On weighted solutions of $\overline{\partial}$-equation in the complex plane $\mathbb{C}$

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Introduction

Definition

Let $1 < p < +\infty$ and $0 < \rho, \sigma < \infty, \gamma > -2$. Denote by $L^p_{\rho,\sigma,\gamma}(\mathbb{C})$ the space of all measurable complex-valued functions $f(z), z \in \mathbb{C}$, satisfying the condition

$$\int_{\mathbb{C}} |f(z)|^p e^{-\sigma|z|^\rho}|z|^\gamma dm(z) < +\infty. \tag{1}$$

Let $H^p_{\rho,\sigma,\gamma}(\mathbb{C})$ be the corresponding subspace of entire functions.

Theorem

If $f \in H^p_{\rho,\sigma,\gamma}(\mathbb{C})$ then f admits an integral representation of the form

$$f(z) = \frac{\rho\sigma^\mu}{2\pi}\int_{\mathbb{C}} f(w)e^{-\sigma|w|^\rho}|w|^\gamma \cdot E_{\rho/2}(\sigma^{2/\rho}z\overline{w};\mu)dm(w), \quad z \in \mathbb{C}, \tag{2}$$

where $\mu = (\gamma+2)/\rho$ and

$$E_{\rho/2}(\eta;\mu) = \sum_{k=0}^{\infty}\frac{\eta^k}{\Gamma(\mu+2k/\rho)}, \quad \eta \in \mathbb{C}, \tag{3}$$

is the Mittag-Leffler type function.

Remark

Note that for $p = 2, \rho = 2, \gamma = 0$, $H^p_{\rho,\sigma,\gamma}(\mathbb{C})$ coincides with the well-known Fock space of entire functions and 2 takes the form

$$f(z) = \frac{\sigma}{\pi}\int_{\mathbb{C}} f(w)\cdot e^{-\sigma|w|^2}\cdot e^{\sigma z\overline{w}}dm(w), \quad z \in \mathbb{C}. \tag{4}$$

Theorem

For functions $f \in C^1(\mathbb{C})$(satisfying certain growth conditions together with $\partial f(z)/\partial\overline{z}$) the following generalization of the formula 4 was established:

$$f(z) = \frac{\sigma}{\pi}\int_{\mathbb{C}} f(w)\cdot e^{\sigma z\overline{w}}\cdot e^{-\sigma|w|^2}dm(w) - \frac{1}{\pi}\int_{\mathbb{C}}\frac{\frac{\partial f(w)}{\partial\overline{w}}}{w-z}\cdot e^{\sigma z\overline{w}}\cdot e^{-\sigma|w|^2}dm(w), \quad z \in \mathbb{C}. \tag{5}$$

From now on we suppose that $\rho > 0, \sigma > 0, \gamma > -2, \mu = \frac{\gamma+2}{\rho}$.

Definition

The kernel Ψ is defined as follows ($z \in \mathbb{C}$):

$$\Psi(z;w) = 1 + \frac{\rho\cdot\sigma^\mu(z-w)}{w}\int_0^{|w|} e^{-\sigma\cdot r^\rho}\cdot r^{\gamma+1}E_{\rho/2}\left(\sigma^{2/\rho}r^2\frac{z}{w};\mu\right)dr$$

$$= 1 + \frac{\rho\cdot\sigma^\mu(z-w)}{w}\sum_{k=0}^{\infty}\frac{\sigma^{2k/\rho}}{\Gamma\left(\mu+\frac{2k}{\rho}\right)}\cdot\frac{z^k}{w^k}\int_0^{|w|}e^{-\sigma\cdot r^\rho}r^{\gamma+1+2k}dr,$$

$$w \in \mathbb{C}\backslash\{0\}, \tag{6}$$

$$\Psi(z;w) = 1, \quad w = 0. \tag{7}$$

Theorem

Assume that a function $f \in C^1(\mathbb{C})$ satisfies the following conditions:

- *for some $\varepsilon \in (0,1)$*
$$f(w) = O(e^{\sigma(1-\varepsilon)|w|^\rho}), \quad \text{when} \quad |w| \to \infty;$$
- *for some $\varepsilon \in (0,1)$*
$$\frac{\partial f(w)}{\partial\overline{w}} = O(e^{\sigma(1-\varepsilon)|w|^\rho}), \quad \text{when} \quad |w| \to \infty.$$

Then we have

$$f(z) = \frac{\rho\sigma^\mu}{2\pi}\cdot\iint_{\mathbb{C}} f(w)\cdot e^{-\sigma|w|^\rho}|w|^\gamma\cdot E_{\rho/2}\left(\sigma^{2/\rho}z\overline{w};\mu\right)dm(w) - \frac{1}{\pi}\cdot\iint_{\mathbb{C}}\frac{\frac{\partial f(w)}{\partial\overline{w}}}{w-z}\cdot\Psi(z;w)dm(w), \quad z \in \mathbb{C}. \tag{8}$$

Weighted solution of $\overline{\partial}$-equation in $\mathbb{C}$

Theorem

Let $g \in C^k_c(\mathbb{C}), k = 1,2,3,...,\infty$ (i.e. g is of class C^k with compact support) and

$$f(z) = -\frac{1}{\pi}\iint_{\mathbb{C}}\frac{g(w)}{w-z}\Psi(z;w)dm(w). \tag{9}$$

Then $f \in C^k(\mathbb{C})$ and

$$\frac{\partial f(z)}{\partial\overline{z}} \equiv g(z), \quad z \in \mathbb{C}. \tag{10}$$

Theorem

Let $g \in C^k(\mathbb{C}), k = 1,2,3,...,\infty$, and

$$|g(w)| \le C_0\cdot e^{\sigma(1-\varepsilon_0)|w|^\rho}, \quad w \in \mathbb{C}, \tag{11}$$

where $C_0 > 0$ and $\varepsilon_0 \in (0;1)$. If we put

$$f(z) = -\frac{1}{\pi}\iint_{\mathbb{C}}\frac{g(w)}{w-z}\Psi(z;w)dm(w), \quad z \in \mathbb{C} \tag{12}$$

then $f \in C^k(\mathbb{C})$ and

$$\frac{\partial f(z)}{\partial\overline{z}} \equiv g(z), \quad z \in \mathbb{C}. \tag{13}$$

Growth Estimates for the solution of $\overline{\partial}$-equation

The formula 9 can be written in the following form:

$$f(z) = -\frac{1}{\pi}\iint_{\mathbb{C}}\frac{g(z+w)}{w}\Psi(z;z+w)dm(w), \quad z \in \mathbb{C}. \tag{14}$$

In addition we have:

$$|g(w)| \le C_0\cdot e^{\sigma(1-\varepsilon_0)|w|^\rho}, \quad w \in \mathbb{C}, \quad 0 < \varepsilon_0 < 1.$$

Fix an arbitrary $\varepsilon_1 \in (0;\varepsilon)$, then, for an arbitrary $R > 0$ we have

$$|\Psi(z;w+z)| \le \frac{const(\rho;\sigma;\gamma;\varepsilon_1)}{e^{\sigma(1-\varepsilon_1)|z+w|^\rho}}, \tag{15}$$

where $|z| \le R$ and $|w| \ge R(1+(4/\varepsilon_1)^{\frac{2}{\rho}}) \equiv R\cdot\varkappa \equiv R_1$, where

$$\varkappa \equiv 1 + \left(\frac{4}{\varepsilon_1}\right)^{2/\rho} > 1. \tag{16}$$

Theorem

Let $g \in C^k(\mathbb{C})$, $k = 1,2,3,\dots,\infty$ and satisfies the condition 11. Also, let the function f is defined by 12 and, as it was proved above, is of class $C^k(\mathbb{C})$ and satisfies 13. For arbitrary $R > 0$ put $M_f(R) = \max_{|z|\le R}|f(z)|$. Then

$$M_f(R) \le \frac{C_0}{\pi}\cdot const(\rho,\sigma,\gamma,\varepsilon_1,\varepsilon_0)\cdot e^{\sigma(\varepsilon_0-\varepsilon_1)\cdot R^\rho} + 2C_0R\varkappa\cdot e^{\sigma(1-\varepsilon_0)(1+\varkappa)^\rho R^\rho}$$
$$\cdot\left(1 + const(\rho;\sigma;\gamma;\varepsilon_2)R^{\gamma+2}(2+\varkappa)(1+\varkappa)^{\gamma+1}e^{\sigma(1+\varepsilon_2)(1+\varkappa)^{\rho/2}R^\rho}\right). \tag{17}$$

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