Integrability, chaos and Galois groups in quantum dynamical systems

Igor Volovich

Steklov Mathematical Institute, Russian Academy of Sciences

Steklov Mathematical Institute Seminar
16 November, 2023

Outlook

- **Integrability of quantum dynamical systems**
- **Classical harmonic oscillators. Ergodicity**
- **Effectiveness. Galois groups**
- **Classical dynamical systems. ODE. PDE**
- **Wave operators. Integrals of motion**
- **Open quantum systems**

Main Statements

- **Any quantum dynamical system is integrable.**
- **It is unitary equivalent to a system of classical harmonic oscillators**
- **Integrals of motion**
- **Prototype - diagonalization of Hermitian matrix**
  - wave operators

Quantum dynamical systems

Quantum dynamical system: $(\mathcal{H},\ U_t)$ $\qquad U_{t+\tau} = U_t U_\tau$

Stone's theorem

Schrödinger equation: $\quad \psi = U_t\,\psi_0 = e^{-itH}\,\psi_0.$

$$i\frac{\partial \psi}{\partial t} = H\psi$$

$$(\mathcal{H},\ U_t) \Longleftrightarrow (\mathcal{H},\ H)$$

Harmonic oscillators

Definition. Complexified system of classical harmonic oscillators

$$\left(L^2(X,\mu),\omega, V_t = e^{-it\omega}\right)$$

$$\omega : X \to \mathbb{R}. \qquad \text{If } f \in L^2(X,\mu) \text{ then } V_t f(x) = e^{-it\omega_x} f(x).$$

X locally compact, μ **Radon measure,** ω **continuous function**

Let $\varphi_x = e^{-it\omega_x} f(x) = q_x + ip_x$

EOM $\quad i\dot{\varphi}_x = \omega_x \varphi_x \Longleftrightarrow \dot{q}_x = \omega_x p_x, \;\; \dot{p}_x = -\omega_x q_x$ **oscillator**

Integrability of Quantum Dynamical Systems

Theorem. **Let** $(\mathcal{H}, \ U_t)$ **be a quantum dynamical systems. Then there exists a complexified system of classical harmonic oscillators** $(L^2(X, \mu), \omega, \ V_t = e^{-it\omega})$ **and a unitary map** $W: \quad \mathcal{H} \to L^2(X, \mu)$ **such that**

$$U_t = W^* \, V_t \, W$$

On an appropriate domain one has

$$H = W^* \omega W$$

Integrals of motion:

$$[H, I_f] = 0, \qquad I_f = W^* f \, W, \qquad [I_f, I_g] = 0, \qquad f, g: \quad X \to \mathbb{R}$$

$$[U_t, W^* e^{i\tau f} W] = 0$$

Finite dimensional case 1/2

$$\mathcal{H} = \mathbb{C}^n, \quad H - \text{Hermitian matrix}$$

$$H = \sum_{j=1}^{n} \omega_j E_j$$

Integrals of motion $[H, E_j] = 0, \quad j = 1, ...n$

$$[E_i, E_j] = 0$$

Historical remarks

Newton, Euler, Lagrange, Liouville, Poincare,...

Liouvilles theorem: *If a Hamiltonian system with n degrees of freedom has n independent integrals in involution, then it can be integrated in quadratures. Canonical transform*.

Classical integrable systems: KdW, Gardner, Green, Kruskal, Miura, Lax, Zakharov, Novikov,...

Quantum integrable systems: Bethe, Yang, Baxter, Faddeev...
Dynamical systems: Bogoliubov, Kolmogorov, Arnold, Moser, Anosov, Sinai, Kozlov, Treschev,...

Non-local currents: Vladimirov, IV

Finite dimensional case 2/2

We first consider the case of a finite-dim. Hilbert space

$$\mathcal{H} = \mathbb{C}^n$$

Theorem 1. The Schrödinger equation $i\dot{\psi} = H\psi$, where

H is a Hermitian operator in $\mathbb{C}^n$, $\psi = \psi(t) \in \mathbb{C}^n$,

regarded as a classical Hamiltonian system, with a

symplectic structure obtained by decomplexification the

Hilbert space, has n independent integrals in involution.

Proof of Theorem 1. 1/2

Let ω_j, $j = 1, ..., n$ be the eigenvalues of the operator H. By diagonalizing the matrix H

$$i\dot{\varphi}_j = \omega_j \varphi_j, \quad \varphi_j = \varphi_j(t) \in \mathbb{C},\ j = 1, ..., n.$$

Passing to the real and imaginary parts of the function $\varphi_j(t) = (q_j(t) + ip_j(t))/\sqrt{2}$,

$$\dot{q}_j = \omega_j p_j,\ \dot{p}_j = -\omega_j q_j.$$

with the Hamiltonian

$$H_{osc} = \sum_{j=1}^{n} \frac{1}{2}\omega_j(p_j^2 + q_j^2) = (\psi, H\psi).$$

Proof of Theorem 1 2/2

Define $I_j(\varphi) = |\varphi_j|^2 = \frac{1}{2}(p_j^2 + q_j^2), \; j = 1, ..., n.$

The functions I_j **are integrals of motion, independent and in involution,** $\{I_j, I_m\} = 0, \; j, m = 1, ..., n.$**. The corresponding level manifold has the form of an** n**-dimensional torus.**
Note that the Hamiltonian is a linear combination of the integrals of motion:

$$H_{osc} = \sum_{j=1}^{n} \omega_j I_j.$$

V.V. Kozlov (nondegenerate spectrum)

Galois theory

If the Galois group of polynomials is solvable, then the roots of polynomials can be expressed in radicals.

H - **Hamiltonian of n-dim. quantum dynamical system**

If the characteristic polynomial of H has the solvable Galois group, then the roots of this polynomials can be expressed in radicals. **One can say that the eigenvalues of the Hamiltonian admits an effective computation**.

In this case the integrals of motion can be computed effectively. **So, the problem of effective computation of integrals of motion for quantum dynamical system is reduced to an effective computation of the roots of polynomials in the so-called main theorem of algebra**.

Abel-Ruffini theorem

$$x^5 - x - 1 = 0$$

Where is quantum chaos?

- **An arbitrary quantum dynamics is unitary equivalent to a system of classical harmonic oscillators**
- **So, what about quantum chaos?**

By the Liouville theorem the dynamics the n-harmonic oscillators is reduced to the motion on the n-dimensional torus T^n with equations of motions

$$\dot{z} = \lambda_i, \; i = 1, ...n$$

If λ_i are not resonance then the flow is ergodic.

Two oscillators

$$H = \frac{\omega_2}{2}(p_1^2 + q_1^2) + \frac{\omega_2}{2}(p_2^2 + q_2^2)$$

$$p_1^2 + q_1^2 = f_1 \qquad p_2^2 + q_2^2 = f_2, \qquad \frac{\omega_1}{\omega_2} = \textbf{irrational}$$

Trajectories on torus $\mathbb{T}^2$. Dense. Ergodic.

Free field. Mixing

$$\mathcal{F} = \oplus_0^\infty L^2(\mathbb{R}^d)^{\otimes_s^n}$$ **Fock space**

$$[a(k), a^*(p)] = \delta(k-p), \qquad a(k)\Psi_0 = 0$$

$$\lim_{t\to\infty}(U_t\Psi, \Phi) = (\Psi, \Psi_0)(\Psi_0, \Phi)$$ **mixing**

$$U_t = e^{-itH_0}, \qquad H_0 = \int dk\,\omega(k)a^*(k)a(k)$$

$$\omega(k) = k^2, \ldots$$

On integrability of classical dynamical systems

Liouville equation (analogue of the Schrödinger eq.)

Particles. ODE. $L^2(\mathbb{R}^{2n})$

$$\partial_t U_t = -i\{H, U_t\} = -i\,\mathcal{L}\,U_t$$

$$W: \quad L^2(\mathbb{R}^{2n}) \to L^2(X, \mu), \qquad \mathcal{L} = W^{-1}\omega W$$

$$\{H, I_f\} = 0, \quad \{I_f, I_g\} = 0 \qquad \text{integrals of motion}$$

Fields. PDE. $L^2(Y)$, $\qquad H = H(\phi(x), \pi(x))$

Sakbaev, I.V.

Integrability in Koopman's approach

Let $(M, \Sigma, \alpha, \tau_t)$ - dynamical system

(M, Σ) - measurable space with measure α

$\tau_t,\ t \in \mathbb{R}$ group of measure-preserving transformations M.

Then the Koopman transform defines a group of unitary operators U_t in $L^2(M, \alpha)$,

$$(U_t f)(m) = f(\tau_t(m)),\ f \in L^2(M, \alpha)$$

U_t **is unitary equivalent to the family of harmonic oscillators** and, in this sense, **any dynamical system is completely integrable in category of Hilbert spaces.**

Complexity

Compare the effectiveness (or complexity) of integrability with the efficiency/complexity of constructing solutions of Hamiltonian systems (that are integrable in the sense of Liouville).

Apparently, the change of variables to action-angle variables in the Liouville theorem,
is an analogue of the unitary transformation to harmonic oscillators.

Wave operators and integrability

- $(\mathcal{H}, H = H_0 + V)$

$$\Omega_\pm = \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}$$

$$\Omega_\pm H_0 = H \Omega_\pm, \qquad H = \Omega_\pm \, H_0 \, \Omega_\pm^{-1}$$

- **Integrals of motion**

$$[H_0, F_0] = 0 \Rightarrow [H, F_\pm] = 0, \quad \text{where} \quad F_\pm = \Omega_\pm \, F_0 \, \Omega_\pm^{-1}$$

Example. QM

- $\mathcal{H} = L^2(\mathbb{R}^n), H = H_0 + V$

$$H_0 = \frac{1}{2}(p_1^2 + ...p_n^2), \qquad V = V(x)$$

- $F_0 = \{F_{0,j\ell}\}, \quad F_{0,j\ell} = p_j^\ell, \quad j = 1, ...n, \ \ell = 1, 2, ...$

- **Integrals of motion**

$$[H_0, F_0] = 0 \Rightarrow [H, F_\pm] = 0, \quad \text{where} \quad F_\pm = \Omega_\pm F_0 \, \Omega_\pm^{-1}$$

Example. QFT

- $\mathcal{H}$ – Fock space, $\qquad H = H_0 + V, \qquad k \in \mathbb{R}^n$

$$H_0 = \int \omega(k) a^*(k) a(k) dk, \qquad [a(k), a^*(p)] = \delta(k-p)$$

$$V = \int g(k,q,p) a^*(k) a^*(q) a(p) dk dq dp + ...$$

- **Integrals of motion**

$$F_{0,j\ell} = \int k_j^\ell a^*(k) a(k) dk$$

$$[H_0, F_0] = 0 \Rightarrow [H, F_\pm] = 0, \quad \text{where} \quad F_\pm = \Omega_\pm F_0 \Omega_\pm^{-1}$$

Wave operators and integrability in QM, 1/2

Consider the Schrödinger equation in $L^2(\mathbb{R}^n)$ of the form

$$i\frac{\partial\psi}{\partial t} = H\psi,$$

where $H = H_0 + V(x)$, $H_0 = -\triangle$ - Laplace operator and the potential V **is short-range**

$$|V(x)| \leq C(1+|x|)^{-\nu}, \;\; x \in \mathbb{R}^n, \;\; \nu > 1.$$

Then H defines a self-adjoint operator and there exist the wave operators $\Omega_\pm$, which are complete and diagonalize H, $H\Omega_\pm = \Omega_\pm H_0$.

Wave operators and integrability in QM, 2/2

- **Higher order integrals of motion**

If $\varphi = \varphi(t, x)$ **is a smooth solution to the free Schrödinger equation** $i\dot{\varphi} + \triangle\varphi = 0$ **then its derivatives** $\partial_x^\alpha \varphi$ **also are solutions of this equation**.

Hence, one gets higher integrals by replacing φ **by** $\partial_x^\alpha \varphi$ **in the known integral** $I_0(\varphi) = \int_{\mathbb{R}^n} |\varphi|^2 dx$.

One gets a set of integrals: $I_\alpha = \int |\partial_x^\alpha \varphi|^2 dx$.

According to Lemma 1a, we obtain integrals

$$J_0, J_\alpha^\pm(\psi) = I_\alpha(\Omega_\pm^{-1}\psi)$$

for the original Schrödinger equation with the potential $V(x)$.

Wave operators and Lippman-Schwinger equation

Constructing wave operators by solving the Lippman-Schwinger integral equation

$$\varphi(x,k) = e^{ikx} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|} V(y)\varphi(y,k)dy$$

$$\Omega_+ = K^{-1}F,$$

$$K^{-1}(x,k) = (2\pi)^{-3/2}\varphi(x,k)$$

Convergent series

Under the condition $\|V\|_R < 4\pi$, the series of perturbation theory for the solution $\varphi(x,k)$ converges:

$$\varphi(x,k) = e^{ikx} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-\xi|}}{|x-\xi|} V(\xi) e^{ik\xi} d\xi + \ldots$$

Rolnik potential. A measurable function V on $\mathbb{R}^3$ is called a Rolnik potential if

$$||V||_R = \int_{\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2} dxdy < \infty.$$

Wave operators and integrability for nonlinear partial differential equations, 1/2

Nonlinear Klein - Gordon equation

$$\ddot{u} - a^2 \triangle u + m^2 u + f(u) = 0, \ t \in \mathbb{R}, \ x \in \mathbb{R}^n, , \qquad (1)$$

where $a > 0, \ m \geq 0$.
Eq. (1) is a Hamiltonian system with the Hamiltonian

$$H = \int_{\mathbb{R}^n} (\frac{1}{2} p(x)^2 + \frac{1}{2} (\nabla u)^2 + \frac{1}{2} m^2 u^2 + V(u(x)) dx = H_0 + V \qquad (2)$$

where $V' = f$ and the the Poisson brackets are
$\{p(x), u(y)\} = \delta(x - y)$

Wave operators and integrability for nonlinear partial differential equations, 2/2

One can take $f(u) = \lambda |u|^2 u$, $\lambda \geq 0$ and $n = 3$. If the initial data $(u(0),\ \dot{u}_t(0)) \in H^{k+1}(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$ then there exists a global solution $u(t)$ of Eq (1) with these initial data. Here $H^k(\mathbb{R}^n)$ is the Sobolev space. For any solution of (1) there exists a unique pair (v_+, v_-) of solutions for the free Klein-Gordon equation

$$\ddot{v} - a^2 \triangle v + m^2 v = 0, \qquad (3)$$

$$\lim_{t \to \pm\infty} \|(u(t),\ \dot{u}(t)) - (v_\pm(t), \dot{v}_\pm(t))\| = 0.$$

Moreover, the correspondence $(u(0),\ \dot{u}(0)) \mapsto (v_\pm(0), \dot{v}_\pm(0))$ defines homeomorphisms (wave operators) $\Omega_\pm$ on $H^{k+1} \times H^k$.

Integrals of motion.

If $v = v(t,x)$ is a smooth solution of Eq. (3) then its partial derivatives $\partial_x^\alpha v$ is also the solution of Eq. (3). Therefore to get higher order integrals of motion from the energy integral

$$E(v) = \int (\dot{v}^2 + a^2(\nabla v)^2 + m^2 v^2)\, dx/2$$

one can just replace v by $\partial_x^\alpha v$. We get higher integrals of motion $I_\alpha(v) = E(\partial_x^\alpha v)$ for Eq. (3). One expects that if $v \in H^{|\alpha|}$ then $J_\alpha^\pm(u) = I_\alpha(\Omega_\pm u)$ will be integrals of motion for Eq. (1).

Particle in a potential field. 1/2

- Let $\Gamma = M \times M^{'}$ phase (symplectic) space, where $M = \mathbb{R}^n$ is the configuration space and $M^{'} = \mathbb{R}^n$ is the dual space of momenta.
- Hamiltonian $H(x, \xi) = \frac{1}{2}\xi^2 + V(x)$, where $\xi \in M'$ and the function $V : M \to \mathbb{R}$ is bounded and the force $F(x) = \nabla V(x)$ is locally Lipschitz.
- Then the solution $(x(t, y, \eta), \xi(t, y, \eta))$ of the equations of motion

$$\dot{x}(t) = \xi(t), \;\; \dot{\xi}(t) = F(x(t))$$

with initial data

$$x(0, y, \eta) = y, \;\; \xi(0, y, \eta) = \eta, \;\; y \in M, \, \eta \in M'$$

exists and is unique for all $t \in \mathbb{R}$.

Particle in a potential field. 2/2

- **Denote** $\phi(t)(y,\eta) = (x(t,y,\eta), \xi(t,y,\eta))$.
  The free Hamiltonian is $H_0(x,\xi) = \xi^2/2$,
  The solution of EOM $\phi_0(t)(y,\eta) = (y + t\eta, \eta)$.
- Suppose $\lim_{|x|\to\infty} V(x) = 0$, $\int_0^\infty \sup_{|x|\geq r} |F(x)| dr < \infty$
- Then for any $(y,\eta) \in M \times M'$ there is a limit

$$\lim_{t\to\infty} \frac{x(t,y,\eta)}{t} = \xi_+(y,\eta),$$

moreover, if $\xi_+(y,\eta) \neq 0$, then there is a limit

$$\lim_{t\to\infty} \xi(t,y,\eta) = \xi_+(y,\eta),$$

and the inverse image of $\mathcal{D} = \xi_+^{-1}(M' \setminus \{0\})$ is the open set of all paths with positive energy unbounded for $t \to \infty$.

Proposition 1 (B.Simon,...)

Let the force satisfy the conditions

$$\int_0^\infty \sup_{|x|\geq r} |\partial_x^\alpha F(x)|(1+r^2)^{1/2} dr < \infty, \ |\alpha| = 0, 1.$$

Then there exists the limit

$$\lim_{t\to\infty} \gamma(-t)\gamma_0(t) = \Omega_+$$

uniformly on compact sets in $M \times (M' \setminus \{0\})$. **The mapping** $\Omega_+ : M \times (M' \setminus \{0\}) \to \mathfrak{D}$ **is symplectic, continuous and one-to-one. There are relations**

$$H\,\Omega_+ = H_0, \ \ \gamma(t)\,\Omega_+ = \Omega_+\,\gamma_0(t).$$

Theorem

The equations of motion of a particle in a potential field satisfying the above conditions determine an integrable system in the sense that the dynamics of $\gamma(t)$ **is symplectically equivalent to the free dynamics of** $\gamma_0(t)$ **on the domain** $\mathcal{D}$ **in** $2n$ **- dimensional phase space,** $\Omega_+^{-1}\gamma(t)\,\Omega_+ = \gamma_0(t)$. **On** $\mathcal{D}$ **there are** n **independent integrals of motion in involution.**

Open quantum systems

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \qquad L^2(X) = L^2(X_1) \otimes L^2(X_2)$$

GKSL equation is quasi-integrable

Summary

- **Any quantum dynamical system is integrable.**
- **It is unitarily equivalent to the system of classical harmonic oscillators.**
- **Higher order integrals of motion**