})\}, \quad \{v_{k_i}\} \mapsto \{-v_{k_i}\}.$$

By transformation of the collections of coordinates by the renumeration: $(\check{x}_1, \check{x}_2) \mapsto (\check{x}_2, \check{x}_1)$, the collection of residues are transformed by the complex conjugated collection: into the antipodal collection, the collection of residues are transformed by the antipodal opposition:

$$\{(\check{x}_{1,k_i}, \check{x}_{2,k_i})\} \mapsto \{(\check{x}_{2,k_i}, \check{x}_{1,k_i})\}, \quad \{v_{k_i}\} \mapsto \{\bar{v}_{k_i}\},$$

where $v \mapsto \bar{v}$ means the complex conjugation. Evidently, the collection of residues remains fixed in the case a different point on the elementary stratum is marked.

Elementary strata $K^{[r-s]}(k_1, \dots, k_s)$, accordingly with collections of coordinates, are divided into 3 types: $\mathbf{I}_b, \mathbf{I}_{a\times\dot{a}}, \mathbf{I}_d$. In the case residues takes values in $\{+\mathbf{i}, -\mathbf{i}\}$ (correspondingly, in $\{+1, -1\}$), one say that an elementary stratum is of the type $\mathbf{I}_b$ (correspondingly, of the type $\mathbf{I}_{a\times\dot{a}}$). In the case there are residues of the complex type $\{+\mathbf{i}, -\mathbf{i}\}$, and the real type $\{+1, -1\}$ simultaneously, one shall say about an elementary stratum of the type $\mathbf{I}_d$. It is easy to check that the restriction of the characteristic mapping $\eta : K_\circ \to K(\mathbf{D}, 1)$ on the elementary stratum of the type $\mathbf{I}_b, \mathbf{I}_{a\times\dot{a}}, \mathbf{I}_d$ correspondingly, is a composition of a mapping into the space $K(\mathbf{I}_b, 1)$ (correspondingly, into the space $\mathbf{I}_{a\times\dot{a}}$, or $K(\mathbf{I}_d, 1)$) with the mapping $i_b : K(\mathbf{I}_b, 1) \to K(\mathbf{D}, 1)$ (correspondingly with the mapping $i_{a\times\dot{a}} : K(\mathbf{I}_{a\times\dot{a}}, 1) \to K(\mathbf{D}, 1)$, or with $i_d : K(\mathbf{I}_d, 1) \to K(\mathbf{D}, 1)$). For strata of the first two types a reduction of the structuring mapping (up to homotopy) is not well-defined; this reduction is well-defined only up to the conjugation of the subgroup $\mathbf{I}_b$ (correspondingly, in the subgroup $\mathbf{I}_{a\times\dot{a}}$).

Recall, the polyhedron $\mathbf{N}_\circ$ is defined by addition elementary anti-diagonal strata to the polyhedron $K_\circ$. On each elementary anti-diagonal stratum $K(k_1, \dots, k_s)$ a residue of each coordinate equals $+\mathbf{i}$.

Define the following open polyhedrons:

$$K_{\mathbf{I}_b\circ} \subset K_\circ \subset \mathbf{N}_\circ \tag{149}$$

$$K_{\mathbf{I}_{a\times\dot{a}}\circ} \subset K_\circ \subset \mathbf{N}_\circ \tag{150}$$

$$K_{\mathbf{I}_d\circ} \subset K_\circ \subset \mathbf{N}_\circ \tag{151}$$

as polyhedrons, which are obtain as union of all elementary strata of the corresponding type. The anti-diagonal stratum (formally) is a stratum of the type $\mathbf{I}_b$, with this convenient the inclusion (149) is replaced into

$$K_{\mathbf{I}_b} \mapsto K_{\mathbf{I}_b;\circ} \cup \Delta_{anti} = K_{\mathbf{I}_b} \subset \mathbf{N}_\circ. \tag{152}$$

### 7.2.6 Coordinates on $\mathbf{N}_\circ$, the mapping $c$ (139)

Let $x \in K^{[r-s]}(k_1, \dots, k_s)$ be a point on an elementary stratum. Let us consider collections of spherical coordinates $\check{x}_{1,i}$, $\check{x}_{2,i}$, $k_1 \le i \le k_s$, $1 \le i \le r_0$ of the point $x$. For each $i$ the following cases are possible: the pairs of $i$-th coordinates coincides; antipodal. Let us define for an ordered pair of coordinates $\check{x}_{1,i}$, $\check{x}_{2,i}$, $1 \le i \le r$ a residue $v_i$ with values $+1$, $-1$: $v_{k_i} = \check{x}_{1,k_i}(\check{x}_{2,k_i})^{-1}$.

By transformation of the collection of coordinates are defined as before in the construction for the mapping $c_0$ (138). Elementary strata $K^{[r-s]}(k_1, \dots, k_s)$, accordingly with collections of coordinates, is only of the type $\mathbf{I}_{a\times\dot{a}}$. In the case residues $v_{k_i}$ takes values in $\{+1, -1\}$). The polyhedron $\mathbf{N}_\circ$ for the mapping $c$ (139) is PL-homeomorphic to the polyhedron (150) for the mapping $c_0$ (138).

### 7.2.7 A prescribed coordinate system on an elementary stratum with odd angles

Let us recall that the polyhedron $K_{\mathbf{I}_b;\circ}$ is decomposed into a disjoint union of elementary strata $K(k_1, \dots, k_s)$, $0 \le s \le r$, residues of angles coordinates take values $\{+\mathbf{i}, -\mathbf{i}\}$. On each elementary stratum $\alpha$ of the type $\mathbf{I_b}$ let us define a coordinate system, which is called a prescribed coordinate system $\Omega(\alpha)$.

Assume that $s$ is odd. Let us call a coordinate system a prescribed system, if a number of coordinates with the residues $+\mathbf{i}$ is greater then the number of coordinates with the residues $-\mathbf{i}$.

The angle coordinate with the residue $-\mathbf{i}$ in the prescribed coordinate system with the smallest index is called a marked coordinate on $K^{[r-s,i]}(k_1, \dots, k_s)$.

An analogous definition for the polyhedron $K_{\mathbf{I}_{a\times\dot{a}};\circ}$ is presented. Let us call a coordinate system a prescribed system, if a number of coordinates with the residues $+1$ is greater then the number of coordinates with the residues $-1$; in the case the residues are divided in the middle (this is possible only if $s$ is even), let us define the prescribed coordinate system such that the residues with the smallest index equals to $+1$.

The angle coordinate with the residue $-\mathbf{i}$ in the prescribed coordinate system with the smallest index is called a marked coordinate on $K^{[r-s,i]}(k_1, \dots, k_s)$.

The prescribed coordinate system on an elementary stratum of the type $\mathbf{I}_{a\times\dot{a}}$, $\mathbf{I}_b$, is determined by the corresponding fixation of a component of the covering over the elementary stratum with respect to the subgroup $\{a, \dot{a}\} \subset \mathbf{D}$, or $\{b\} \subset \mathbf{D}$ correspondingly.

### 7.2.8 Admissible boundary strata of elementary strata

For an arbitrary elementary stratum $\beta \subset \hat{K}^{[r-s,i]}(k_1, \dots, k_s)$ of the type $\mathbf{I_{b,c}}$ (correspondingly, of the type $\mathbf{I_{a\times\dot{a},c}}$) of $\hat{K}_\circ$ let us consider a smallest stratum in its boundary $\alpha$ of the same type, $\beta \ne \alpha$, this means that $\alpha$ in inside the closure of $\beta$: $\alpha \subset Cl(\beta) \subset Cl(\hat{K}^{[r-s,i]}(k_1, \dots, k_s))$.

Prescribed coordinate system $\Omega(\beta)$ on the stratum $\beta \subset \hat{K}(k_1, \dots, k_s)$ is restricted to the elementary stratum $\alpha \subset Cl(\beta)$ inside its boundary. We shall write: $\alpha \prec \beta$. In the case when the restriction of a prescribed coordinate system

$\Omega(\beta)|_\alpha$ coincide with the prescribed coordinate system $\Omega(\alpha)$ on the smallest stratum, we shall call that the pair of strata $(\alpha,\beta)$ is admissible. In the opposite case, the pair of strata $(\alpha,\beta)$ is not admissible.

In the case a pair $(\alpha,\beta)$ is not admissible, the transformation $\Omega(\beta)|_\alpha$ into $\Omega(\alpha)$ are described below for all types of strata.

For pair of strata of the type $\mathbf{I_{b,c}}$ a non-admissible of the pair means that the system of coordinates $\Omega(\beta)|_\alpha$ are related with $\Omega(\alpha)$ by one of the following transformations:

$$(\check{x_1},\check{x_2}) \mapsto (\check{x_2},\check{x_1}), \tag{153}$$

$$(\check{x_1},\check{x_2}) \mapsto (-\check{x_2},-\check{x_1}), \tag{154}$$

$$(\check{x_1},\check{x_2}) \mapsto (-\check{x_1},\check{x_2}), \tag{155}$$

$$(\check{x_1},\check{x_2}) \mapsto (\check{x_1},-\check{x_2}). \tag{156}$$

### 7.3 An additional structure on the polyhedron $K_\circ$ of the mapping $c_0$ (138): an involution $\tau_c$

An elementary stratum, except the anti-diagonal stratum, is a 2-sheeted covering with respect to a free involution, denoted by $\tau_c$ (see the formula below) over the corresponding elementary stratum of a polyhedron $\hat{K}_\circ = K_\circ/\tau_c$, a stratum of this polyhedron is denoted by

$$\hat{K}^{[r-s]}(k_1,\dots,k_s), \qquad 1 \le s \le r. \tag{157}$$

Define the involution $\tau_c$ on the space $K_\circ$ by the formula:

$$\tau_c : ([x,y],\lambda) \mapsto ([ix,iy],\lambda).$$

This involution is free outside the antidiagonal (recall that by the definition of $k_\circ$ the antidiagonal is removed) This is followed from the coordinate description of elementary strata of $K_\circ$ (see below)), coordinates are invariant by the involution $\tau_c$ and a factorization by the involution is well-defined.

**Lemma 31.** *There exists a mapping $\eta_c$, which is included into the following diagram, where vertical mappings are 2-sheeted coverings:*

$$\begin{array}{ccc} K_\circ & \xrightarrow{\ \eta\ } & B\mathbf{D} \\ \downarrow & & \downarrow \\ \hat{K}_\circ = K_\circ/\tau_c & \xrightarrow{\ \eta_c\ } & BC \end{array}$$

*Доказательство.* In each elementary stratum, which consists of points $((x,y),\lambda)$ on the sphere, let us define the action of the dihedral group $\mathbf{D} = \{a,b : [a,b]=b^2; b^4=e\}$, by the formulas:

$$a\colon ((x,y),\lambda) \mapsto ((y,x),\lambda)$$
$$b\colon ((x,y),\lambda) \mapsto ((y,-x),\lambda).$$

Define the group $C$ as the quadratic central extension of the group $\mathbf{D}$ by the element $c$, where $c^2 = b^2$. Define the action of $C$, generated by elements $a,b,c$, by the formulas:

$$a\colon ((x,y),\lambda) \mapsto ((y,x),\lambda)$$
$$b\colon ((x,y),\lambda) \mapsto ((y,-x),\lambda)$$
$$c\colon ((x,y),\lambda) \mapsto ((ix,iy),\lambda).$$

This action is free and one may define $\hat{K}_\circ = K_\circ/\tau_c$. The involution $\tau_c$, defined above, are given by the action on the element $c \in C \setminus \mathbf{D}$. $\square$

All the statements above are formulated for elementary strata of the polyhedron $\hat{K}_\circ = K_\circ/\tau_c$. Elementary strata of this polyhedrons (157) are divided into the following types: $\mathbf{I}_{b,c}, \mathbf{I}_{a\times\dot{a},c}, \mathbf{I}_{d,\mathbf{c}}$, respectively to the type of the subgroup in $C$. Surprisingly, strata of the type $\mathbf{I}_{b,c}$, $\mathbf{I}_{a\times\dot{a},c}$ are homeomorphic, see Proposition 34. (A homeomorphism in prescribed coordinate systems is given by the formula: $\Phi : ([x,y],\lambda) \mapsto ([ix,y],\lambda)$. Homeomorphism $\Phi$ induces an isomorphism of the corresponding structuring subgroups $\mathbf{I}_{b,c} \mapsto \mathbf{I}_{a\times\dot{a},\mathbf{c}}$, which is given on the generators by the formula: $b \mapsto ac$, $c \mapsto c$.)

Analogously to (149),(150),(151) let us define the following open subpolyhedrons:

$$K_{\mathbf{I}_{b,c};\circ}/\tau_c = \hat{K}_{\mathbf{I}_{b,c};\circ} \subset \hat{K}_\circ, \tag{158}$$

$$K_{\mathbf{I}_{a\times\dot{a},c};\circ}/\tau_c = \hat{K}_{\mathbf{I}_{a\times\dot{a},c};\circ} \subset \hat{K}_\circ, \tag{159}$$

$$K_{\mathbf{I}_{d,c};\circ}/\tau_c = \hat{K}_{\mathbf{I}_d;\circ} \subset \hat{K}_\circ. \tag{160}$$

The polyhedra (158), (159) are homeomorphic.

Definition of prescribed coordinate system and admissible boundary strata for the subpolyhedrons (158), (159) is analogous to (149), (150).

## 7.4 Rezolutions $C_{\mathrm{Im}} \mapsto \mathbf{I}_{b,c}$, $C_{\mathrm{Re}} \mapsto \mathbf{I}_{a\times\dot{a},c}$

Recall, the group $C$ is generated by 3 generators $\{a,b,c\}$. This group admits two subgroups $\{b,c\}$, $\{a\times\dot{a},c\}$ described by the collection of generators. Let us construct epimorphisms $\mathbf{I}_{b,c}$ by $C_{\mathrm{Im}} \longrightarrow \mathbf{I}_{b,c}$, $C_{\mathrm{Re}} \longrightarrow \mathbf{I}_{a\times\dot{a},c}$, which are called resolutions.

Let us consider the following diagram:

$$
\begin{CD}
K_{b;\circ} @>>> B\mathbb{Z}_4(b) \\
@VVV @VVV \\
K_{\mathbf{I}_b;\circ} @>>> B\mathbf{D},
\end{CD}
\tag{161}
$$

where $K_{b;\circ} \to K_{\mathbf{I}_b;\circ}$ is a double covering over the polyhedron (149), which is described by the subgroup $\{b\} \subset \mathbf{D}$. this diagram is extended into the right:

$$
\begin{CD}
K_{b;\circ} @>>> B\mathbb{Z}_4(b) @>>> * \\
@VVV @VVV @VVV \\
K_{\mathbf{I}_b;\circ} @>>> B\mathbf{D} @>>> B\mathbf{D}/\mathbb{Z}_4(b) @= B\mathbb{Z}_2.
\end{CD}
\tag{162}
$$

the composition of horizontal mappings classifies the double covering $K_{b;\circ} \to K_{\mathbf{I}_b;\circ}$. The generator of the right-bottom group $\mathbb{Z}_2$ is the image of an arbitrary element from the non-trivial residue class of the subgroup $\mathbb{Z}_4(b)$ in $B\mathbf{D}$. Let us take the element $\dot{A}$.

The space $B\mathbb{Z} = S^1$ admits the positive generator in $\mathbb{Z}$, which is denoted by $\hat{A}$. Then we have the following commutative square, the vertical mappings are epimorphisms $\hat{A} \mapsto \dot{A}$, the semi-direct product $\mathbf{D} = \mathbb{Z}_4(b) \rtimes \mathbb{Z}_2(\dot{A})$ in the bottom left term determines the analogous semi-direct product $\mathbb{Z}_4(b) \rtimes \mathbb{Z}(\hat{A})$ in the upper left term of the diagram:

$$
\begin{CD}
B\mathbb{Z}_4(b) \rtimes \mathbb{Z}(\hat{A}) @>>> B\mathbb{Z}(\hat{A}) \\
@VVV @VVV \\
B\mathbf{D} = B\mathbb{Z}_4(b) \rtimes \mathbb{Z}_2(\dot{A}) @>>> B\mathbb{Z}_2(\dot{A}).
\end{CD}
\tag{163}
$$

Let us relates the diagram with the structured group of the polyhedron $\hat{K}_{\mathbf{I}_b;\circ} = K_{\mathbf{I}_{b,c};\circ}/\tau_c$. For this reason let us remark that there exist diagrams, which are analogous to (161) and (162):

$$
\begin{CD}
\hat{K}_{b,c;\circ} = K_{b,c;\circ}/\tau_c @>>> B\mathbf{I}_{b,c} @>>> * \\
@VVV @VVV @VVV \\
\hat{K}_{\mathbf{I}_{b,c};\circ} = K_{\mathbf{I}_{b,c};\circ}/\tau_c @>>> BC @>>> BC/C_1 @= B\mathbb{Z}_2
\end{CD}
\tag{164}
$$

moreover, the compositions of the horizontal bottom mappings is classified the left vertical double covering. Recall, in the diagram above $\mathbf{I}_{b,c}$ is a subgroup in $C$, which is generated by elements $b$ and $c$. It is convenient to write-down this subgroup as $\mathbb{Z}_4(b) \times \mathbb{Z}_2(cb^3)$. Then $C$ can be represented as a semi-direct product $\mathbb{Z}_4(b) \times \mathbb{Z}_2(cb^3) \rtimes_{\theta_{\mathrm{Im}}} \mathbb{Z}_2(\dot{A})$, where the action $\theta_{\mathrm{Im}}$ на $\mathbb{Z}_4(b) \times \mathbb{Z}_2(cb^3)$

is defined by the formula: $(x, y) \mapsto (-x + 2y, y)$. This fact is followed from the formulas:

$$\dot{A} \cdot b = b^3 \cdot \dot{A} \tag{165}$$

$$\dot{A} \cdot b^3 c = bc \cdot \dot{A} = b^2 \cdot b^3 c \cdot \dot{A} \tag{166}$$

Let us denote by $C_{\mathrm{Im}}$ the group $\mathbb{Z}_4(b) \times \mathbb{Z}_2(cb^3) \rtimes_{\theta_{\mathrm{Im}}} \mathbb{Z}(\hat{A})$, then we get the following Cartesian square:

$$\begin{array}{ccc} BC_{\mathrm{Im}} & \longrightarrow & B\mathbb{Z}(\dot{A}) \\ \downarrow & & \downarrow \\ BC & \longrightarrow & B\mathbb{Z}_2(\dot{A}) \end{array} \tag{167}$$

A theorem, which is proved in Theorem 41, and is called the resolution construction is the following.

**Theorem 32.** *There exists a mapping (a rezolution) of $\hat{K}_{\mathbf{I}_{b,c};\circ} = K_{\mathbf{I}_{b,c};\circ}/\tau_c$ of the bottom left space of the diagram (164) into the space $BC_{Im}$ in the upper left vertex of the diagram (167), such that this mapping commutes with the left vertical projection $BC_{Im} \longrightarrow BC$ in the diagram.*

Let us describe analogous diagrams to the diagrams (168),(167) for the space $\hat{K}_{\mathbf{I}_{a\times\dot{a}};\circ}$ instead of $\hat{K}_{\mathbf{I}_b;\circ}$.

$$\begin{array}{ccccccc} \hat{K}_{a\times\dot{a};\circ} = K_{a\times\dot{a},c;\circ}/\tau_c & \longrightarrow & B\mathbf{I}_{a\times\dot{a},c} & \longrightarrow & * & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \hat{K}_{\mathbf{I}_{a\times\dot{a}};\circ} = K_{\mathbf{I}_{a\times\dot{a},c};\circ}/\tau_c & \longrightarrow & BC & \longrightarrow & BC/C_1 & = & B\mathbb{Z}_2 \end{array} \tag{168}$$

$$\begin{array}{ccc} BC_{\mathrm{Re}} & \longrightarrow & B\mathbb{Z}(\dot{A}) \\ \downarrow & & \downarrow \\ BC & \longrightarrow & B\mathbb{Z}_2(\dot{A}) \end{array} \tag{169}$$

The following theorem is a direct analog of Theorem 32 for the group $C_{Re}$.

**Theorem 33.** *There exists a mapping (a rezolution) of $\hat{K}_{\mathbf{I}_{a\times\dot{a}};\circ} = K_{\mathbf{I}_{a\times\dot{a},c};\circ}/\tau_c$ of the bottom left space of the diagram (168) into the space $BC_{Re}$ in the upper left vertex of the diagram (169), such that this mapping commutes with the left vertical projection $BC_{Re} \longrightarrow BC$ in the diagram.*

Let us prove that Theorem 33 is a corollary of Theorem 32, let us use the following proposition.

**Proposition 34.** *The mapping* $\Phi : \hat{K}_{\mathbf{I}_b\circ} = K_{\mathbf{I}_{b,c}\circ}/\tau_c \to \hat{K}_{\mathbf{I}_{a\times\dot{a}};\circ} = K_{\mathbf{I}_{a\times\dot{a},c};\circ}/\tau_c$ *between the upper left polyhedron in the diagrams (167), (169), which is given by the formula* $\Phi : ([x,y],\lambda) \mapsto ([ix,y],\lambda)$*, is a homeomorphism.*

*Доказательство.* A correctness of the definition of $\Phi$ and existence of an inverse mapping is straightforward exercise. It is convenient to note that the formula for $\Phi$ gives a well-defined mapping only after a factorization by $\tau_c$. $\square$

By Theorem 32 we have the mapping

$$\hat{K}_{\mathbf{I}_b;\circ} \to B\mathbb{Z}(\dot{A}) = S^1, \tag{170}$$

which is defined by the resolution. Thus, we get a mapping:

$$\hat{K}_{\mathbf{I}_{a\times\dot{a}};\circ} \to S^1, \tag{171}$$

which is given by the composition:

$$K_{\mathbf{I}_{a\times\dot{a},c};\circ}/\tau_c \stackrel{\Phi^{-1}}{\to} K_{\mathbf{I}_{b,c};\circ}/\tau_c \to S^1. \tag{172}$$

We have the following commutative diagram:

$$\begin{array}{ccc} \hat{K}_{\mathbf{I}_b;\circ} & \longrightarrow & BC \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ \hat{K}_{\mathbf{I}_{a\times\dot{a}};\circ} & \xrightarrow{\eta_c} & BC \end{array} \tag{173}$$

where the mapping $\Psi$ is induced by the corresponding automorphism of the group $C$, which will be denoted by the same:

$$\begin{aligned} \Psi &: C \to C \\ \Psi &: a \mapsto bc \\ \Psi &: b \mapsto ac \\ \Psi &: c \mapsto c \end{aligned}$$

Then there exist a commutative diagram:

$$\begin{array}{ccccc} \hat{K}_{\mathbf{I}_{a\times\dot{a}};\circ} & \xrightarrow{\Phi^{-1}} & \hat{K}_{\mathbf{I}_b;\circ} & \longrightarrow & B\mathbb{Z}(\hat{A}) \\ \downarrow & & \downarrow & & \downarrow \\ BC & \xrightarrow{\Psi^{-1}} & BC & \longrightarrow & B\mathbb{Z}_2(\dot{A}) \end{array} \tag{174}$$

The kernel of the composition of the bottom mappings coincides with $\Psi^{-1}(C_{Im})$, this kernel is generated by elements: $c$ and $ac^3$, this can be represented as a semi-direct product $\mathbb{Z}_4(c) \times \mathbb{Z}_2(\dot{a})$. The group $C$ itself is represented by a semi-direct product: $\mathbb{Z}_4(c) \times \mathbb{Z}_2(\dot{a}) \rtimes_{\theta_{\mathrm{Re}}} \mathbb{Z}_2(\dot{A})$, where the action of the generator $\dot{A}$ on

$\mathbb{Z}_4(c) \times \mathbb{Z}_2(\dot{a})$ is well-defined by the formula: $(x, y) \mapsto (x + 2y, y)$. This follows from the formula:

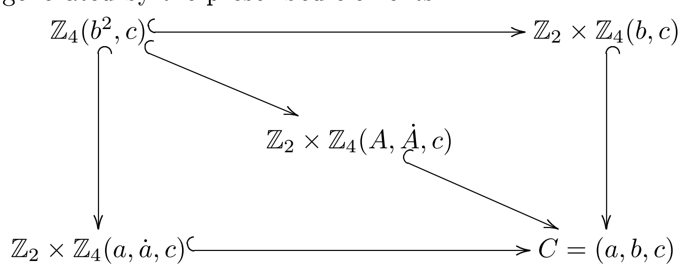
$$\dot{A} \cdot c = c \cdot \dot{A} \tag{175}$$

$$\dot{A} \cdot \dot{a} = \dot{a} \cdot c^2 \cdot \dot{A} \tag{176}$$

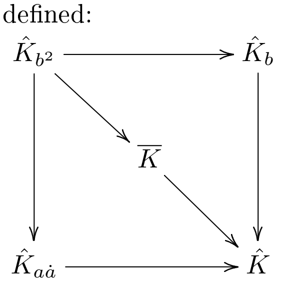
Therefore the classifying mapping $\hat{K}_{\mathbf{I}_{a\times\dot{a}};\circ} \to BC$ is lifted into $BC_{\mathrm{Re}} = B\mathbb{Z}_4(c) \times \mathbb{Z}_2(\dot{a}) \rtimes_{\theta_{\mathrm{Re}}} \mathbb{Z}(\dot{A})$. Proposition 34 is proved. $\square$

### 7.5 The polyhedron $\hat{K}_{\mathbf{I}_b;\circ}$, the resolution of its structuring mapping

Let us consider the polyhedron $\hat{K}_{\mathbf{I}_b;\circ}$, whit coordinate system of the type $\mathbf{I}_{b,c}$, for short re-denote this polyhedron by $\hat{K}$. Consider in $C$ subgroups, which are generated by the prescribed elements:




In this diagram all mappings are induced by inclusions of subgroups of the index 2. The diagram of the 2-sheeted coverings over the polyhedron $\hat{K}$ is well-defined:




In this section points of the space $\hat{K}_{b^2} = K_{b^2,c}$, denoted by $[(x, y), \lambda]$, and the momentum coordinate $\lambda$ should be omitted. In the space $\hat{K}_{b^2}$ a point $(x, y)$ coincides with points $(\mathbf{i}x, \mathbf{i}y)$,$(-x, -y)$,$(-\mathbf{i}x, -\mathbf{i}y)$.

#### 7.5.1 Construction of equivariant morphism of a bundle over $\hat{K}_{b^2}$ with layers non-ordered pairs of oriented real line into a bundle with layers $\mathbb{C}$

Let us divide the space $\hat{K}_{b^2}$ into two spaces $\hat{K}_{b^2,R}$, $\hat{K}_{b^2,L}$ with the common boundary. This common boundary let us denote by $\Sigma \subset \hat{K}_{b^2}$. Define the subspaces in all elementary strata separately. Let us consider the momenta coordinate $\lambda$, which takes value in the simplex $\Delta^{r-j}$, where $j$ is the deep of the strata. Let us consider two faces of simplex $\Delta^{r-j}$, which are defined by momenta, corresponded to angles with positive and negative imaginary residues. The simplex itself is the join of this two faces, let us assume that a parameter of the join $t \in [0,1]$ takes the value 0 on the negative face and the value 1 to the positive face.

Define $\Delta^{r-j} = \Delta_R \cup \Delta_L$, where the first subspace $\Delta_R$ is defined as points of the join, which are attached to positive faces, with $t$ coordinate on the segment $[0,1/2]$. The second subspace $\Delta_L$ is defined as points, which are attached to negative face, with coordinate on the segment $[1/2,1]$.

**Definition 35.** Let us consider polyhedron $\hat{K}^1_{b^2}$, which consists of all strata in $\hat{K}^1_{b^2}$ of the deep 0 and 1. Let us define the following $\tau_a$-involutive disjoint decomposition for maximal strata:

$$\hat{K}^0_{b^2,t} \cup \hat{K}^0_{b^2,s} = \hat{K}^0_{b^2}. \tag{177}$$

We may extend this decomposition the decomposition (177) and define the $\tau_a$-involutive polyhedrons:

$$\hat{K}^1_{b^2,t} \cup \hat{K}^1_{b^2,s} = \hat{K}^1_{b^2}. \tag{178}$$

Take in $\hat{K}_{b^2,t}$ all strata with prescribed coordinate system, which is attached to $\Delta_+$. Take a strata in $\hat{K}_{b^2,s}$ in the opposite case, when the prescribed coordinate system is in the $\tau_a$-involutive strata. The decomposition (177) is well-defined. The decomposition (178) is defined by the closure of strata of codimension 1.

The each component in the decomposition (178) is extended to the hole space $\hat{K}^1_{b^2}$. A stratum in $\hat{K}^0_{b^2}$ in $\hat{K}^0_{b^2,t}$ looks as the stratum equipped by a prescribed collection of positive (right) angles, and in $\hat{K}^0_{b^2,s}$ looks as the same stratum with the $\tau_a$-involutive prescribed collection of negative (left) angles. To keep the globally order of the lines in $E_{\mathbb{R}}$ we get a mixed denotations $t_+$ and $t_-$, in maximal strata for the first line. The each bundle $E_R$, $E_{\mathbb{C}}$ constructed below is a pair of trivial bundles over different spaces, to get a global definition the pairs are identify by $t_\pm \mapsto t_\mp$, $z \mapsto z$ on the intersection.

Over the space $\hat{K}^1_{b^2} = \hat{K}^1_{b^2,t} \cup \hat{K}^1_{b^2,s}$ let us define a pair of bundles equipped with involutions $\tau_i, \tau_b, \tau_i$ and an equivariant morphism between them. If we omit the involution $\tau_a$, the bundles and the morphism become trivial. The involution $\tau_a$ changes the subspace $\hat{K}^1_{b^2,t}$ into $\hat{K}^1_{b^2,s}$ in (178), this involution translate a prescribed angle into its conjugated. We will use this involution on the space (178) is we assume that this involution is extended by the involution in the source bundle $E_{\mathbb{R}}$, which changes the lines in $E_{\mathbb{R}}$, and denote the same involution by $\tilde{\tau}_a$ when the extension preserves the lines.

Define a residue of an angle coordinate in $\hat{K}^0_{b^2,s}$ as the prescribed residues of the corresponding $\tilde{\tau}_a$-involutive coordinate in $\hat{K}^0_{b^2,t}$. (Apriori one may define an alternative residue of an angle in an $s$-stratum, using negative angles in the $s$-strata itself. By the construction an alternative residue is conjugated to the $t$-residue, defined above using $\tau_a$-antipodal $t$-stratum. The alternative $\tau_a$-antisymmetric residues of angles coordinates are unrequired in the construction. The $\tau_a$-residues is used below as a total resedue. Note that $\tilde{\tau}_a$-residues are additive, the total $\tau_a$-residue is multiplicative. )

A fiber of $E_{\mathbb{R}}$ is a disjoin union of the two real lines $\mathbb{R} \sqcup \mathbb{R}$: $E_{\mathbb{R}} = \hat{K}^1_{b^2} \times (\mathbb{R} \sqcup \mathbb{R})$. The unit vectors in the fibers $\mathbb{R} \sqcup \mathbb{R}$ over $\hat{K}^1_{b^2,t}$ is denoted by $\pm t_+$, $\pm t_-$. The basis vectors $+1$ and $-1$ determine a decomposition of the layers into two rays.

For a $t$-stratum the lower superscript $\pm$ of a line, which will be used in the formulas, is associated with the inner positive-prescribed (negative-non-prescribed) angles collection. For a $s$-stratum tht lower superscript $\pm$ of lines is associated with inner right-prescribed (left-prescribed) angles in a $\tilde{\tau}_a$-conjugated $t$-stratum (recall the corresponding $\tilde{\tau}_a$-conjugated left $s$-angle to a right $t$-angle is collected with the opposite sign with respect to the angle in the $s$-stratum itself). Geometrically, when we pass by a path from a maximal stratum $\alpha_1$ in $\hat{K}^0_{b^2,t}$ to a neighboring maximal stratum $\alpha_2$ in $\hat{K}^0_{b^2,s}$ trough a common stratum of deep 1, the fiber $t_+$ is transformed to $t_-$; because in a $t$-stratum $\alpha_1$ the fiber $t_+$ is associated with inner right (prescribed) residue in the stratum; the extension of the same fiber in the neighboring $s$-stratum $\alpha_2$ is denoted by $t_-$, because this fiber is associated with the inner left (non-prescribed) residue in the $\tau_a$-conjugated stratum.

Let us describe the target bundle of the morphism. This bundle denote by $E_{\mathbb{C}}$, without equivariant $\tau_a$-structure this bundle is trivial with the complex line layer: $E_{\mathbb{C}} = \hat{K}^1_{b^2} \times \mathbb{C}$. Assume that the trivialization of the bundle $E_{\mathbb{C}}$ is described by the variable $z$ over $\hat{K}^1_{b^2,t}$. Over $\hat{K}^1_{b^2,s}$ the trivialization is given by the coordinate $z$ denoted the same. Let us describe equivariant structures on the constructed bundles.

Let us define 3 involutive transformations of $E_{\mathbb{R}}$ (the involution $\tau_a$ changes the component in (178) ) by the following formula:

$$\tau_a : ((x,y), t_\pm) \mapsto ((y,x), t_\pm) \tag{179}$$

$$\tau_b : ((x,y), t_\pm) \mapsto ((-y,x), t_\pm) \tag{180}$$

$$\tau_i : ((x,y), t_\pm) \mapsto ((x,y), -t_\mp) \tag{181}$$

By this formula $\tau_b$ is given by the trivial transformation of layers over the base $\hat{K}_{b^2}$. The involution $\tau_i$ acts on the base $\hat{K}_{b^2}$ by the identity, this transformation translates in the target bundle $E_{\mathbb{R}}$ lower superscripts and reveres directions of the layers.

The action of the corresponding involutions on $E_{\mathbb{C}}$ in the $t$-system is given by the formulas:

$$\tau_a : ((x,y), z) \mapsto ((y,x), \bar{z}) \tag{182}$$

$$\tau_b : ((x,y), z) \mapsto ((-y,x), z) \tag{183}$$

$$\tau_i : ((x,y), z) \mapsto ((x,y), \bar{z}). \tag{184}$$

The involutions defined above are pairwise commuted. The transformation $\tau_a$ changes the components in (177) and we may prove that the formulas corresponds each other with respect to the identification of the fibres in a common point on $\hat{K}^1_{b^2}$. Two fibers of the bundle $E_{\mathbb{C}}$ are transformed by $z \mapsto z$, because the bundle $E_{\mathbb{C}}$ is non-trivial (is $\tau_a$-twisted) and the involution $\tau_a$ changes the $z$ coordinate. The transformation $\tau_a$ changes lines in the fibre of the bundle $E_{\mathbb{R}}$. Because a line of $E_{\mathbb{R}}$ is equipped with different lower superscript, which are corresponded with respect to $\tau_a$, and the formulas for $\tau_a$ preserves the lower superscript of lines, geometrically we get the transposition of lines by $\tau_a$.

The result of the section is the following.

**Theorem 36.** *There exist $(\tau_a, \tau_b, \tau_i)$-equivariant mapping $F$ of the bundle $E_{\mathbb{R}}$ into the bundle $E_{\mathbb{C}}$, which is fibrewized monomorphism.*

*Доказательство.* The mapping $F$ is constructed by the induction over the codimension of simplex momenta. More precisely, let us start the construction of $F$ over centers of maximal simplexes, where the total collection of residues are defined. Over each maximal simplex in $\hat{K}_{b^2}$ this mapping is denote by $F^{(0)}$. Then let us extend the mapping on the segment, which join a pairs of centers of the simplexes and cross a codimension 1 face, or which join pairs of points of $\hat{K}_{b^2,t}$ and $\hat{K}_{b^2,s}$ in a common strata. This extension is denoted by $F^{(1)}$.

Additionally, to conclude the proof, one has to check that obstructions for equivariant extension of the mapping $F^{(1)}$ on $\hat{K}^{(1)}_{b^2}$ to the hole complex $\hat{K}_{b^2}$ is trivial.

**Lemma 37.** *There exists a $(\tau_a, \tau_b, \tau_i)$-equivariant mapping $F^{(0)}$ on simplexes of the complex $\hat{K}_{b^2}$ of the deeps $0$ and $1$.*

### 7.5.2 Total $t$-residue, total $s$-residue, integer total $t$-residue, integer total $s$-resedue

The base of the induction is the construction of the mapping $F^{(0)}$, the restriction of $F$ on the subspace, which consists of points $((x,y),\lambda) \in \hat{K}_{b^2}$, for which all momenta coordinate $\lambda_j \neq 0$. Assume we get the case $\hat{K}^0_{b^2,t}$, the residues collection $\{\delta_j\}$ exists for all numbers $j$ of coordinates. Denote in this $t$-stratum $\alpha_1$ the total $t$-residue by $\theta = \prod_{j=1}^{r} \delta_j$. We associate this residue with the opposite base vector $-1 \in t_+$ on the line. The number $r$ is odd, therefore $\theta$ is an imaginary.

In a stratum $\alpha_2 \subset \hat{K}_{b^2,s}$ a residue $\delta'_j$ is defined by the corresponding angle in the conjugated stratum $\tilde{\tau}_a(\alpha_2)$), for the corresponding pair of strata $(\alpha_2, \tilde{\tau}_a(\alpha_2))$ we have a one-to-one $\tilde{\tau}_a$-involutive products: $\prod_{j=1}^{r} \delta_j|_{\tau_a(\alpha_2)} = \prod_{j=1}^{r} \delta'_j|_{\alpha_2}$. In an $s$-stratum $\alpha_2$ the product $\theta = \prod_{j=1}^{r} \bar{\delta'}_j$ is associated with the positive base vector $+1 \in t_-$ on the line. This residue $\theta$ is called the total $s$-residue for $\alpha_2$.

The following sums for a $t$-stratum $\alpha_1$:

$$\Theta_1^R = \sum_{j=1}^{r} \delta_j; \quad \Theta_1^L = \sum_{j=1}^{r} (-\bar{\delta}_j) \tag{185}$$

are integer lifts of $\theta$, let us call the first sum the inner-integer (the total) resedue.

For an $s$-stratum $\alpha_2$ the following sums:

$$\Theta_2^R = \sum_{j=1}^{r} (\delta'_j); \quad \Theta_2^L = \sum_{j=1}^{r} (-\bar{\delta}'_j). \tag{186}$$

are integer lifts of the conjugated total residue $\bar{\theta}$ in $\alpha_2$, let us call the first sum the inner-integer resedue. The second sums in (185), (186) are not required in the formulas, but with the expressions is possible one-to-one rewrite all the formula in the construction.

A residue $\delta_j$ is $\tilde{\tau}_a$-invariant, an integer residue $\Theta^R$ is $\tilde{\tau}_a$-invariant, the total resedue $\theta$ is $\tau_a$-antisymmetric: we get for the corresponding two points $\mathbf{x} \in \alpha_1, \tau_a(\mathbf{x}) \in \tau_a(\alpha_1)$:

$$\bar{\theta}|_{\mathbf{x}} = \theta|_{\tau_a(\mathbf{x})},$$

$$\sum_{j=1}^{r} \delta_j|_{\mathbf{x}} = \sum_{j=1}^{r} \delta'_j|_{\tilde{\tau}_a(\mathbf{x})}.$$

In a maximal stratum of an arbitrary type on $\hat{K}_{b^2}$ let us define the morphism $F^{(0)}$ by the formula:

$$F^{(0)} : ((x, y), t_+) \mapsto ((x, y), -\theta t). \tag{187}$$

$$F^{(0)} : ((x, y), t_-) \mapsto ((x, y), \bar{\theta} t). \tag{188}$$

The formulas (187), (188) are well defined on the intersection of components (177). Take a maximal $t$-stratum $\alpha_1$ and join this stratum with a neighbour stratum $\alpha_2$ in the $s$-component trough a common strata of the deep 1 in $\hat{K}^{(1)}$. The total inner residue $\theta_1$ in $\alpha_1$ is well-defined. But, if we get the extension of the $t$-coordinate in a neighboring stratum $\alpha_2$ (this means the local extension of the coordinate system in $\alpha_1$ to $\hat{K}_{b^2;s}$)), this coordinate system is not regular.

By this extension, we define a deformation of $\Theta_1^R$ (185) into $\Theta_2^R$ (186). We get the following relation:

$$\theta_1 = \bar{\theta}_2; \quad \Theta_1^R = \Theta_2^R. \tag{189}$$

To calculate $\theta_2$ we have to look at all prescribed collection of angles at the $t$-stratum $\alpha_1$. Denote by $\{\delta_j\}_{\alpha_1}$ the prescribed collection of resedues. When we pass to a (neighbour) $s$-strata $\alpha_2$, we may to look at the extended collection: $\{\delta'_j\}_{\alpha_2}$, which is the conjugated collection $\{\bar{\delta}_j\}_{\alpha_1}$ for all residues instead the residue that is deformed. The product $\theta_2 = \prod_{j=1}^{r} \bar{\delta'}_j$ in $\alpha_2$ coinsids with $\bar{\theta}_1 = \prod_{j=1}^{r} \delta_j$ in $\alpha_1$.

The collection $\{\delta'_j\}_{\alpha_2}$ is not the prescribed collection of resedues in $\alpha_2$. To get the prescribed collection $\{\bar{\delta'}_j\}_{\alpha_2}$, we have to change all angles by the conjugation. As the conclusion, to get the first formula (189). The second formula is deduced from (186): the sum $\sum_{j=1}^{r}(\delta'_j)$ in $\alpha_2$ coinsids to the sum $\sum_{j=1}^{r} \delta_j$ in $\alpha_1$.

We define the deformation of $\Theta^R_1$ for $t$-strata $\alpha_1$ into $-\Theta^R_2$ for $s$-strata $\alpha_2$ along the segment, which agree with the complex structure of $E_{\mathbb{C}}$ on the $t$-stratum $\alpha_1$. This is a linear extrapolation over the inner segment by the first formula (185), (186). In the stratum of the deep 1 one positive inner residue is omitted, we get the clockwise rotation of the negative target base vector on $t_+$ trough the angle $\frac{\pi}{2}$ (algebraically, trough the angle $-\frac{\pi}{2}$). When we get a back deformation in the $s$-stratum $\alpha_2$ of the target of the same geometrically negative base vector on $t_-$, we have contr-clockwise rotation by the same conjugated angle. Algebraically formulas for integer total resedues are defined as inner extrapolation with no conjudated terms. The second formula (189) defines the opposite boundary conditions of the deformation; this gives the deformation of the common base vector on the common line: $-1 \in t_+$-line in $\alpha_1$ and $-1 \in t_-$-line in $\alpha_2$.

In the middle of the path the target of the base vector $-1 \in t_+$ from $\alpha_1$ coincids with the target of the base vector $+1 \in t_-$ from $\alpha_2$. As the result we get the common target by $F$ (for $t_+$-line in the $t$-stratum $\alpha_1$ and for $t_-$-line in the $s$-stratum $\alpha_2$) over the middle point of the deformation, which belongs to a stratum of the deep 1.

The deformation of the second common line (for $t_-$-line in the $t$-stratum $\alpha_1$ and for $t_+$-line in the $s$-stratum $\alpha_2$) is defined as the real axis conjugated deformation. The explicite definition uses the interpolation of $\overline{\Theta_1}^R$ into $-\overline{\Theta_2}^R$. This deformation determines the real-conjugated deformation of the targets of the base vectors $+1 \in t_-$ in $\alpha_1$ and $+1 \in t_+$ in $\alpha_2$.

There is no obstruction for extension over the full collection of segments over intersection of the components of the complex (178), because the halph extension, constructed above, is translated by the involution $\tau_a$ with fixed boundary condition to the extension over the complex. When residues and the comlex structure in $E_{\mathbb{C}}$ are conjugated, we get the gauge $\Theta^R \mapsto \Theta^L$ in the construction.

### 7.5.3 Proof of Lemma 37

Let us check that the mapping defined above by (187), (188) is $(\tau_a, \tau_b, \tau_i)$-equivariant.

In fact, for the base vectors $t = +1$:

$$F^{(0)}\tau_i((x,y),t_+) = F^{(0)}((x,y),-t_-) = ((x,y),-\overline{\theta})$$
$$\tau_i F^{(0)}((x,y),t_+) = \tau_i((x,y),-\theta t) = ((x,y),-\overline{\theta})$$

$$F^{(0)}\tau_a((x,y),t_+) = F^{(0)}((y,x),t_+) = ((y,x),-\bar{\theta}t)$$
$$\tau_a F^{(0)}((x,y),t_+) = \tau_a((x,y),-\theta t) = ((y,x),-\bar{\theta}t)$$

$$F^{(0)}\tau_b((x,y),t_+) = F^{(0)}((-y,x),t_+) = ((-y,x),-\theta)$$
$$\tau_b F^{(0)}((x,y),t_+) = \tau_b((x,y),-\theta t) = ((-y,x),-\theta)$$

Lemma 37 is proved. □

Let us extend the mapping $F^{(0)}$ on segments in the origin $E_{\mathbb{R}}$ over $\hat{K}_{b^2}$, which join centers of maximal simplexes, the centers in the common coordinate $\hat{K}_{b^2,t}$, or, $\hat{K}_{b^2,s}$.

Let us consider two simplexes in $\hat{K}_{b^2,+t}$, with non-degenerated coordinates $\lambda_j$, which have a common face of the codimension 1. This means, in particular, that centers of the simplexes are distinguished by the only residue. Let us denote by $\hat{\theta}'$ the product of all residues for the first simplex, and by $\hat{\theta}''$ the product of all residues of the second simplex. Then $\theta' = \exp(\pi\hat{\theta}'/2)$, $\theta'' = \exp(\pi\hat{\theta}''/2)$.

Let us consider a segment of the homotopy, which is inside $\hat{K}_{b^2,t}$, or inside $\hat{K}_{b^2,s}$ (Case $I$). On the segment with $r$, which is join centers $((x',y'),(y',x'))$, $((x'',y''),(y'',x''))$ of the first and second simplexes correspondingly, let us denote $F^{(1)}$ by the following formula, where the value $t = \pm$ in the target is defined by the corresponding ray in the layer of the bundle:

$$F^{(1)} : ((1-r)(x',y') + r(x'',y''),t_+) \mapsto$$
$$\mapsto ((1-r)(x',y') + r(x'',y''), -\exp(((1-r)\hat{\theta}' + r\hat{\theta}'')\pi/2)t) \qquad (190)$$

$$F^{(1)} : ((1-r)(x',y') + r(x'',y''),t_-) \mapsto$$
$$\mapsto ((1-r)(x',y') + r(x'',y''), \overline{\exp(((1-r)\hat{\theta}' + r\hat{\theta}'')\pi/2)}t) \qquad (191)$$

Let us consider a segment of the homotopy $F^{(1)}$, which joins a stratum in $\hat{K}_{b^2,t}$ with $\hat{K}_{b^2,s}$ (Case $II$). In this case the formula for $F^{(1)}$ vas described above using (189). A lift $\Theta^R$ into the imaginary axis of the total resedues $\theta$, which describes the target of the negative base vector of $t_+$ in $\alpha_1 \subset \hat{K}_{b^2,t}$ and the target of the negative base vector of $t_-$ in $\alpha_2$ (Case $II;+$) is defined by the formula for differential of the mapping: $d(\theta) = +\mathbf{i}$. The target of the positive base vector of $t_-$ in $\alpha_1 \subset K_{b^2,t}$ and the target of the positive base vector of $t_+$ in $\alpha_2$ are deformed by the equivalent formula for the differential: $d\bar{\theta} = -\mathbf{i}$ (Case $II;-$).

Let us check that the formula gives $(\tau_a, \tau_b, \tau_i)$-equivariant mapping. Let us assume that the $t$-component in the source of $\tau_a$ and $s$-component in the target of $\tau_a$ are considered (Case I).

Namely, for the base vectors $s = t = +1$,

$$F^{(1)}\tau_a :((1-r)(x',y') + r(x'',y''), t_+) = F^{(1)}((1-r)(y',x') + r(y'',x''), s_-) \mapsto$$
$$\mapsto ((1-r)(y',x') + r(y'',x''), -\exp(((1-r)\bar{\hat{\theta}}' + r\bar{\hat{\theta}}'')\pi/2)).$$

On the other hand,

$$\tau_a F^{(1)} :((1-r)(x',y') + r(x'',y''), t_+) \mapsto$$
$$\mapsto \tau_a((1-r)(x',y') + r(x'',y''), -\exp(((1-r)\hat{\theta}' + r\hat{\theta}'')\pi/2)) =$$
$$= ((1-r)(y',x') + r(y'',x''), -\overline{\exp(((1-r)\hat{\theta}' + r\hat{\theta}'')}\pi/2)).$$

Because $\overline{\exp(z)} = \exp(\bar{z})$, the $\tau_a$-equivariance is proved.

For Case $II;+$ (the $t_+$-line strats from $\alpha_1 \subset \hat{K}_{b^2,t}$ and ends by $t_-$-line to $\alpha_2 \subset \hat{K}_{b^2,t}$) in $\alpha_1$ we get the homotopy of the target by the vector $-1 \in t_+$ with infinitesemal $d(\Theta^R_{\alpha_1}) = +\mathbf{i}$. This homotopy is given by the interpolation of $\Theta^R_1$ into $-\Theta^R_2 = -\Theta^R_1$.

The $\tau_a$-conjugated deformation starts by $t_+$-line from $\tau_a(\alpha_1) \subset \hat{K}_{b^2,s}$ and ends by $t_-$-line to $\tau_a(\alpha_2) \subset \hat{K}_{b^2,t}$. This homotopy $\tau_a(F)$ is given by the interpolation of targets of boundaries over $-\Theta^R_1$ and $\Theta^R_2$. When $t$ is outside the segment of the interpolation near the point $\Theta^R_2 = \Theta^R_1$ in $\alpha_2$, we get the boundary condition: $\tau_a(F(\Theta^R_2 - 0)) = \tau_a(F(\Theta^R_2 + 0))$. This fact is: the targets of base vectors $t_+, t_-$ on a common line in $E_{\mathbb{R}}$ in the case the segments join different components of (178) have common real $z$-coordinates over the middles of the segments.

This implies the homotopy $\tau_a(F)$ of the target by the vector $+1 \in t_-$ in $\tau_a(\alpha_2)$ with infinitesemal $d(\Theta^R_{\alpha_2}) = -\mathbf{i}$.

Because the complex structure in the image is changed, we have the gauge $\Theta^R \mapsto \Theta^L$.

Then we assume that the case $I$ for $t = +1$ is considered:

$$F^{(1)}\tau_i :((1-r)(x',y') + r(x'',y''), t_+) = F^{(1)}((1-r)(x',y') + r(x'',y''), -t_-) \mapsto$$
$$\mapsto ((1-r)(x',y') + r(x'',y''), -\overline{\exp(((1-r)\hat{\theta}' + r\hat{\theta}'')\pi/2)}).$$

On the other hand,

$$\tau_i F^{(1)} :((1-r)(x',y') + r(x'',y''), t_+) \mapsto$$
$$\mapsto \tau_i((1-r)(x',y') + r(x'',y''), -\exp(((1-r)\hat{\theta}' + r\hat{\theta}'')\pi/2)t) =$$
$$= ((1-r)(x',y') + r(x'',y''), -\overline{\exp(((1-r)\hat{\theta}' + r\hat{\theta}'')\pi/2)}) =$$

The deformation in Case $II$ of the two lines are real-conjugated and starting targets of the two lines coincide. This proves $\tau_i$-equivariance.

For the The $\tau_i$-equivariance is proved. The $\tau_b$-equivariance is evident. Lemma 37 is proved. □

Let us conclude with the proof of Theorem 36. The obstruction of an extension of the morphism, which is constructed in Lemma 37 on strata of the deep 3 and greater are trivial. The obstruction to extend of the morphism on strata of the deep 2 and greater are calculated explicitly. Take a stratum $\hat{K}_{b^2}^{[r-2]}$ of the deep 2, to this stratum strata of the deeps 1 and 0 are bounded. Because on each such a stratum the $t$-system is well-defined, the morphism $F$ on the hall collection of the strata is given by a common formula. Below we say that in the construction of the morphism (197) the monodromy of the morphism $F$ over each closed loop on the boundary of the codimension 2-strata equals to zero. Theorem 36 is proved. □

#### 7.5.4 A morphism of a line bundle $\tilde{\tau}_a$-skew-invariant and $\tilde{\tau}_b$-invariant into the trivial $\mathbb{C}$-bundle over $\hat{K}_{b^2}$

Let us consider a $(\tau_a, \tau_b, \tau_i)$-morphism $F$ of the bundle $E_{\mathbb{R}}$ into the bundle $E_{\mathbb{C}}$, which is constructed in Theorem 36. Let us extend this equivariant morphism by a tensor product, using an additional coordinate $w$: $(\tau_a, \tau_b, \tau_i) \mapsto (\tilde{\tau}_a, \tilde{\tau}_b, \tilde{\tau}_i)$.

Define $(\tau_a, \tau_b, \tau_i)$-equivariant $S^0$-covering over $\hat{K}_{b^2}$ (a double covering), which is considered as a bundle $E_S$ over $\hat{K}_{b^2}$ with the sphere $S^0 = \{\pm 1\}$ in a layer. Without equivariant structure we get: $E_S = \hat{K}_{b^2} \times S^0$. The involutions are defined by the following formula:

$$\tau_a : ((x, y), w) \mapsto ((-x, y), -w) \qquad (192)$$

$$\tau_b : ((x, y), w) \mapsto ((-y, x), w) \qquad (193)$$

$$\tau_i : ((x, y), w) \mapsto ((x, y), -w). \qquad (194)$$

Involutions $\tau_b$, $\tau_i$ are now defined in an extended bundle $E_{\mathbb{R}} \otimes E_S$, let us denote the involutions by $\tilde{\tau}_b$, $\tilde{\tau}_i$ correspondingly. The involution $\tau_a$ is also defined in the extended bundle, let us denote $\tilde{\tau}_a = \tau_i \circ \tau_b \circ \tau_a$, more detailed, let us define $\tilde{\tau}_a$ by the formulas:

$$\tilde{\tau}_a : ((x, y), t_{\pm}) \mapsto ((-x, y), -t_{\mp})$$

$$\tilde{\tau}_a : ((x, y), z) \mapsto ((-x, y), z)$$

and by the formula:

$$\tilde{\tau}_a : ((x, y), w) \mapsto ((x, y), w),$$

which is followed from (192),(79) and from the formula below (179). Let us note that after re-denotation, if $(x, y) \in \hat{K}_{b^2,t}$, then $(-x, y) \in \hat{K}_{b^2,s}$, but we use $t$-system and the formula (179).

Involutions $(\tilde{\tau}_b, \tilde{\tau}_a, \tilde{\tau}_i)$ are defined in the bundle $E_{\mathbb{C}}$ by a natural way. The involutions $\tilde{\tau}_b$, $\tilde{\tau}_a$ on $\mathbb{C}$–bundle are trivial, the involution $\tilde{\tau}_i$ acts in each layer by the conjugation. The involution $\tau_a$ invariantly transforms each oriented layer of the line subbundle $t_+$ in the bundle $E_{\mathbb{R}}$ into the oriented layer of another line subbundle $t_-$ over the corresponding point of the base, by an orienting preserved morphism of the layers. The same way, the involution $\tilde{\tau}_a$ preserves the orienting layers of the bundle $E_{\mathbb{R}}$ and changes the orientation of the base vectors. Involution in extended bundle allows to define an equivariant morphism $\tilde{F}$, as a natural extension of the equivariant morphism $F$. Thus, we define a $(\tilde{\tau}_a, \tilde{\tau}_b, \tilde{\tau}_i)$-equivariant morphism $\tilde{F}$.

In the collection of the involutions only the involution $\tilde{\tau}_i$ is non-trivial on $E_S$ and conjugates the complex structure in $E_{\mathbb{C}}$. The involution $\tilde{\tau}_i$ is an exceptional, because this involution changes the $w$-coordinate (the coordinate of the covering).

Let us formulate the result in the following lemma.

**Lemma 38.** *Consider the $(\tilde{\tau}_a, \tilde{\tau}_b, \tilde{\tau}_i)$-equivariant tensor product $E_{\mathbb{R}} \otimes E_S$, $E_{\mathbb{C}} \otimes E_S$.*

*1. The formula of $F$ is defined $(\tilde{\tau}_a, \tilde{\tau}_b, \tilde{\tau}_i)$-equivariant morphism*

$$\tilde{F} : E_{\mathbb{R}} \otimes E_S \mapsto E_{\mathbb{C}} \otimes E_S.$$

*2. The $\tilde{\tau}_i$-equivariant factormorphism*

$$F^{\downarrow} : E_{\mathbb{R}} \otimes_{\tilde{\tau}_a, \tilde{\tau}_b} E_S \mapsto E_{\mathbb{C}} \otimes_{\tilde{\tau}_a, \tilde{\tau}_b} E_S$$

*is well-defined.*

**Proof of Lemma 38**

This is well-known fact from linear algebra. □

### 7.5.5 A calculation of the factor-bundles

**Lemma 39.** *The bundle $S(E_{\mathbb{C}}) \otimes_{(\tilde{\tau}_a, \tilde{\tau}_b)} E_S$ is isomorphic to the trivial $\tilde{\tau}_i$-equivariant bundle $K \times S^1 \to K$.*

*Доказательство.* Define the mapping $F_{\mathbb{C}} : S(E_{\mathbb{C}}) \otimes_{(\tau_a, \tau_b, \tau_i)} E_S$ into $K \times S^1$ by the formula:

$$F_{\mathbb{C}} : ((x,y), z, w) \mapsto ([x,y], c_w(z)).$$

In this formula $z \in \mathbb{C}, |z| = 1, w \in S^0 = \{\pm 1\}$. Let us recall that $(x,y) \in K_{b^2}$ и $(x,y) = (-x,-y)$, $[x,y] \in K$, where $[x,y] = a[x,y] = b[x,y]$. Additionally, here we have used

$$c_w(z) = \begin{cases} z, & \text{if} w = 1; \\ \overline{z}, & \text{if} w = -1. \end{cases}$$

It is clear that $c_w(z) = c_{-w}(\overline{z})$.

Let us check that $F_{\mathbb{C}}$ is well-defined. In fact,

$$\tau_a((x,y),z,w) = ((y,x),\overline{z},-w) \overset{F_{\mathbb{C}}}{\mapsto} ([y,x],c_{-w}(\overline{z}))$$
$$\tau_b((x,y),z,w) = ((-y,x),\overline{z},-w) \overset{F_{\mathbb{C}}}{\mapsto} ([-y,x],c_{-w}(\overline{z}))$$
$$\tau_i((x,y),z,w) = ((x,y),\overline{z},-w) \overset{F_{\mathbb{C}}}{\mapsto} ([x,y],c_{-w}(\overline{z}))$$

In the result in all cases we get the same point $([x,y],c_w(z)) \in K \times S^1$.

The inverse mapping $G_{\mathbb{C}} : K \times S^1 \to S(E_{\mathbb{C}}) \otimes_{(\tau_a,\tau_b,\tau_i)} E_S$ is given by the formula:

$$G_{\mathbb{C}} : ([x,y],z) \mapsto ((x,y),z,1),$$

where $z \in S^1 \subset \mathbb{C}$. To check that the formula is well-defined it has to be checked that when a representative of the point $([x,y],z) \in K \times S^1$ is replaced, the point $((x,y),z,1) \in S(E_{\mathbb{C}}) \otimes_{(\tau_a,\tau_b,\tau_i)} E_S$ remains the same. A gauge is the following: $([y,x],z) = ([-y,x],z) = ([x,-y],z)$. It is easy to check that the points $((y,x),z,1)$, $((-y,x),z,1)$, $((x,-y),z,1)$ coincide to the point $((x,y),z,1) \in S(E_{\mathbb{C}}) \otimes_{(\tau_a,\tau_b,\tau_i)} E_S$.

Additionally, it is evident that $F_{\mathbb{C}}$ and $G_{\mathbb{C}}$ are commuted with the projections onto the base $K$ and that this morphisms are inversed. $\square$

**Lemma 40.** *The bundle $S(E_{\mathbb{R}}) \otimes_{(\tilde{\tau}_a,\tilde{\tau}_b,\tilde{\tau}_i)} E_S$ isomorphic to the wedge covering $K_b \to K$.*

*Доказательство.* The bundle $S(E_{\mathbb{R}}) \otimes_{(\tau_a,\tau_b,\tau_i)} E_S$ after a gauge $(\tau_a,\tau_b,\tau_i) \mapsto (\tilde{\tau}_a,\tilde{\tau}_b,\tilde{\tau}_i)$ is splitted into a $\tilde{\tau}_a$ equivariant wedge of the $(\tilde{\tau}_b,\tilde{\tau}_i)$-equivariant non-trivial line -bundles. $\square$ $\square$

### 7.5.6 The convolution of the tensor product

By Lemma 40 and Lemma 38, Statement 2 the factormorphism $F^{\downarrow}$ is the pair of morphisms of line bundles:

$$F_{+1} : \hat{K}_b^1 \otimes E_S \to S^1 \otimes E_S, \tag{195}$$

$$F_{-1} : \hat{K}_b^1 \otimes E_S \to S^1 \otimes E_S. \tag{196}$$

The morphism (195), restricted to the subbundle $\hat{K}_b^1 \otimes \{+1\} \subset \hat{K}_b^1 \otimes E_S$, takes values in $E_S \otimes \{+1\}$. This morphism is the required in Theorem 32 mapping, on the subpolyhedron consists of all strata of deeps 0 and 1. The morphism (196), restricted to the subbundle $\hat{K}_b^1 \otimes \{-1\} \subset \hat{K}_b^1 \otimes E_S$ in the target, is the second (conjugated) copy of (195), it takes values in $E_S \otimes \{-1\}$.

### 7.5.7 An extension of the morphism (195) over $\hat{K}^1$ to a morphism over $\hat{K}^2$ and over $\hat{K}$

Let us extend the constructed morphism onto a morphism of the prescribed bundles 2-skeleton. Because over $\hat{K}_b$ the boundary of of a strata of the deep 2 is inside one map, the problem of the extension is local.

In a prescribe map the formula of $F$ has the trivial monodromy over a path $l$ in the boundary of a 2-cell. In a neighborhood of a deep 2 strata $\alpha^2$ there are 4 maximal strata, which are destinguished by various inner values of the angles, which are degenerated on the deep 2 stratum. Because $\alpha^2$ is a stratum with an odd number $d-2$ of angles, a $\pm$ type of the strata is well-defined by the sign of prescribed angles. Also we may assume that the covering $\tau_a$ admits a prescribed trivialization, which is agree with the trivialization on $\alpha^2$. Assume for simplicity that $\alpha^2$ is a $t$-stratum.

To prove that monodromy along $l$ is trivial, let us consider the case when the only maximal stratum $\beta_1$ is $s$-stratum, and there are 3 $r$-strata. In the case all maximal strata around $\alpha^2$ are of the same $t$-type, included $\beta_1$, the statement is clear: the monodromy around $\alpha^2$ is trivial. In the case $\beta_1$ is an $s$-stratum, we do a formal elementary flips, this flip changes a type of the maximal strata $\beta_1$ into its opposite. By this agreement, a maximal stratum $\beta_1$ has the positive inner type with a number of prescribed positive inner angles less then $\frac{r}{2}$. The formal elementary flip preserves the monodromy.

Denote by

$$\hat{\pi} : \hat{K}_{\mathbf{I}_b;\circ} \to S^1 \tag{197}$$

the mapping, which is determined by the constructed morphism (195), (restricted over the 2-skeleton of the polyhedron), recall, the polyhedron $\hat{K}_{\mathbf{I}_b;\circ}$ is defined by (158).

The extension of the morphism (197) over $\hat{K}^2$ to a morphism over the polyhedron $\hat{K}$ is obvious, because obstructions belong to $\pi_i(S^1)$, $i \geq 2$. Let us formulate this fact as a theorem.

**Theorem 41.** *There exists a morphism (197), which is used in Theorem 32.*

## 7.6 Additional properties of the resolution (197)

Recall, $dim(\hat{K}_{\mathbf{I}_b;\circ}) = n-k+1, \quad n-k+1 \equiv 1 \pmod 2$. The boundary of the polyhedron $\hat{K}_{\mathbf{I}_b;\circ}$ consists of points on all regular strata, which are closed to antidiagonal (242). Let us consider of the open polyhedron, a thin regular neighborhood of the boundary of $\hat{K}_{\mathbf{I}_b;\circ}$, which is denoted by $\hat{U}_\circ \subset \hat{K}_{\mathbf{I}_b;\circ}$. Analogously, define $U_\circ \subset K_{\mathbf{I}_b;\circ}$.

Consider the mapping $\hat{U}_\circ \to S^1$, which is the restriction of the mapping $\hat{\pi} : \hat{K}_{\mathbf{I}_b;\circ} \to S^1$, the quotient of the equivariant mapping (197). Because the structuring group $\hat{U}_\circ$ is reduced to the subgroup $(b,c) \subset C$, the mapping $\hat{U}_\circ \to S^1$ contains values in the even index 2 subgroup $2[\pi_1(S^1)] \subset \pi_1(S^1)$.

**Lemma 42.** *1. There exists a subpolyhedron $\hat{U}_{reg,\circ} \subset \hat{U}_\circ$, such that the restriction of the mapping $\hat{\pi}$ to the complement $\hat{U}_\circ \setminus \hat{U}_{reg,\circ}$ is homotopic to the mapping into a point, and the polyhedron itself admits the following mappings:*

$$\hat{U}_{reg,\circ} \xrightarrow{\hat{\eta}_{reg}} \hat{P} \xrightarrow{\hat{\zeta}_{reg}} K(\{b,c\},1), \tag{198}$$

*where $\hat{P}$ is a polyhedron of the dimension $\frac{(3n-3k+6)}{4}$ (assuming $k \equiv 0 \pmod 4$)) (i.e. of the dimension a little greater then $\frac{3}{4}dim(\hat{K}_{\mathbf{I}_b;\circ}) = \frac{3(n-k+1)}{4}$); $\hat{\eta}_{reg}$, $\hat{\zeta}_{reg}$ are mappings, which are called the control mapping and the structuring mapping, correspondingly, for $\hat{P}$.*

*2. The covering subpolyhedron $U_{reg,\circ} \subset U_\circ$, such that the restriction of the mapping $\pi$ to the complement $U_\circ \setminus U_{reg,\circ}$ is homotopic to the mapping into a point, and the polyhedron itself admits the following mappings:*

$$U_{reg,\circ} \stackrel{\eta_{reg}}{\longrightarrow} P \stackrel{\zeta_{reg}}{\longrightarrow} K(\{b\},1), \tag{199}$$

*where $P$ is a polyhedron of the dimension $\frac{n-k+2}{2}$ (i.e. of the dimension a little greater then $\frac{1}{2}dim(K_{\mathbf{I}_b;\circ}) = \frac{n-k+1}{2}$), $\eta_{reg}$, $\zeta_{reg}$ are mappings, which are called the control mapping and the structuring mapping, correspondingly, for $P$.*

### Proof of Lemma 42

Proof of Statement 1. Consider in the polyhedron $\hat{U}_{reg,\circ} \subset \hat{K}_{\mathbf{I}_b;\circ}$: the polyhedron of elementary strata with prescribes coordinate system, see Susbsubsection (7.2.7). Let us define a "nice"coordinate system on each stratum of the polyhedron. A "nice"coordinate system is a coordinate system, which is attached to the anti-diagonal boundary, in which coordinates with residues $-\mathbf{i}$ are singular and associated momenta are small, and coordinates with residue $+\mathbf{i}$ are regular and associated momenta is defined a hyperface of the corresponding hyperface of the $\lambda$-simplex.

The prescribed coordinate system on an elementary stratum could be different with respect to the "nice"coordinate system. Such a stratum let us call regular. Let us consider all strata for which the "nice"coordinate system coincides with prescribed coordinate system. This implies that for a regular stratum the number of "nice"residue $+\mathbf{i}$ less or equals to the number of "nice"angles with the residue $-\mathbf{i}$. Denote the polyhedron of all "nice"strata by $\hat{Q}_{nice} \subset \hat{U}_{reg,\circ}$ Let us define a $l$-polyhedron, denoted by $\hat{Q}^{(l)} \subset \Delta_{antidiag}$, which is defined as the closure of all maximal strata in $\hat{U}_{reg,\circ}$, with the antidiagonal boundary a full subpolyhedron strata of the deep $l$. Obviously,

$$\bigcup_{l, l \le \frac{n-k+2}{4}} \hat{Q}^{(l)} \supset \hat{Q}_{nice}. \tag{200}$$

Each polyhedron in (200) admits a mapping onto a polyhedron of the dimension $\frac{3n-3k+6}{4}$, because $\frac{n-k-2}{4}$ momenta coordinates, which is attached to residues $-\mathbf{i}$ can be omitted. The degeneration of momenta in $\hat{Q}^{(l)}$ are possible only with residues $+\mathbf{i}$, the polyhedron has only "nice"faces.

The mapping $\pi$ (197), restricted on $\hat{K}_{\mathbf{I}_b;\circ} \setminus \hat{Q}_{nice}$, is homotopic to constant mapping, because maximal strata have only admissible boundaries. Lemma 42 Statement 1 is proved.

Proof Statement 2. Consider in the polyhedron $U_{reg,\circ} \subset \hat{K}_{\mathbf{I}_b;\circ}$ elementary strata with prescribes coordinate system, see Susbsubsection (7.2.7). Let us define a "nice"coordinate system on each stratum of the polyhedron as in Statement 1. Let us consider all strata for which the "nice"coordinate system coincides with prescribed coordinate system. This implies that for a regular stratum the number of "nice"residue $+\mathbf{i}$ less or equals to the number of "nice"angles with the residue $-\mathbf{i}$. In particular, for a regular stratum the number of "nice"residue $+\mathbf{i}$ less, or equals to the number of all coordinates of the stratum.

Denote by $P \subset \Delta_{antidiag}$ the subpolyhedron, consists of boundaries of all regular strata. The polyhedron $P$ is equipped by the projection $U_{reg;\circ} \longrightarrow P$ as a projection, which keeps all angles with the residue $+\mathbf{i}$ and all associated momenta. By definition $P$ is a subpolyhedron in $S^{n-k+1}/\mathbf{i}$. This implies that the structured mapping $\zeta_{reg}$ exists. Lemma 42 Statement 2 is proved. □

Let us consider the mapping $c_0$ and consider the polyhedrons $\mathbf{N}_\circ$. Accordingly to (149), (150), (149) a decomposition of this polyhedron into subpolyhedrons is well-defined. This decomposition corresponds to reductions of the structured group of elementary strata to the corresponding subgroups in $D$. For $k' \geq 2$ the statement about two disjoint components on the polyhedrons by a formal deformation of equivariant holonomic mapping $c_0^{(2)}$, is deduced from the following lemma.

**Lemma 43.** *The classifying mapping*

$$\pi_d = p_{A,\dot{A}} \circ \eta_{\mathbf{d}} : \hat{K}_{\mathbf{I}_d} \to K(\mathbf{D}, 1) \to K(\mathbb{Z}/2, 1) \tag{201}$$

*for the canonical covering over $\hat{K}_{\mathbf{I}_d}$, defined by (160), where $\eta_d$ is the structuring mapping, $p_{A,\dot{A}}$ is the projection on the residue class of the subgroup $(A, \dot{A}) \subset \mathbf{D}$, is lifted to a mapping:*

$$\bar{\pi}_d : \hat{K}_{\mathbf{I}_d} \to S^1, \tag{202}$$

*where on each elementary strata of the polyhedrons the mapping $\bar{\pi}$ is homotopic to a constant mapping.*

## Proof of Lemma 43

Let us assume $N >> n$ ($N = n + 4$ is sufficient) and let us consider a "stabilization"of the mapping $c_0$, by a replace in the definition of this mapping the parameter $n \mapsto N$. Then an inclusion of polyhedrons

$$\hat{K}_{\mathbf{I}_d}(n) \subset \hat{K}_{\mathbf{I}_d}(N). \tag{203}$$

is well-defined. Let us consider a subpolyhedron in $\hat{K}_{\mathbf{I}_d}(N)$, which is denoted the same, consists of elementary strata with a complete collection of residues af all types $+1, -1, +\mathbf{i}, -\mathbf{i}$. It is clear, that the inclusion (244) is well-defined after by this "stabilization". Moreover, the inclusion (244) is naturally with respect to structuring mappings, therefore the statement of Lemma is sufficiently to prove for the polyhedrons $\hat{K}_{\mathbf{I}_d}(N)$.

The projections $p_b : \hat{K}_{\mathbf{I}_d}(N) \to \hat{K}_{\mathbf{I}_b}(N)$, $p_{a\times\dot{a}} : \hat{K}_{\mathbf{I}_d}(N) \to \hat{K}_{\mathbf{I}_{a\times\dot{a}}}(N)$ are well-defined, moreover, the classifying mapping $\bar{\pi}_{\mathbf{d}}$ is the sum of the two mappings (the mappings into the circle are summed in the right hand side of the formula):

$$\bar{\pi}_d(N) = \pi_{a\times\dot{a}} \circ p_{a\times\dot{a}} + \pi_b \circ p_b,$$

where $\pi_{a\times\dot{a}}$ is obtained from (197), $\pi_b$ is (197), $\pi_{a\times\dot{a}}$ is analogous mapping, constructed in Theorem 33. Lemma 43 is proved. □

Let us conclude investigations of properties of polyhedrons $\mathbf{N_b}$, $\mathbf{N_{a\times\dot{a};\circ}}$ by the following calculation of characteristic numbers, which is required for the next subsection and Theorem 22. The fundamental class of the polyhedrons $\mathbf{N}_{\mathbf{I}_{a\times\dot{a}};\circ}$, $\mathbf{N}_{\mathbf{I}_{a\times\dot{a}};\circ}$ are well-defined, because the polyhedrons is a union of maximal strata, and each maximal strata is equipped with the fundamental class. The same is satisfied for an arbitrary submanifold of an elementary strata, which is generic with respect to the boundary of its maximal strata.

**Lemma 44.** *1. The Hurewich image of the fundamental class of the polyhedrons $\mathbf{N}_{\mathbf{I}_{a\times\dot{a}};\circ}$ in the group $H_{n-2k+1}(K(\mathbf{D},1))$, $dim(\mathbf{N}_\circ) = n-2k+1$ is trivial. The image of the fundamental class of the closed polyhedrons $\mathbf{N}_{\mathbf{I}_b}$ (152) by the structuring mapping $\eta$ is translated into the generator in $H_{n-2k+1}(K(\mathbf{D},1);\mathbb{Z}/2)$.*

*2. The Hurewich image of the fundamental class of an arbitrary codimension $2s$ subpolyhedron in $\mathbf{N}_{\mathbf{I}_{a\times\dot{a}};\circ}$ (which is invariant with respect to the involution of the double covering $\mathbf{N}_{\mathbf{I}_{a\times\dot{a}};\circ} \longrightarrow \hat{\mathbf{N}}_{\mathbf{I}_{a\times\dot{a}};\circ}$ ($K_{\mathbf{I}_b;\circ} \to \hat{K}_{\mathbf{I}_b;\circ}$) in the group $H_{n-2k-2s+1}(K(\mathbf{D},1))$ is trivial.*

**Proof of Lemma 44**

Proof of Statement 1. The polyhedron $\mathbf{N_b}$ is decomposed into a collection of maximal strata, each stratum is a manifold and contains the fundamental class. The structure of maximal strata is agree to the structure of the polyhedron $K$. In $K$ there exists the only anti-diagonal stratum (with residues $_{\mathbf{i}}$) and some number of maximal strata with a mixed structure of the imaginary residues.

It is sufficiently to consider the transfer of the structuring mapping $\eta$ over each stratum separately and calculate the image the the fundamental class by the transfer $\eta^!_{A\times\dot{A}}$. For the antidiagonal maximal strata of the polyhedron $\mathbf{N_{b;\circ}}$ we get that the image of the fundamental cycle in $K(A\times\dot{A},1)$ by the projection into $K(A\times\dot{A},1) \to K(A,1)$ is translated into the fundamental class in $H_{n-k}(K(A,1);\mathbb{Z}_2)$, $k \equiv 0 \pmod 2$. The generic elementary stratum (which is not the antidiagonal maximal stratum) is the double covering over the corresponding stratum of the polyhedron $\hat{K}_b$. The transfer with respect to the subgroup $(A,\dot{A}) \subset \mathbf{D}$ is shrieked into the covering in the tower of the subgroups $(A,\dot{A}) \subset C \to \mathbf{D}$, therefore the fundamental; class of the stratum is mapped by $p_{A\times\dot{A}\to A} \circ \eta^!_{A,\dot{A}}$ in $H_{n-k}(K(A,1);\mathbb{Z}_2)$ to the trivial homology class.

Statement 2 is proved analogously. Lemma 44 is proved. □

### 7.7 Proof of Lemma 9

Let us use the description of the self-intersection polyhedrons of the formal extension of the mapping $c$, described in Subsubsection 7.2.6. By Theorem 33 there exists a mapping (202), the covering over this mapping gives the integer lift of the structuring mapping, Claim –1, Definition 7. The control condition, Claim –2, Definition 7, is a corollary of Theorem 42 Claim –1 and Proposition 34. Lemma 9 is proved. $\square$

# 8 Classifying space for iterated self-intersections of $\mathbf{I}_{\mathbf{a}\times\dot{\mathbf{a}}}$-framed immersions

## 8.1 Preliminary constructions and definitions

Let $n_1$, $n_2$ be two odd positive integers, $n_1 + n_2 = n$, $k_1, k_2$ be two even positive integers, $n_1 > k_1$, $n_2 > k_2$. Consider the Cartesian product

$$\begin{array}{c} \bar{G} = c_{0,1} \times c_{0,2} : \mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2} \to \\ S^{n_1-k_1} \times S^{n_2-k_2} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^n \end{array} \qquad (204)$$

of two mappings (138) in the corresponding dimensions. Let us denote by $I_1 \times I_2$ the pair of the involution by the **i**-multiplication on the factors $\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2}$; the involutions commute.

By the pair of involutions $(I_1, I_2)$ define another pair of involutions $(\bar{I}, \hat{I})$. The first involution $\bar{I} = diag(I_1, I_2)$ in the pair is the diagonal involution. Denote $\hat{I} = I_1$. Obviously, $I_1 \pmod{\bar{I}} = I_2 \pmod{\bar{I}}$ and one may consider the involution $\hat{I}$ as an involution on the factorspace of $\bar{I}$-involution, This point of view looks natural, because the pair of the involution $(\bar{I}, \hat{I})$, where $\hat{I}$ is a bottom (secondary) involution, $\bar{I}$ in a top (primary) involution, depends not of an order of the factor. The mapping (204) is the $\bar{I}$-double covering over the following mapping:

$$\begin{array}{c} G : (\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2})/\bar{I} \to \\ S^{n_1-k_1} \times S^{n_2-k_2} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^n \end{array} \qquad (205)$$

The mapping (205) is the $\hat{I}$-double covering over the following mapping:

$$\begin{array}{c} \hat{G} : ((\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2})/\bar{I})/\hat{I} \to \\ S^{n_1-k_1} \times S^{n_2-k_2} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^n \end{array} \qquad (206)$$

#### 8.1.1 Configuration space

Denote by $\bar{\Gamma}_\circ(\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2})$ be the canonical 2-sheeted covering over the delated product $\Gamma_\circ(\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2})$ of the space $\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2}$. There is a $\chi$-equivariant mapping, which is a formal extension of the mapping $\bar{G}$ (205):

$$\bar{G}^{(2)} : \bar{\Gamma}_\circ(\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2}) \to \mathbb{R}^n \times \mathbb{R}^n.$$

The involution $\chi$ permutes the factors in the source delated product an in the target. The involution on the covering $\bar{\Gamma}_{\circ}(\mathbb{R}\mathrm{P}^{n_1-k_1} \times \mathbb{R}\mathrm{P}^{n_2-k_2})$ over $\Gamma_{\circ}(\mathbb{R}\mathrm{P}^{n_1-k_1} \times \mathbb{R}\mathrm{P}^{n_2-k_2})$ is denoted by $T$. On the delated product $\bar{\Gamma}_{\circ}(\mathbb{R}\mathrm{P}^{n_1-k_1} \times \mathbb{R}\mathrm{P}^{n_2-k_2})$, equipped with the involution $T$, the involutions $(I_1, I_2)$ are defined as extensions of the corresponding involutions on the space. An involution and its extension is denoted by the same for short. Additionally, the involution $\hat{T}$ on the factor of the delated product with respect to $\bar{T}$-involution is defined.

The Cartesian products $G^{(2)}$ is defined by two copies of mappings (205). The mapping $\bar{G}^{(2)}$ is $(I_1, I_2, T)$-equivariant mapping $\bar{G}^{(2)}$).

### 8.1.2 Iterated self-intersection polyhedron of $\bar{G}$ (204), diagram description

On the secondary delated product $\bar{\Gamma}_{\circ}(\bar{\Gamma}_{\circ}(\mathbb{R}\mathrm{P}^{n_1-k_1} \times \mathbb{R}\mathrm{P}^{n_2-k_2}))$, we get an additional (secondary) involution $T^{ext}$, which is defined analogously to the primary involution. We get the following collection

$$(\bar{I}, \hat{I}; T, T^{ext}) \tag{207}$$

of the involutions on the secondary delated product.

Let us define a $(\bar{I}, \hat{I}; T, T^{ext})$-equivariant polyhedron

$$\overline{\mathbf{NN}}_{\circ} \subset \bar{\Gamma}_{\circ}\bar{\Gamma}_{\circ}(\mathbb{R}\mathrm{P}^{n_1-k_1} \times \mathbb{R}\mathrm{P}^{n_2-k_2}), \tag{208}$$

which is called the iterated self-intersections of $(I_1, I_2)$-equivariant mapping (204).

At the first step of the definition, let us define the $T$-equivariant polyhedron of of formal self-intersection point of the top mapping (204):

$$\bar{\mathbf{N}}_{\circ} \subset \bar{\Gamma}_{\circ}(\mathbb{R}\mathrm{P}^{n_1-k_1} \times \mathbb{R}\mathrm{P}^{n_2-k_2}), \tag{209}$$

where the bar over the targets means that this polyhedron is $T$-equivariant:

$$\begin{gathered}\bar{\mathbf{N}}_{\circ} = \{(\mathbf{x}_1 \times \mathbf{y}_1; \mathbf{x}_2 \times \mathbf{y}_2) \in \bar{\Gamma}_{\circ} : \mathbf{x}_1 \times \mathbf{y}_1 \neq \mathbf{x}_2 \times \mathbf{y}_2, \\ G(\mathbf{x}_1 \times \mathbf{y}_1) = G(\mathbf{x}_2 \times \mathbf{y}_2)\}.\end{gathered} \tag{210}$$

The definition is analogously to the formula (142). Accordingly with a previous denotation, we get $(\mathbf{y}, \mathbf{x}) = \chi(\mathbf{x}, \mathbf{y})$.

At the second step of the definition, let us consider the equivariant polyhedron (209) as the origin of the mapping inside the external (secondary) delated product. We use the formula as (142) again: we restrict the mapping $\bar{G}^{(2)}/\bar{T}$ on the polyhedron (209) at the origin. Define the (207)-equivariant polyhedron (212):

$$\begin{gathered}\overline{\mathbf{NN}}_{\circ} = \\ \{((\mathbf{x}_{1,1} \times \mathbf{y}_{1,1}; \mathbf{x}_{1,2} \times \mathbf{y}_{1,2}) \times (\mathbf{x}_{2,1} \times \mathbf{y}_{2,1}; \mathbf{x}_{2,2} \times \mathbf{y}_{2,2})) \subset \bar{\Gamma}_{\circ}\bar{\Gamma}_{\circ}(\mathbb{R}\mathrm{P}^{n_1-k_1} \times \mathbb{R}\mathrm{P}^{n_2-k_2}) : \\ (\mathbf{x}_{1,1} \times \mathbf{y}_{1,1}; \mathbf{x}_{1,2} \times \mathbf{y}_{1,2}) \neq (\mathbf{x}_{2,1} \times \mathbf{y}_{2,1}; \mathbf{x}_{2,2} \times \mathbf{y}_{2,2}); \\ G^{(2)}(\mathbf{x}_{1,1} \times \mathbf{y}_{1,1}; \mathbf{x}_{1,2} \times \mathbf{y}_{1,2}) = G^{(2)}(\mathbf{x}_{2,1} \times \mathbf{y}_{2,1}; \mathbf{x}_{2,2} \times \mathbf{y}_{2,2})\}.\end{gathered} \tag{211}$$

Of course, conditions (210) in the formula (211) for points in $\bar{\Gamma}_{\circ}(\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2})$ should be satisfied. Recall $T^{ext}$ is the involution in the external delated product, we get the quotient:

$$\mathbf{NN}_{\circ} \subset \bar{\Gamma}_{\circ}(\bar{\Gamma}_{\circ}(\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2})/\bar{T})/\bar{T}^{ext}. \qquad (212)$$

The polyhedron (208) is the $(T;T^{ext})$-covering over (212).

### 8.1.3 Iterated self-intersection polyhedron of $\bar{G}$ (204), coordinate description

The coordinate description is the following. Using the formula (211), let us consider all ordered quadruples in $\overline{NN}$:

$$\begin{array}{c}((\mathbf{x}_{1,1},\mathbf{x}_{1,2}),(\mathbf{y}_{1,1},\mathbf{y}_{1,2});(\mathbf{x}_{2,1},\mathbf{x}_{2,2}),(\mathbf{y}_{2,1},\mathbf{y}_{2,2}));\\ \mathbf{x}_{i,j} \in \mathbb{RP}^{n_1-k_1}; \quad \mathbf{y}_{i,j} \in \mathbb{RP}^{n_2-k_2}, \quad i=1,2;\ j=1,2\end{array} \qquad (213)$$

with conditions:

$$\bar{G}(\mathbf{x}_{1,1}) = \bar{G}(\mathbf{x}_{1,2}) = \bar{G}(\mathbf{x}_{2,1}) = \bar{G}(\mathbf{x}_{2,2}) \in \mathbb{R}^{n_1};$$

$$\bar{G}(\mathbf{y}_{1,1}) = \bar{G}(\mathbf{y}_{1,2}) = \bar{G}(\mathbf{y}_{2,1}) = \bar{G}(\mathbf{y}_{2,2}) \in \mathbb{RP}^{n_2}.$$

Define (208) as the space of all ordered quadruples (213) with the condition: $\mathbf{x}_{i,1} \ne \mathbf{x}_{i,2}$, $\mathbf{y}_{i,1} \ne \mathbf{y}_{i,2}$, $i=1,2$ (the internal diagonal condition), $\mathbf{x}_{1,j} \ne \mathbf{x}_{2,j}$, $\mathbf{y}_{1,j} \ne \mathbf{y}_{2,j}$, $j=1,2$ (the external diagonal condition) The formulas (213), (211) coincide.

## 8.2 Stratifications

### 8.2.1 The stratification of the Cartesian product of two spheres $J_1 \times J_2$, $dim(J_1) = n_1 - k_1$, $dim(J_2) = n_2 - k_2$

Let $n_1, n_2$ be odd, $k_1, k_2$ be even positive number, $n_1+n_2=n$, $k_1+k_2=k$. This stratification is the Cartesian product of the two stratifications of the factors, described in Subsubsection 214. Let us define the space $J_1^{[s]} \times J_2^{[t]}$ as a subspace in $J$ as the union of all subspaces $J(x_{\mathfrak{k}_1},\dots,x_{\mathfrak{k}_s}) \times J(y_{\mathfrak{k}_1},\dots,y_{\mathfrak{k}_t}) \subset J_1 \times J_2$.

Therefore, the following double stratification

$$J_1^{(r_1)} \times J_2^{(r_2)} \subset \dots \subset J_1^{(0)} \times J_2^{(0)}, \qquad (214)$$

of the space $J_1 \times J_2$ is well-defined. For a given stratum the number $r_1-s+r_2-t$, $2(r+s)-1=n-k$, of omitted coordinates is called the deep of the stratum.

### 8.2.2 The stratification of $\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2}$

Let us define the stratification of the product $\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2}$ of projective spaces is defined by the product of the stratifications of factors defined in Subsubsection (7.2.2). Denote the maximal open cell $p^{-1}(J(x_1,\dots,x_{r_1}) \times J(y_1,\dots,y_{r_2})$ by

$$U(x_1,\dots,x_{r_1}) \times U(y_1,\dots y_{r_2}) \subset S^{n_1-k_1}/-1 \times S^{n_2-k_2}/-1. \qquad (215)$$

This cell is called an elementary stratum of the deep 0. A point on an elementary stratum of an arbitrary deep $(r_1-s, r_2-t)$ is defined by the collection of spherical coordinates $(\check{x}_{\mathfrak{k}_1},\dots,\check{x}_{\mathfrak{k}_s},\check{y}_{\mathfrak{k}_1},\dots\check{y}_{\mathfrak{k}_t}; l_1 \times l_2)$, where $\check{x}_{\mathfrak{k}_i}$, $\check{y}_{\mathfrak{k}_j}$ are coordinates on a circle, which is covered the circles of the join with the numbers $\mathfrak{k}_i, \mathfrak{k}_j$; $l_1 \times l_2$ are coordinates on $(s-1)\times(t-1)$-dimensional subsimplex of the product of the joins.

### 8.2.3 The stratification of the polyhedron $\overline{\mathbf{NN}}_\circ$ (208)

The polyhedron $\overline{\mathbf{NN}}_\circ$ is the disjoin union of elementary strata, which are defined as inverse images of the strata (215). Let us denote tis strata by

$$\begin{array}{c} \overline{KK}^{[r_1-s]}(\mathfrak{k}_{1,1},\mathfrak{k}_{2,1}\dots,\mathfrak{k}_{1,s},\mathfrak{k}_{2,s}) \times \overline{KK}^{[r_2-t]}(\mathfrak{k}_{1,1},\mathfrak{k}_{2,1},\dots,\mathfrak{k}_{1,t},\mathfrak{k}_{2,t}) = \\ \overline{KK}^{[r_1-s;r_2-t]}(\mathfrak{k}_{1,1},\mathfrak{k}_{2,1}\dots,\mathfrak{k}_{1,s},\mathfrak{k}_{2,s};\mathfrak{k}_{1,1},\mathfrak{k}_{2,1},\dots,\mathfrak{k}_{1,t},\mathfrak{k}_{2,t}) \\ 1 \le s_i \le r_i, i=1,2; \quad r_1+r_2=r. \end{array} \qquad (216)$$

Let us describe an elementary stratum (216) by means of coordinate system. For the simplicity of denotation, let us consider the case $s_1 = r_1$, $s_2 = r_2$ this is the case of maximal elementary stratum. Analogous formula exists for points on deeper elementary strata (216).

Assume a quadruple of points $((\mathbf{x}_{1,1},\mathbf{x}_{1,2}),(\mathbf{y}_{1,1},\mathbf{y}_{1,2});(\mathbf{x}_{2,1},\mathbf{x}_{2,2}),(\mathbf{y}_{2,1},\mathbf{y}_{2,2}))$ determines a point on (216), let us fix for a quadruple a lift $((\check{\mathbf{x}}_{1,1},\check{\mathbf{x}}_{1,2}),(\check{\mathbf{y}}_{1,1},\check{\mathbf{y}}_{1,2});(\check{\mathbf{x}}_{2,1},\check{\mathbf{x}}_{2,2});(\check{\mathbf{y}}_{2,1},\check{\mathbf{y}}_{2,2}))$ on the covering sphere $S^{n_1-k_1}$, each point is mapped into the corresponding point of the quadruple by the projection $S^{n_1-k_1} \times S^{n_2-k_2} \to \mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2}$. With respect to constructions above, denote by

$$\begin{array}{l} (\check{x}_{1,1;i},\check{x}_{1,2;i}),(\check{y}_{1,1;j},\check{y}_{1;2;j}); \\ (\check{x}_{2,1;i},\check{x}_{2,2;i});(\check{y}_{2,1;j},\check{y}_{2,2;j}), \\ i=1,\dots,r_1; \quad j=1,\dots,r_2 \end{array} \qquad (217)$$

the collection of spherical coordinates of the corresponding point. A spherical coordinate determines a point on the circle with the same number $i,j$, the circle is a covering over the corresponding coordinate $J_1(i) \subset J_1$ $(J_2(j) \subset J_2)$ in the join.

Recall, that a coordinate in a quadruple with a prescribed number $i$ $(j)$ of the corresponding angle coordinate determines a point in a prescribed fiber of the standard cyclic $\mathbf{i}$-covering $S^1 \to S^1/\mathbf{i}$.

Collections of spherical coordinates (217) of points are considered with respect to common elementary transformations. A common transformation of $x, y$-collections with respect to the first and second superscripts is possible, this transformation is described by a monodromy over a closed path and describes the representation of the fundamental group of the configuration space, which is called the structuring group. Recall, that each $\mathbf{x}$ and $\mathbf{y}$ coordinate of a point is defined up to undependably multiplication by $-1$. Denote by $\mathbf{DD}$ the group of the order 32 which is the product of the two dihedral groups $\mathbf{DD} = \mathbf{D_x} \times \mathbf{D_y}$ with a common residue class of the permutation preserving subgroup $A \times \dot{A} \subset \mathbf{D}$ in the each factor. Collection of spherical coordinates of a point on the configuration space is transformed up to $8 \cdot 4 \times 8 \cdot 4 \times 2 = 128 \cdot 16$ different transformations. (Up to transformation of the group $\mathbf{DD} \rtimes \mathbf{DD}$ of the order 2048.)

### 8.2.4 A classification of strata of the polyhedron $\overline{KK}_\circ$ (211) by the residues collection

Elementary strata of the polyhedron (211), accordingly with collections of coordinates, are divided into several types, a full description of types is unrequired. Let us introduce types

$$\{\mathbf{II}_b, \mathbf{II}_{a\times\dot{a}}, \mathbf{II}_d, \mathbf{I}_b\mathbf{I}_{a\times\dot{a}}, \mathbf{I}_{a\times\dot{a}}\mathbf{I}_b\} \tag{218}$$

of strata of the polyhedron (211).

For an arbitrary elementary strata (216) a residues collection of pairs (a number of an $x$ or $y$ residues in a pair is defined by a first superscript and a pair itself in the collection is considered over a second superscript):

$$\begin{array}{c}\{v_{1,\mathfrak{k}_i} = \check{x}_{1,1;\mathfrak{k}_i}\check{x}^{-1}_{1,2;\mathfrak{k}_i}, v_{2,\mathfrak{k}_i} = \check{x}_{2,1;\mathfrak{k}_j}\check{x}^{-1}_{2,2;\mathfrak{k}_j};\\ w_{1,\mathfrak{k}_j} = \check{y}_{1,1;\mathfrak{k}_j}\check{y}^{-1}_{1,2;\mathfrak{k}_j}, w_{2,\mathfrak{k}_j} = \check{y}_{2,1;\mathfrak{k}_j}\check{y}^{-1}_{2,2;\mathfrak{k}_j}\},\\ 1 \le \mathfrak{k}_i \le r_1 - k_1, \quad 1 \le \mathfrak{k}_j \le r_2 - k_2.\end{array} \tag{219}$$

is well defined. The residues collection (219) for an elementary stratum of the polyhedron (212) is well-defined up to a common conjugation of $v, w$-collection and up to the multiplication of each $v, w$-collection by $-1$ undependably. A relation between residues in the collection (219) for an elementary stratum depends of a type of the stratum, see below the classification (223) - (231).

In the case residues collection for all coordinates of an elementary stratum takes values in $\{+\mathbf{i}, -\mathbf{i}\}$ (correspondingly, in $\{+1, -1\}$), one say that an elementary stratum is of the subtype $\mathbf{II}_b$ (correspondingly, of the subtype $\mathbf{II}_{a\times\dot{a}}$). In the case there are at least $v$, or $w$ residues in the collection of the complex type $\{+\mathbf{i}, -\mathbf{i}\}$, and of the real type $\{+1, -1\}$ simultaneously, one shall say about an elementary stratum of the subtype $\mathbf{II}_d$. In the case all $v$-residues in $\{+1, -1\}$ and all $w$-residues in $\{+\mathbf{i}, -\mathbf{i}\}$, or, oppositely all $x$-residues in $\{+\mathbf{i}, -\mathbf{i}\}$ and all $w$-residues in $\{+1, -1\}$ we say on a mixed-type stratum of the subtype $\mathbf{I}_b\mathbf{I}_{a\times\dot{a}}$ or, of the subtype $\mathbf{I}_{a\times\dot{a}}\mathbf{I}_b$.

Let us introduce subtypes

$$\{\mathbf{II}^\circ_b,\ \mathbf{I}_{a\times\dot{a}}\mathbf{I}^\circ_b,\ \mathbf{II}_{a\times\dot{a}}\mathbf{I}^\circ_b,\ \mathbf{II}^\circ_d\} \tag{220}$$

of strata of the type $\mathbf{II}_b$. For an arbitrary elementary strata (216) of the type $\mathbf{II}_b$ let us define the collection:

$$\{vv_{1;\mathfrak{k}_i}, vv_{2;\mathfrak{k}_i}, ww_{1;\mathfrak{k}_j}, ww_{2;\mathfrak{k}_j}\} \tag{221}$$

of cross-residues:

$$\begin{gathered} \{vv_{s;\mathfrak{k}_i} = \check{x}^{-1}_{1,s;\mathfrak{k}_i}\check{x}_{2,s;\mathfrak{k}_i}, \\ ww_{s;\mathfrak{k}_j} = \check{y}^{-1}_{1,s;\mathfrak{k}_j}\check{y}_{2,s+1;\mathfrak{k}_j}\}, \\ s = 1,2; \quad 1 \le \mathfrak{k}_i \le r_1 - k_1, \quad 1 \le \mathfrak{k}_j \le r_2 - k_2. \end{gathered} \tag{222}$$

Apriory for an elementary strata the collection (221) is defined up to a general gauge $vv_{2;\mathfrak{k}_i} \mapsto vv'_{2;\mathfrak{k}_i}$, $ww_{2;\mathfrak{k}_j} \mapsto ww'_{2;\mathfrak{k}_j}$. The cross-residues collection (222) for an elementary stratum of the polyhedron (223) will be considered on an appropriate covering over the polyhedron, which is called the regularization, see Subsubsection 8.2.6.

In the case the cross-residues collection for an elementary stratum of the type $\mathbf{II}_b$ takes values in $\{+\mathbf{i}, -\mathbf{i}\}$ (correspondingly, in $\{+1, -1\}$), one say that an elementary stratum is of the subtype $\mathbf{II}^{\circ}_b$ (correspondingly, of the subtype $\mathbf{II}^{\circ}_{a\times\dot{a}}$). In the case there are $vv$, or cross-residues in the $vv$-collection, or in the $ww$-collection of the complex residues $\{+\mathbf{i}, -\mathbf{i}\}$, and of the real residues $\{+1, -1\}$ simultaneously, one shall say about an elementary stratum of the subtype $\mathbf{II}^{\circ}_d$. The $vv$-collection $\{+\mathbf{i}, -\mathbf{i}\}$ and $ww$-collection $\{+1, -1\}$ determines a stratum of the subtype $\mathbf{I}_b\mathbf{I}^{\circ}_{a\times\dot{a}}$; the $vv$-collection $\{+1, -1\}$ and $ww$-collection $\{+\mathbf{i}, -\mathbf{i}\}$ determines a stratum of the subtype $\mathbf{I}_{a\times\dot{a}}\mathbf{I}^{\circ}_b$.

Define the following open polyhedrons (the first one is outside the thin antidiagonal):

$$KK_{\mathbf{II}_b;\circ} \subset KK_{\circ}, \tag{223}$$

$$KK_{\mathbf{II}_{a\times\dot{a}}\circ} \subset KK_{\circ}, \tag{224}$$

$$KK_{\mathbf{II}_d} \subset KK_{\circ}, \tag{225}$$

$$KK_{\mathbf{I}_b\mathbf{I}_{a\times\dot{a}}} \subset KK_{\circ}, \tag{226}$$

$$KK_{\mathbf{I}_{a\times\dot{a}}\mathbf{I}_b} \subset KK_{\circ}. \tag{227}$$

as polyhedrons, which are obtain as union of all elementary strata of the given type. A classification for subpolyhedrons in $\widehat{KK}_{\circ}$ is analogous.

Take a marked point on the maximal thin elementary stratum of the polyhedron $KK_{\circ}$ . At a point on the antidiagonal (242) we get the following relations:

$$vv_{1;\mathfrak{k}_i} = vv_{2;\mathfrak{k}_i} = \mathbf{i}; \quad ww_{1;\mathfrak{k}_i} = ww_{2;\mathfrak{k}_i} = \mathbf{i}. \tag{228}$$

A general transformation of coordinates does not preserves this relation.

In the polyhedron (223) the following subpolyhedrons are defined:

$$KK_{\mathbf{II}_b,\mathbf{II}_b^\circ} \subset KK_{\mathbf{II}_b;\circ}, \tag{229}$$

$$KK_{\mathbf{II}_b,\mathbf{II}_{a\times\dot{a}}^\circ} \subset KK_{\mathbf{II}_b;\circ}, \tag{230}$$

$$KK_{\mathbf{II}_b,\mathbf{I}_d} \subset KK_{\mathbf{II}_b;\circ}. \tag{231}$$

$$KK_{\mathbf{II}_b,\mathbf{I}_b\mathbf{I}_{a\times\dot{a}}^\circ} \subset KK_{\mathbf{II}_b;\circ}. \tag{232}$$

$$KK_{\mathbf{II}_b,\mathbf{I}_{a\times\dot{a}}\mathbf{I}_b^\circ} \subset KK_{\mathbf{II}_b;\circ}. \tag{233}$$

using the corresponding subtypes for $\mathbf{II}_b$-type classification.

### 8.2.5 A regularization of the polyhedron (229)

Let us describe the structuring group $\mathbf{DD}\tilde{\times}\mathbf{DD}$ of (212). The factor $\mathbf{DD}$ is the structuring group of the primary $T$-covering of the internal delated product in the formula. This is an index 2 subgroup in $\mathbf{D} \times \mathbf{D}$ (the subgroup of the order 32), which is given by the formula $(x, y), x \cong y \pmod{A \times \dot{A}}$, $A \times \dot{A} \subset \mathbf{D}$. The group $\mathbf{DD}\tilde{\times}\mathbf{DD}$ of the order 512 is defined as the skew-product of the two subgroups $\mathbf{DD}$, a permutation of the factor corresponds to the external double covering.

Let us define the index 2 subgroup of the order 256, the structuring group of the subpolyhedron (229):

$$R' \subset \mathbf{DD}\tilde{\times}\mathbf{DD}, \tag{234}$$

which is defined by elements $[d_1 \times d_2, \tau] \in \mathbf{DD}\tilde{\times}\mathbf{DD}$, $d_1, d_2 \in \mathbf{DD}$, $d_1 = d_2 \pmod{b^2}$, $\tau$ is given by the cyclic permutation of the dihedral blokes, namely, by the twisted permutation which changes a primary $\mathbf{x}$-coordinates in a secondary pair and a primary $\mathbf{y}$-coordinate in the pair with the scaling by the element $b \in \mathbf{I}_b$.

Define the regularization polyhedron of (229)

$$RKK_{\mathbf{II}_b,\mathbf{II}_b^\circ} \subset KK_{\mathbf{II}_b,\mathbf{II}_b^\circ} \tag{235}$$

as the double covering with respect to the index 2 subgroup (234). The standard denotation

$$\overline{RKK}_{\mathbf{II}_b,\mathbf{II}_b^\circ} \subset \overline{KK}_{\mathbf{II}_b,\mathbf{II}_b^\circ} \tag{236}$$

for the secondary canonical double covering over (235) is used.

Let us describe the structuring group of the subpolyhedron (235) more detailed. Take a marked point on the thin antidiagonal inside (235). At this point all cross-residues (221) are imaginary. Take a generic closed loop $l : S^1 \subset RKK_{\mathbf{II}_b,\mathbf{I}_b;\circ}$ and consider an evolution of cross-residues along it. An arbitrary elementary perturbation of the collection (221) keeps this property, because a transformation of an imaginary cross-residues into a real contradicts with the property of imaginary primary residues in (219). Therefore, in particular, an element $b \times 1$ is not realizable as a transformation in the group (234), because this element changes a collection of imaginary cross-residues into the corresponding real collection. The structuring group of (235) over $\mathbf{x}$ - coordinate is inside the following subgroup of the order 16:

$$R_{\mathbf{x}} = \mathbf{I}_b(diag) \rtimes Z/2 \subset \mathbf{D}\tilde{\times}\mathbf{D}. \tag{237}$$

Because transformations over $\mathbf{x}$ and $\mathbf{y}$ coordinates coincides modulo $\mathbf{I}_d(diag) \rtimes 1$, we get an index 2 relation of $\mathbf{x}$ and $\mathbf{y}$ coordinates, and a common transformation of the dihedral $x$ and $y$-blokes.

### 8.2.6 Relations for cross-residues for polyhedron $\overline{RKK}_{\mathbf{II}_b,\mathbf{I}_b;\circ}$

Moreover, for the polyhedron (236) there is a following relation for cross-residues:

$$vv_{1;\mathfrak{k}_i} = vv_{2;\mathfrak{k}_i}; \quad ww_{1;\mathfrak{k}_i} = ww_{2;\mathfrak{k}_i}, \tag{238}$$

which are imaginary. This relation is satisfied, because the regularization condition (8.2.5) is satisfied globally.

From this the following property for the subpolyhedron (236) is deduced. Take the quotient $(\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2})/\bar{I}$ (for the definition $\bar{I}$, see Subsubsection 8.1), take the standard complex conjugation involution of this quotient, denote it by

$$Conj_1 \times Conj_2 : (\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2})/\{I_1, I_2\} \to (\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2}).$$

Denote by

$$C_{Conj_1 \times Conj_2} = ((\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2})/\bar{I}) \rtimes_{Conj_1 \times Conj_2} S^1 \times S^1,$$

the torus of the involution $Conj_1 \times Conj_2$.

There exist a mapping:

$$s : \overline{KK}_{\mathbf{II}_b,\mathbf{I}_b} \setminus D_{anti} \longrightarrow C_{Conj_1 \times Conj_2} \tag{239}$$

with the source is outside the antidiagonal (see (241)), the coordinate mappings (197) near the antidiagonal are disconnected. The mapping is defined as following. For an elementary strata the full subcollection of the angles-coordinates is divided by residues into two collections: $V_+ = \{v_{1;\mathfrak{k}_i} = +\mathbf{i}\}$ and $V_- = \{v_{1;;\mathfrak{k}_i} = -\mathbf{i}\}$, $V = V_+ \cup V_-$. This subcollection of the coordinates with the momenta determines the coordinate projections (239) along $(\mathbb{RP}^{n_1-k_1} \times \mathbb{RP}^{n_2-k_2})$; restricted on an elementary stratum by identity on angles $V_+$ and by the conjugation on angles $V_-$. The mapping (197) considered for each coordinate along the circle coordinate $S^1$ is not well-defined near antidiagonal, but the monodromy of this mapping takes even values.

The mappings (239) are defined using mappings (197), see Theorem 46, for each $\mathbf{x}$, $i=1;\mathbf{y}$, $i=2$-projection. Along the circle generator the mapping $\pi_i$, $i=1,2$ changes the collection $V$ to its complement conjugated subcollection. This gives the mapping (239), where the projection on the torus $p_C : C_{Conj_1 \times Conj_2} \to S^1 \times S^1$ corresponds to the composition $p_C \circ s_i = \pi_i$, $i=1,2$.

The mappings $s_1 \times s_2$ are not covering mappings over a common mapping (235). Theorem 46 below defines a (partial defined) "secondary"mapping:

$$\sigma : RKK_{\mathbf{II}_b,\mathbf{I}_b} \setminus \Delta_{antidiag} \longrightarrow CC_{Conj^2}, \quad i=1,2, \tag{240}$$

where

$$Conj^2 : \mathbb{RP}^{n_1-k_1}/I_1 \times \mathbb{RP}^{n_2-k_2}/I_2 \to \mathbb{RP}^{n_1-k_1}/I_1 \times \mathbb{RP}^{n_2-k_2}/I_2$$

is the pair of the complex conjugations over the coordinates, $CC_{Conj}$ is the torus of this two involutions. The source of the mapping (240) is considered outside the thin antidiagonal (241).

### 8.2.7 Antidiagonal strata

For the polyhedron (223) let us define the thick antidiagonal, using the first coordinate on the secondary covering (because for the second coordinate we have the same equation, see 238):

$$\begin{array}{c} D^{[0]}_{anti} = \{(\mathbf{x}_{1,1},\mathbf{x}_{1,2}),(\mathbf{y}_{1,1},\mathbf{y}_{1,2});(\mathbf{x}_{2,1},\mathbf{x}_{2,2}),(\mathbf{y}_{2,1},\mathbf{y}_{2,2})\} : \\ \mathbf{i}\mathbf{x}_{1,1}=\mathbf{x}_{1,2},\mathbf{i}\mathbf{y}_{1,1}=\mathbf{y}_{1,2},\mathbf{i}\mathbf{x}_{2,1}=\mathbf{x}_{2,2},\mathbf{i}\mathbf{y}_{2,1}=\mathbf{y}_{2,2}; \\ \mathbf{i}\mathbf{x}_{1,1}=\mathbf{x}_{2,1},\ \mathbf{i}\mathbf{y}_{1,1}=\mathbf{y}_{2,2} \pmod{\pm 1}. \end{array} \tag{241}$$

Let us define exceptional (thin) antidiagonal strata. The maximal thin antidiagonal stratum is given by the formula:

$$\begin{array}{c} \Delta^{[0]}_{anti} = \{(\mathbf{x}_{1,1},\mathbf{x}_{1,2}),(\mathbf{y}_{1,1},\mathbf{y}_{1,2});(\mathbf{x}_{2,1},\mathbf{x}_{2,2}),(\mathbf{y}_{2,1},\mathbf{y}_{2,2})\} : \\ \mathbf{i}\mathbf{x}_{1,1}=\mathbf{x}_{1,2},\ \mathbf{i}\mathbf{x}_{2,1}=\mathbf{x}_{2,2},\ \mathbf{i}\mathbf{x}_{1,1}=\mathbf{x}_{2,1}, \\ \mathbf{i}\mathbf{y}_{1,1}=\mathbf{i}\mathbf{y}_{1,2},\ \mathbf{i}\mathbf{y}_{2,1}=\mathbf{y}_{2,2},\ \mathbf{i}\mathbf{y}_{1,1}=\mathbf{y}_{2,2}. \end{array} \tag{242}$$

The relation on $\Delta_{anti}$ shows that of $\mathbf{x}$, $\mathbf{y}$-coordinates are not symmetric with primary 1 and 2-coordinates on the configuration space. For thin antidiagonal deeper strata formulas are analogous. Obviously, the polyhedron (242) is inside (241).

The union of all antidiagonal thin (thick) strata with the polyhedron (229) determines the corresponding sequence of polyhedrons:

$$\Delta_{anti} \subset D_{anti} \subset \overline{RKK}_{\mathbf{II}_b,\mathbf{I}_b} \subset \overline{KK}_{\mathbf{II}_b,\mathbf{I}_b} \subset \overline{KK}_{\circ}. \tag{243}$$

A general structuring group $\mathbf{DD} \rtimes \mathbf{DD}$ over the subpolyhedron (242) admits a reduction into the diagonal subgroup $\{b \times 1; 1 \times \dot{b}\} = \mathbb{Z}/4 \times \mathbb{Z}/4 \subset$

$\mathbf{DD} \rtimes \mathbf{DD}$, $b, \dot{b} \in \mathbf{I}_b$. A structure group over the subpolyhedrons (241) outside the thin antidiagonal is represented in $\mathbf{x}$ and $\mathbf{y}$ form accordingly to indexes in antidiagonal equation. The $\mathbf{x}$-pice admits the structuring group of the order 16.

### 8.2.8 A stabilization

The subgroup $\{b \times 1, 1 \times b, \tau\} \subset \mathbf{DD} \rtimes \mathbf{DD}$ isomorphic to $\mathbf{D} \times \mathbb{Z}/4$. Using this fact, one may stabilized the polyhedron

$$\overline{KK}_{\mathbf{II}_b,\mathbf{I}_b}(n) \subset \overline{KK}_{\mathbf{II}_b,\mathbf{I}_b}(N) \qquad (244)$$

for wery large $n \mapsto N$, $n, N$ are numbers of coordinates. In this case one may assume that the image of the thick diagonal of the polyhedron $\overline{KK}_{\mathbf{II}_b,\mathbf{I}_b}(n)$ is outside the thick diagonal of $\overline{KK}_{\mathbf{II}_b,\mathbf{I}_b}(N)$ and contains only deep strata.

Note that the group $\{b \times 1, 1 \times \dot{b}\}$ is a covering translation group of the antidiagonal polyhedrons $\Delta_{antidiag}(n)$ and $\Delta_{antidiag}(N)$. This group is the structured translation covering subgroup over the polyhedrons $\overline{KK}_{\mathbf{II}_b,\mathbf{I}_b}(n)$ and $\overline{KK}_{\mathbf{II}_b,\mathbf{I}_b}(N)$, the same group in the target and in the source of (244). A small shift of the inclusion (244) gives extension of origin strata, such that a result of the shift for an arbitrary stratum is a target stratum with a full collection of residues $v, w \in \{-\mathbf{i}, \mathbf{i}\}$. The structured group of the thick antidiagonal in a stabilized polyhedron is clear. The generator $\omega$ admits the diagonal coordinate representation.

## 8.3 A resolution

Let us prove an analog of Lemma 43 for the subpolyhedrons of the polyhedron $KK_\circ$. Denote by $\overline{KK}_{\mathbf{II}_d;\circ}$ the canonical (secondary) double covering over the polyhedron (225). Denote by

$$\pi_{\mathbf{II}_d} : \overbrace{KK_{\mathbf{II}_d;\circ}} \longrightarrow \overline{KK}_{\mathbf{II}_d;\circ} \qquad (245)$$

the canonical primary double covering over the polyhedron $\overline{KK}_{\mathbf{II}_d;\circ}$.

**Theorem 45.** *1. The canonical primary covering* (245) *over the polyhedron* $\overline{KK}_{\mathbf{II}_d;\circ}$ *is a pull-back of the coordinate covering over the circles. The formula of the circle covering is invariant with respect to the involution* $\chi : \mathbf{x} \mapsto \mathbf{y}$ *(Subsubsection 8.1.1).*

*2. The canonical secondary covering* $\overline{KK}_{\mathbf{II}_b,\mathbf{I}_d} \longrightarrow KK_{\mathbf{II}_b,\mathbf{I}_d}$ *is a pull-back of the product of the two coordinate standard coverings over the circle. The* $\chi$*-involution. The construction is skew-invariant with respect to the involution* $\chi : \mathbf{x} \mapsto \mathbf{y}$.

#### Proof of Theorem 45

Statements are corollaries of Lemma 43. Apply a stabilization, described in Subsubsection 8.2.8. We may prove the lemma with the assumption that all elementary strata admits $\mathbf{x}$ and $\mathbf{y}$-residues of all different type. Let us consider the coordinate projections

$$p_1 : \overbrace{KK_{\mathbf{II}_d;\circ}} \longrightarrow \overline{K}_{\circ;1} \supset \overline{K_{\mathbf{I}_d;1}},$$

$$p_2 : \overbrace{KK_{\mathbf{II}_d;\circ}} \longrightarrow \overline{K}_{\circ;2} \supset \overline{K_{\mathbf{I}_d;2}},$$

on the polyhedrons (142), where $\overline{K_{\mathbf{I}_d;i}} \subset \overline{K}_{\circ;i}$, $i = 1, 2$ are the subpolyhedrons, defined by the formula (151). Define $\overbrace{KK_{\mathbf{II}_d;i}} \subset \overline{K}_{\circ;i}$ the inverse images of the subpolyhedron $\overline{K_{\mathbf{I}_d;i}}$ by projections $p_i$. Obviously,

$$\overbrace{KK_{\mathbf{II}_d;1}} \cup \overbrace{KK_{\mathbf{II}_d;2}} = \overbrace{KK_{\mathbf{II}_d;\circ}}.$$

Because of the stabilization property we get:

$$\overbrace{KK_{\mathbf{II}_d;1}} = \overbrace{KK_{\mathbf{II}_d;2}} = \overbrace{KK_{\mathbf{II}_d;\circ}}.$$

With this condition the canonical (primary) covering over $\overbrace{KK_{\mathbf{II}_d;i}}$ is the pull-back of the canonical covering over $\overline{K_{\mathbf{I}_d;1}}$. By Lemma 43 this canonical covering is the pull-back of the covering over the circle.

The construction of the covering is given by the primary residues (219), which are invariant with respect to the involution $\mathbf{x} \mapsto \mathbf{y}$. The formula of the residues used two coordinates primary indexes, which are invariant. Statement 1 is proved.

Proof of Statement 2. Consider the canonical secondary covering $\overline{KK}_{\mathbf{II}_b} \longrightarrow KK_{\mathbf{II}_b}$ and restrict this covering over the subpolyhedron $KK_{\mathbf{II}_b,\mathbf{I}_d} \subset KK_{\mathbf{II}_b}$. The total space of this restriction coincides with the secondary covering $\overline{KK}_{\mathbf{II}_b,\mathbf{I}_d} \longrightarrow KK_{\mathbf{II}_b,\mathbf{I}_d}$ over the polyhedron (231). Let us apply a stabilization and decompose, analogous with the proof Statement 1, the polyhedron as following:

$$\overline{KK}_{\mathbf{II}_b,\mathbf{I}_d} = \overline{KK}_{\mathbf{II}_b,\mathbf{I}_d;1} \cup \overline{KK}_{\mathbf{II}_b,\mathbf{I}_d;2}, \tag{246}$$

where $\overline{KK}_{\mathbf{II}_b,\mathbf{I}_d;i}$, $i = 1, 2$ is a polyhedron defined as the inverse image of the subpolyhedron $\bar{K}_{\mathbf{I}_d} \subset \overline{K}_\circ$ by the coordinate projection

$$p_i : \overline{KK}_{\mathbf{II}_b,\mathbf{I}_d} \to \bar{K}_{\mathbf{I}_d} \subset \overline{K}_\circ.$$

Because of a stable condition we get:

$$\overline{KK}_{\mathbf{II}_b,\mathbf{I}_d} = \overline{KK}_{\mathbf{II}_b,\mathbf{I}_d;1} = \overline{KK}_{\mathbf{II}_b,\mathbf{I}_d;2}, \tag{247}$$

Each polyhedron in the right-hand side of (246) is a double covering of the projection on the first index 1 in (139). This decomposition is well-defined, because for $\overline{KK}_{\mathbf{II}_b}$ a cross-residue $vv_{1,\mathfrak{k}_i}$ is real iff the cross-residue $vv_{2,\mathfrak{k}_i}$ is real. This proves that the first index 1 is global over the subpolyhedron (246) with external $T^{ext}$-equivariant structure (207), see the equation (228) over the marked point. An analogous construction is possible using $ww_{1,\mathfrak{k}_j}$, $ww_{2,\mathfrak{k}_j}$ cross-residues.

The canonical (secondary) covering over $\overline{KK_{\mathbf{II}_d;i}}$, $i = 1, 2$ is the pull-back of the canonical covering over $\overline{K_{\mathbf{I}_d}}$; by Lemma 43 this covering is the pull-back of the covering over the circle. The construction of the covering is given by the secondary cross-residues (222), which are skew-invariant with respect to the involution $\mathbf{x} \mapsto \mathbf{y}$. The formula of the cross-residues used two secondary coordinates, the involution $\chi$ permutes the secondary coordinates. Statement 2 is proved. Theorem 45 is proved. □

### 8.3.1 Properties of $\overline{KK}_{\mathbf{II}_b}$, $\overline{RKK}_{\mathbf{II}_b,\mathbf{I}_b}$

Let us consider a sequence (243). Let us define a splitting

$$SP : \overline{KK}_{\mathbf{II}_b,\mathbf{I}_b} \setminus D_{anti} \longrightarrow \bar{K}_{\mathbf{I}_b,1;\circ} \times \bar{K}_{\mathbf{I}_b,2;\circ}, \tag{248}$$

which is $T$-equivariant with respect to the canonical (secondary) involution in the preimage and the canonical involution in the image. The mapping (248) (as usual, the formula is presented for a maximal strata) is given by:

$$\begin{gathered}(\check{x}_{1,1;i}, \check{x}_{1,2;i}), (\check{y}_{1,1;j}, \check{y}_{1;2;j}); (\check{x}_{2,1;i}, \check{x}_{2,2;i}); (\check{y}_{2,1;j}, \check{y}_{2,2;j}) \mapsto \\ (\check{x}_{1,1;i}, \check{x}_{2,1;i}) \times (\check{y}_{1,1;j}, \check{y}_{2;1;j}).\end{gathered} \tag{249}$$

The splitting (248) is a well-defined $PL$-mapping, because the permutation of the second index (the permutation by $\tau$ in the subgroup (237)) gives the $b$-image of the same point because of the identity:

$$-\mathbf{i}\mathbf{x}_{1,1} = \mathbf{x}_{2,1}, \ -\mathbf{i}\mathbf{x}_{1,1} = \mathbf{x}_{1,2}, \ \mathbf{x}_{1,1} = \mathbf{x}_{2,2}; \ or$$

$$\mathbf{i}\mathbf{x}_{1,1} = \mathbf{x}_{2,1}, \ -\mathbf{i}\mathbf{x}_{1,1} = \mathbf{x}_{1,2}, \ -\mathbf{x}_{1,1} = \mathbf{x}_{2,2}.$$

The equation for the $\mathbf{y}$ coordinate is analogous. Recall, that polyhedrons in the the source and the target of (248) are considered outside the thick antidiagonal.

**Theorem 46.** *There exists the $\hat{I}, T^{ext}$-equivariant resolution mapping*

$$\pi : \overline{KK}_{\mathbf{II}_b,\mathbf{I}_b} \setminus D_{anti} \to S^1 \times S^1, \tag{250}$$

*which is the Cartesian product of the two coordinate mappings (197).*

**Proof of Theorem 46**

The required mapping (250) outside the antidiagonal is induced from coverings over the coordinate mappings (197) by the equivariant mappings (248). Let us prove that the mapping (250) is continues outside the thin antidiagonal. Regular coordinates for each projection near the antidiagonal is defined as in the proof of Lemma 42. The singular stratum is defined by the following condition: there exist a $\mathbf{x}$-, or $\mathbf{y}$-coordinate, for which the number of singular coordinates is greater then half of the coordinate. Outside the polyhedron of singular strata the global regular $\mathbf{x}$ or, $\mathbf{y}$–coordinate system is well-defined. Singularities for the $\mathbf{x}$ projection of (250) is not possible inside a subpolyhedron $P_{\mathbf{x}} \cup P_{\mathbf{y}}$, where $P_{\mathbf{x}}$ ($P_{\mathbf{y}}$) is a polyhedron of all strata, where $\mathbf{x}$-angles ($\mathbf{y}$-angles) are regular. Analogously, the statement is true for the $\mathbf{y}$-projection. This is given by arguments of Lemma 42 (see also an analogous proof Lemma 49). The regular condition for $\mathbf{x}$-projection is completely determined the $\mathbf{x}$-coordinate system; the $\mathbf{x}$-coordinate system on the thick antidiagonal is related with the $\mathbf{y}$-coordinate system, see 241, and near $D_{antidiag}$ the prescribed $\mathbf{x}$-coordinate system determines $\mathbf{y}$-coordinate system. An arbitrary closed loop near the thick antidiagonal is homologous to two loops outside $P_{\mathbf{x}}$ and outside $P_{\mathbf{y}}$. □

Let us formulate an analog of Lemma 44. An analog of the involution $\tau_c$ from Subsection 7.3 on the polyhedron $KK_{\mathbf{II}_b,\mathbf{I}_b;\circ}$ is required. Consider the projection (248). Coordinate involutions $\tau_{c;i}$, $i = 1, 2$, determine the involution on $KK_{\mathbf{II}_b,\mathbf{I}_b;\circ}$ outside the thick antidiagonal. The diagonal of $\tau_{c;i}$-involutions is the involution on $KK_{\mathbf{II}_b,\mathbf{I}_b;\circ}$, which is free outside the thin antidiagonal. Denote this involution by $\tau\tau_c$. On the polyhedron $KK_{\mathbf{II}_b,\mathbf{I}_{a\times\dot{a}};\circ}$, the involution $\tau\tau_c$ is analogously defined.

**Corollary 47.** *There exists a mapping (240) with singularities described by Lemma 42 for the pair of the mapping (197) in the target of (248).*

**Lemma 48.** *1. The Hurewicz image of the fundamental class of the antidiagonal polyhedrons $\Delta_{antydiag}$ (243) in $KK_{\mathbf{II}_{a\times\dot{a}};\circ}$ in the group $H_{n_1+n_2-k_1-k_2}(K(\mathbf{D} \rtimes \mathbf{D}, 1); \mathbb{Z}/2)$, $dim(KK_{\mathbf{II}_{a\times\dot{a}};\circ}) = n_1+n_2-k_1-k_2$ is the generator $t_{n_1-k_1} \otimes t_{n_2-k_2} \in H_{n_1+n_2-k_1-k_2}(K(\mathbb{Z}/4 \times \mathbb{Z}/4; \mathbb{Z}/2)) \to H_{n_1+n_2-k_1-k_2}(K(\mathbf{D} \rtimes \mathbf{D}, 1); \mathbb{Z}/2)$.*

*2. The Hurewicz image by the structured homomorphism of the fundamental class of an arbitrary codimension $2s$ subpolyhedron in $\mathbf{NN}_{\mathbf{II}_b,\mathbf{I}_b;\circ}$ (in $\mathbf{NN}_{\mathbf{II}_b,\mathbf{I}_{a\times\dot{a}};\circ}$ ), which is invariant with respect to the involution $\tau\tau_c$ of the double covering $\mathbf{NN}_{\mathbf{II}_b,\mathbf{I}_b;\circ} \longrightarrow \widehat{\mathbf{NN}}_{\mathbf{II}_b,I_b;\circ}$ ($\mathbf{NN}_{\mathbf{II}_b,\mathbf{I}_{a\times\dot{a}};\circ} \longrightarrow \widehat{\mathbf{NN}}_{\mathbf{II}_b,\mathbf{I}_{a\times\dot{a}};\circ}$) in the group $H_{n_1+n_2-k_1-k_2-2s}(K(\mathbf{D} \rtimes \mathbf{D}, 1))$ is trivial.*

### Proof of Lemma 48

Proof is analogous to Lemma 44. □

Let us prove an analog of Lemma 42 Statement 2 for polyhedrons $RKK_{\mathbf{II}_b,\mathbf{I}_b;\circ}$.

Recall, $dim(KK_{\mathbf{II}_b,\mathbf{I}_b;\circ}) = n_1-k_1+n_2-k_2, \quad n_1, n_2 \equiv 1 \pmod 2$, $k_1, k_2 \equiv 0 \pmod 2$. The boundary of the polyhedron $RKK_{\mathbf{II}_b,\mathbf{I}_b;\circ}$ consists of points on all regular strata, which are closed to anti-diagonal (242). Let us consider of the open polyhedron, a small regular neighborhood of the thin antidiagonal polyhedron $\Delta_{anti} \subset RKK_{\mathbf{II}_b,\mathbf{I}_b}$, which is denoted by $U_\circ \subset RKK_{\mathbf{II}_b,\mathbf{I}_b} \setminus \Delta_{anti}$.

Consider the mapping $U_\circ \to S^1 \times S^1$, which is the restriction of the equivariant mapping (250). Because the structuring group $U_\circ$ is reduced to the subgroup $\mathbf{I}_b \times \mathbf{I}_{\dot{b}} = (b, \dot{b})$, the mapping $U_\circ \to S^1$ contains values in the even index $2 \times 2$ subgroup $2[\pi_1(S^1)] \times 2[\pi_1(S^1)] \subset \pi_1(S^1) \times \pi_1(S^1)$.

**Lemma 49.** *There exists a subpolyhedron $U_{reg,\circ} \subset U_\circ$, such that the restriction of the mapping $\pi$ to the complement $U_\circ \setminus U_{reg,\circ}$ is homotopic to the mapping into a point, and the polyhedron itself admits the following mappings:*

$$U_{reg,\circ} \stackrel{\eta_{reg}}{\longrightarrow} P_1 \times P_2 \stackrel{\zeta_{reg}}{\longrightarrow} K(\{b, \dot{b}\}, 1), \qquad (251)$$

*where $P_i$ is a polyhedron of the dimension $\frac{n_i - k_i + 2}{2}$, $i = 1, 2$ (i.e. of the dimension a little greater then $\frac{1}{2} dim(K_{\mathbf{I}_b,i;\circ}) = \frac{n_i - k_i + 1}{2}$), $\eta_{reg}$, $\zeta_{reg}$ are mappings, which are called the control mapping and the structuring mapping, correspondingly, for $P_1 \times P_2$ the structuring group $\{b, \dot{b}\}$, in the target of the structured mapping is described in Subsubsection* 8.2.8.

**Proof of Lemma 49**

Let us consider regular elementary strata (216) of $RKK_{\mathbf{II}_b,\mathbf{I}_b}$ near the thick antidiagonal. For each $\mathbf{x}, \mathbf{y}$-coordinate denote a notion of regular coordinate as in Lemma 42. The definition is a straightforward analog by the splitting (248).

Let us difine the polyhedron $Q \subset U_{reg;\circ}$ consists of antidiagonal boundaries of all regular strata for any $\mathbf{x}, \mathbf{y}$-coordinate. An elementary stratum in $U_{reg;\circ}$ is called a singular stratum, if for both $\mathbf{x}$ and $\mathbf{y}$-coordinates at least half cross-residues in one of the two full total collections (238) are different from the antidiagonal cross-residues collection. The regular polyhedron is the union $Q = Q_{\mathbf{x}} \cup Q_{\mathbf{y}}$. The restriction of the mapping (250) to $Q$ is homotopic to the constant mapping iff the monodromy of this mapping along an arbitrary closed loop inside $Q_{\mathbf{x}}$, or inside $Q_{\mathbf{y}}$ is the identity. Inside the polyhedrons the mapping (250) everywhere is defined as the difference of the variation of singular coordinates with the referred constant mapping inside the antidiagonal. The complement to $Q$ in $U_{reg;\circ}$ is a polyhedron $P$, for which the both coordinates are singular. This polyhedron is dominated by the $P_1 \times P_2$, where $P_i$ is the polyhedron of all regular $\mathbf{x}$-strata for $i = 1$, $\mathbf{y}$-strata for $i = 2$.

The existence of the mappings (251) is proved analogously to (199). The dimension of such a polyhedron is indicated in the statement of the lemma. Lemma 49 is proved. $\square$

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