Finitely presented expansions of semigroups, groups, and algebras

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What do we use in this lecture?

We travel back to classics:

1. 1910: Finite presentations (Dehn)
2. 1940: Residually finite algebras (Malcev, Mackenzie)
3. 1944: Construction of a simple set (Post)
4. 1964: The theorem of Golod–Shafarevich,

all with computability in the background.

Plan

- Definitions and historical background
- Statement of the problem
- Immune algebras
- Meta-theorem: Non finite presentability
- Applications of the NFP theorem

Algebras

The word algebra is used in two ways.

One is **algebras** are rings that are vector spaces over fields.

The other is this:

Definition

An **algebra** is an algebraic structure of the form:

$$\mathcal{A} = (A; f_1, \ldots, f_n, c_1, \ldots, c_r), \text{ where } A \neq \emptyset, \text{ and}$$

$f_1, \ldots, f_n$ are operations and $c_1, \ldots, c_r$ are constants on A.

The sequence of symbols $f_1, \ldots, f_n, c_1, \ldots, c_r$ is called the **signature** of the algebra.

Examples

1. Groups $\mathcal{G} = (G; \ \cdot)$ are algebras of signature $\cdot$.
2. Rings $\mathcal{R} = (R; \ +, \times, 0)$ are algebras of signature $+$, $\times$, 0.
3. The arithmetic $(\omega; +, \times, S, 0)$ is of signature $+$, $\times$, S, and 0.
4. Expand the group $\mathcal{G}$ by adding the unary operation $^{-1}$.
   This is an expanded algebra of signature $\cdot$, $^{-1}$.
5. Let $\mathcal{G} = (G; \ \cdot)$ be a f. g. group with generators a, b.
   Let us expand this group: $(G; \ \cdot, \ ^{-1}, \ a, \ b)$.
   This expanded algebra has signature $\cdot$, $^{-1}$, a, b.

Term algebra $\mathcal{F}$

Let $\mathcal{F}$ be the term algebra built from constants.

For instance, the terms of $\mathcal{F}$ over the signature f, c are

$$c,\ f(c,c),\ f(c,f(c,c)),\ f(f(c,c),c),\ f(f(c,c),f(c,c)),$$
$$f(c,f(c,f(c,c))),\ f(f(c,f(c,c)),c),\ f(c,f(f(c,c),c)),\ \ldots$$

The operation f on the term algebra $\mathcal{F}$ is defined as follows. Given two terms t_1 and t_2,

The value of f on (t_1, t_2) is $f(t_1, t_2)$

Computability of $\mathcal{F}$

The following program generates the domain of $\mathcal{F}$:

- **Initialisation**: The constants c are terms.
- **Loop**: If t_1, ..., t_n are terms and f is n-ary operation symbol then $f(t_1, \ldots, t_n)$ is a term.

The following program computes n-ary operation f of $\mathcal{F}$:

- On input $(t_1, \ldots, t_n)$ output $f(t_1, \ldots, t_n)$.

Universality and uniqueness of $\mathcal{F}$

- **Universality:** Any algebra generated by the constants is a homomorphic image of $\mathcal{F}$.
- **Uniqueness**: Universality of $\mathcal{F}$ determines the algebra $\mathcal{F}$ up to isomorphism.

Dehn: Finite presentations (1910)

Definition

A **finite presentation** S is a **finite** set of equations of the form

$$t = p,$$

where t and p are terms (that might contain variables).

Let $E(S)$ be the congruence relation on $\mathcal{F}$ generated by S. Set:

$$\mathcal{F}_S = \mathcal{F}/E(S).$$

Definition

The algebra $\mathcal{F}_S$ is called **finitely presented** by S.

Examples

1. Let S be the group axioms in the signature $\cdot$, $^{-1}$, a, b.
   - The algebra $\mathcal{F}_S = \mathcal{F}/E(S)$ is the free group.
   - Let S' be S together with $ba^2b^{-1} = a$. The algebra $\mathcal{F}_{S'} = \mathcal{F}/E(S')$ is a Baumslag-Solitar group.
2. Let S be the equation $fg(x) = gf(x)$ in the signature f, g, a. The algebra $\mathcal{F}_S = \mathcal{F}/E(S)$ is isomorphic to

$$(\omega \times \omega; f, g, a),$$

where $a = (0, 0)$, $f(i, j) = (i + 1, j)$ and $g(i, j) = (i, j + 1)$.

Properties of $\mathcal{F}_S$:

1. The algebra $\mathcal{F}_S$ satisfies S.
2. The equality on $\mathcal{F}_S$ is $E(S)$.
3. The operations of $\mathcal{F}_S$ are computable and respect $E(S)$.
4. The algebra $\mathcal{F}_S$ is universal and unique.
5. The equality relation $E(S)$ in $\mathcal{F}_S$ is computably enumerable (semidecidable).

Malcev and MacKenize: Residual finiteness (1940)

Definition

An algebra $\mathcal{A}$ is **residually finite** if for all elements $x, y \in \mathcal{A}$, $x \neq y$, there is a homomorphism h of $\mathcal{A}$ onto a finite algebra $\mathcal{B}$ such that $h(x) \neq h(y)$.

Example

1. The successor structure $(Z; S, 0)$ is residually finite.
2. The structure $(\omega; S, S^{-1})$ is not residually finite.
3. Finitely generated abelian groups are residually finite.
4. The term algebra $\mathcal{F}$ is residually finite.

Theorem of Malcev and MacKenzie (1940)

Theorem

Let $\mathcal{A}$ be algebra finitely presented by S. If $\mathcal{A}$ is residually finite, then the word problem for $\mathcal{A}$ is decidable.

Proof.

List all finite algebras satisfying S: $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \ldots$. These algebras are homomorphic images of $\mathcal{A}$.

For all distinct $[x], [y] \in \mathcal{A}$ there is a finite algebra $\mathcal{B}_i$ in which images of $[x]$ and $[y]$ are distinct.

So, non-equality in $\mathcal{A}$ is c.e.. Hence, the word problem is decidable. □

Post: Immune and simple sets (1944)

Definition

An infinite set $Y \subset \omega$ is **immune** if Y has no infinite c.e. subset.
A c.e. set $A \subset \omega$ is **simple** if $\omega - A$ is immune.

Post's construction: Start enumerating all c.e. sets uniformly:

$$W_0, W_1, \ldots \ldots$$

Put x into A if (1) x is enumerated in W_i, (2) $x > 2i$, and (3) no element in W_i (you have seen so far) is put into A.

The constructed set A is simple!

We almost finished our journey back

We traveled through:

1. 1910: Finite presentations (Dehn)
2. 1940: Residually finite algebras (Malcev, Mackenzie)
3. 1944: Construction of a simple set (Post)

Later we will travel back to Golod–Shafarevich theorem.

The finite presentability problem

The general statement:

Given an algebra $\mathcal{A}$, is $\mathcal{A}$ finitely presented?

Finite presentability necessarily implies the following:

- The algebra $\mathcal{A}$ must be f.g. by the constants.
- So, the algebra $\mathcal{A}$ must be of the form $\mathcal{A} = \mathcal{F}/E$.
- The equality E must be computably enumerable.
- The operations of $\mathcal{A}$ must be computable and respect E.

Assumptions for the rest of the talk:

1. We view algebras $\mathcal{A}$ as quotients $\mathcal{A} = \mathcal{F}/E$.
2. The atomic operations of $\mathcal{A}$ are computable and respect E.
3. The equality relation E is computably enumerable (unless we say otherwise).

We call these **computably enumerable (c.e.) algebras**.

Examples of non-finitely presented algebras:

- The groups $Z_k \wr Z^n$, with $k, n > 1$, are not finitely presented (Baumslag, 1960).
- The algebra $(\omega; x+1, 2^x, 0)$ is not finitely presented (Bergstra and Tucker, 1979).

Expansions

Definition

An **expansion** of $\mathcal{A} = (A; f_1, \ldots, f_n, c_1, \ldots, c_r)$ is any algebra of the form $\mathcal{A}' = (A; f_1, \ldots, f_n, g_1, \ldots, g_s, c_1, \ldots, c_r)$.

Note:

1. Since we view $\mathcal{A}$ as $\mathcal{F}/E$, the functions g_j respect E.
2. The functions g_j are necessarily computable.

Theorem (Bergstra-Tucker, $\approx$ 1980)

Every algebra with decidable word problem has a finitely presented expansion.

Example

A finitely presented expansion of $(\omega;\ x+1,\ 2^x,\ 0)$ is

$$(\omega;\ x+1,\ x+y,\ x\times y,\ 2^x,\ 0)$$

given by

$$x+0=x,\ x+(y+1)=(x+y)+1$$

$$x\times 0=0,\ x\times 1=x,\ x\times(y+1)=(x+y)+x,$$

$$2^0=1,\ 2^1=2,\ 2^{x+1}=2^x\times 2.$$

Statement of the problem

Bergstra and Tucker, and independently, Goncharov (1980s):

Does every f.g. and c.e. algebra have
a finitely presented expansion?

Answer: *No*.

Theorem (Kassymov (1988), Khoussainov(1994))

There exist f.g. and c.e. algebras that have no finitely presented expansions.

Reformulation of the problem

Can such examples be found in typical algebraic structures such as semigroups, groups, and algebras?

Definition

An infinite algebra $\mathcal{A} = \mathcal{F}/E$ is **effectively infinite** if $\mathcal{A}$ has an infinite c.e. sequence $t_0, t_1, \ldots$ of pairwise distinct elements.

Otherwise, call the algebra $\mathcal{A}$ **immune**.

Immune semigroups

Theorem (with D. Hirschfeldt)

There exists a f.g., c.e., and immune semigroup.

Proof (Outline). Consider $\mathbf{A} = (\{a,b\}^\star; \circ)$ the free semigroup. Let $X \subseteq \{a,b\}^\star$ be a nonempty subset. Define:

$$u \equiv_X v \iff u = v \vee u \text{ and } v \text{ have substrings from } X.$$

The semigroup $\mathbf{A}(X)/\equiv_X$ is well defined.

Lemma

There is a simple set X such that $\mathbf{A}(X)/\equiv_X$ is infinite, and hence immune.

Proof of the Lemma:

Use Post's adapted construction:

Put y into X if y is the first string of length $\geq i+5$ that appeared in W_i for some i. The set X is simple.

An argument, similar to the argument by Post, shows there are infinitely many strings that contain no substrings from X.

Hence, $\mathbf{A}(X)/\equiv_X$ is infinite. This proves the theorem.

The case of groups

A natural question arises:

Are there f.g. immune groups?

Such groups answer the generalised Burnside problem.

Properties of immune algebras

Let $\mathcal{A} = \mathcal{F}/E$ be an immune algebra. Then:

1. Each expansion of $\mathcal{A}$ is immune.
2. Each f.g. subalgebra of $\mathcal{A}$ is immune or finite.
3. For every term $t(x)$ the trace
$$a, t(a), tt(a), \ldots$$
is eventually periodic.
4. Infinite homomorphic images of $\mathcal{A}$ are immune.

Separator Lemma

Lemma (Separator Lemma)

If $\mathcal{A} = \mathcal{F}/E$ is **residually finite** *then for all distinct $x, y \in A$ there is a subset $S(x, y) \subset F$ such that:*

1. *$S(x, y)$ is computable and E-closed.*
2. *$x \in S(x, y)$ and $y \in F \setminus S(x, y)$.*

Comments:

- Immunity is not used in the proof. Also, there are no computability-theoretic assumptions on E.
- The set $S(x, y)$ is called a separator set.

Proof of the Separator Lemma

Let h be a homomorphism from $\mathcal{A} = \mathcal{F}/E$ onto finite algebrta $\mathcal{B}$

$$h : \mathcal{A} \to \mathcal{B}.$$

such that $h(x) \neq h(y)$. Consider the set of all pre-images of $h(x)$ in the free algebra $\mathcal{F}$. This set is the desired $S(x, y)$.

In other words, $S(x, y)$ is the set of all ground terms that map onto $h(x)$ under the natural homomorphism from $\mathcal{F}$ onto $\mathcal{B}$.

The Main Lemma

Lemma

If $\mathcal{A} = \mathcal{F}/E$ is immune and residually finite, then so are all expansions of $\mathcal{A}$.

Proof: Let x, y be distinct elements of an expansion $\mathcal{A}'$ of $\mathcal{A}$.

Define:

$a \equiv_{(x,y)} b \iff$ no elements in $S(x, y)$ and its complement are identified by the congruence relation on $\mathcal{A}'$ generated by (a, b).

Malcev's lemma (1954)

To understand properties of $\equiv_{(x,y)}$ we need Malcev's lemma.

Lemma

Let $E(a,b)$ be the least congruence of an algebra $\mathcal{A}$ containing (a,b). Then $(c,d) \in E(a,b)$ if and only if the following is true:

There exists a sequence of elements $e_0, \ldots, e_n$, and the sequence of terms $t_0(x,\bar{g}), \ldots, t_{n-1}(x,\bar{g})$ such that

- $e_0 = c$, $e_n = d$, *and*
- $\{e_0, e_1\} = \{t_0(a,\bar{g}), t_0(b,\bar{g})\}$,
  $\{e_1, e_2\} = \{t_1(a,\bar{g}), t_1(b,\bar{g})\}$,
  $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$
  $\{e_{n-1}, e_n\} = \{t_{n-1}(a,\bar{g}), t_{n-1}(b,\bar{g})\}$.

Algebraic terms

Definition

Let $\mathcal{A}$ be an algebra. A term over $\mathcal{A}$ is **algebraic** if it is of the form $t(x, \bar{g})$ where x is a variable and $\bar{g}$ are parameters from $\mathcal{A}$.

Lemma

An equivalence relation E on $\mathcal{A}$ is a congruence relation if and only if every algebraic term over $\mathcal{A}$ respects E. □

Properties of $\equiv_{(x,y)}$

Recall:

$a \equiv_{(x,y)} b \iff$ no element in $S(x, y)$ and its complement is identified by the congruence relation on $\mathcal{A}'$ generated by (a, b).

Property 1: The relation $\equiv_{(x,y)}$ forms an equivalence relation.

Proof.

Assume $a \equiv_{(x,y)} b$, $b \equiv_{(x,y)} c$ but not $a \equiv_{(x,y)} c$.

$a \equiv_{(x,y)} b$, $b \equiv_{(x,y)} c$ imply $a, b, c \in S(x, y)$ or $a, b, c \notin S(x, y)$.

$\neg(a \equiv_{(x,y)} c)$ implies that there is an algebraic term with $t(x)$ such that one of $t(a)$, $t(c)$ is in $S(x, y)$ and the other is not.

Assume $t(a) \in S(x, y)$. Then $t(b) \in S(x, y)$ since $a \equiv_{(x,y)} b$. Hence, $t(c) \in S(x, y)$ since $b \equiv_{(x,y)} c$. Contradiction. □

Properties of $\equiv_{(x,y)}$

Recall:

$a \equiv_{(x,y)} b \iff$ no element in $S(x, y)$ and its complement is identified by the congruence relation on $\mathcal{A}'$ generated by (a, b).

Property 2: $\equiv_{(x,y)}$ is a congruence relation on $\mathcal{A}'$.

Proof.

Otherwise, there exist $a, b \in \mathcal{A}'$ and an algebraic term $t(x)$ such that $a \equiv_{(x,y)} b$ and not $t(a) \equiv_{(x,y)} t(b)$.

So, there is an algebraic term $p(y)$ such that either
$p(t(a)) \in S(x, y)$ and $p(t(b)) \notin S(x, y)$ or
$p(t(a)) \notin S(x, y)$ and $p(t(b)) \in S(x, y)$. □

Recall:

$a \equiv_{(x,y)} b \iff$ no element in $S(x,y)$ and its complement is identified by the congruence relation on $\mathcal{A}'$ generated by (a,b).

Property 3: The relation $\equiv_{(x,y)}$ is a co-c.e. relation.

Proof.

$a \not\equiv_{(x,y)} b$ iff $\exists (c,d) \in E(a,b)(c \in S(x,y)\ \&\ d \notin S(x,y))$

if and only if

there is an algebraic term $t(x,\bar{g})$ such that $c = t(a,\bar{g})$ and $d = t(b,\bar{g})$ and $t(a,\bar{g}) \in S(x,y)$ and $t(b,\bar{g}) \notin S(x,y)$.

This is a c.e. event. □

Property 4: The quotient $\mathcal{A}'/\equiv_{(x,y)}$ is finite.

Proof.

Suppose not. Then $Tr(\equiv_{(x,y)})$ is a computably enumerable set.

Indeed, t is a minimal element in its equivalence class

if and only if

$z \not\equiv_{(x,y)} t$ for all $z < t$.

Therefore, $\mathcal{A}' / \equiv_{(x,y)}$ is effectively infinite. Hence $\mathcal{A}$ is effectively infinite. Contradiction. □

Finally note that in $\mathcal{A}'/\equiv_{(x,y)}$ the images of x and y are distinct. We proved the main lemma.

The non-finite presentability theorem

Theorem (The NFP Theorem)

Let $\mathcal{A} = \mathcal{F}/E$ be a f.g., c.e., immune, and residually finite algebra. Then $\mathcal{A}$ has no finitely presented expansions.

Proof.

Let $\mathcal{A}'$ be a finitely presented expansion.

Since $\mathcal{A}'$ is residually finite, by Malcev/McKenzie theorem, the equality E in $\mathcal{A}'$ is decidable. □

Application of the NFP Theorem: Semigroups case

Corollary (Semigroups case)

There exists a f.g. c.e. and immune semigroup that has no finitely presented expansion.

Proof.

The semigroup $\mathbf{A}(X)$ that we already built is f.g., c.e., immune, and residually finite. Apply the NFP theorem. □

How about algebras and groups?

Now we want to construct finitely generated, computably enumerable, residually finite, and immune algebras and groups.

We use Golod-Shafarevich Theorem.

We travel back: Golod-Shafarevich (1964)

Let K be a finite field. Consider the algebra

$$\mathcal{P} = K\langle x_1, x_2, \ldots, x_s \rangle$$

of polynomials in non-commuting variables.

Represent $\mathcal{P}$ as the direct sum

$$\mathcal{P} = \sum_n \mathcal{P}_n$$

where $\mathcal{P}_n$ is the vector space spanned over s^n monomials of degree n.

Golod-Shafarevich Theorem (1964)

Let H be a set of homogeneous polynomials.

Let $I = \langle H \rangle$ be the ideal generated by H.

Theorem (Golod Shafarevich)

Let r_n be the number of polynomials in H of degree n.

Let ϵ be such that $0 < \epsilon < s/2$ and for all n we have:

$$r_n \leq \epsilon^2 \cdot (s - 2\epsilon)^{n-2}.$$

Then the algebra

$$\mathcal{A} = \mathcal{P}/I = \sum_n \mathcal{P}_n/I$$

is infinite dimensional.

Application of the NFP Theorem: Algebras case

Corollary (with A. Miasnikov)

There exists a f.g. c.e. and immune algebra that has no finitely presented expansion.

Proof (Outline)

Use Post's type of construction to build a simple set H of homogeneous polynomials. Also, satisfy the assumption of Golod-Shafarevich theorem. The algebra

$$\mathcal{A} = \mathcal{P}/I$$

is residually finite. Apply the NFP theorem.

Application of the NFP Theorem: Groups case

Corollary (with A. Miasnikov)

There exists a f.g., c.e., and immune group G that has no finitely presented expansion.

For the algebra $\mathcal{A}$ built above (over two variables x and y), the semigroup $G = G(\mathcal{A})$ generated by

$$(1+x)/I \ \textit{and} \ (1+y)/I,$$

by Golod's theorem, is a group. The group G is residually finite. Apply the NFP theorem.

Is immunity important?

Answer: *No*.

Theorem

There exists a f.g., c.e., effectively infinite, and residually finite algebra that has no finitely presented expansion.

Answer: *Yes*.
The transversal of the equality relation built in the algebra is immune. This immunity is used in the proof.

Open questions

Are there semigroups, algebras, and groups that have no quasi-equational finite presentations?

Are there finitely presented immune algebras?

THANK YOU