Complete separation of variables in the Hamilton-Jacobi equation for geodesics

M. Katanaev
Steklov Mathematical Institute, Moscow

Notation: $(\mathbb{M}, g)$ - (pseudo)Riemannian manifold $x^\alpha, \ \alpha = 1, \ldots, n$ - coordinates

$\gamma = \left(x^\alpha(\tau)\right), \ \tau \in \mathbb{R}$ - worldline (trajectory) of a point particle

$S = \int_\gamma d\tau \frac{1}{2} g_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta$ - the action $\dot{x}^\alpha := \frac{dx^\alpha}{d\tau}$

Equations of motion $\ddot{x}^\alpha = -\Gamma_{\beta\gamma}{}^\alpha \dot{x}^\beta \dot{x}^\gamma$ - geodesics

There is always the integral of motion (the energy) $E := \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$

The Hamilton-Jacobi equation

The action function $W(x)$

$$\frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta = E = \text{const}$$

where $p_\alpha(x) := \partial_\alpha W(x)$

Separation of variables in the Hamilton-Jacobi equation

Definition. Coordinates x^α, if they exist, are called separable if the Hamilton-Jacobi equation admits complete separation of variables, that is the action function is a sum

$$W = \sum_{\alpha=1}^{n} W_\alpha(x^\alpha, c), \qquad c = (c_a) = (c_1, \ldots, c_n) \in \mathbb{R}^n$$

where functions $W_\alpha(x^\alpha, c)$ depend only on single coordinate and, possibly, a full set of independent parameters which satisfy the inequality

$$\det \frac{\partial^2 W}{\partial x^\alpha \partial c_a} \neq 0.$$

Functions $W_\alpha(x^\alpha, c)$ are called separating.

Definition. Two separable coordinate systems are called equivalent if there is a canonical transformation $(x, p) \mapsto (X, P)$ such that new coordinates X depend only on the old ones $X(x)$ but not on momenta. Moreover, separable coordinate systems are equivalent in the case when parameters are related by nondegenerate transformation of parameters $c \mapsto \tilde{c}(c)$ which do not involve coordinates

The problem

The Hamilton-Jacobi equation

$$g^{\alpha\beta} p_\alpha p_\beta = 2E \qquad (**)$$

where $p_\alpha(x^\alpha, c) = \partial_\alpha W_\alpha(x^\alpha, c) := W'_\alpha(x^\alpha, c)$

The problem is to find all metrics $g_{\alpha\beta}(x)$ depending only on coordinates, constant $E(c)$, and all possible functions $p_\alpha(x^\alpha, c)$ depending on single coordinates and, possibly, the full set of parameters such that

$$\det \frac{\partial p_\alpha}{\partial c_a} \neq 0.$$

P. Stackel (1893) Necessary and sufficient conditions for complete separation of variables for diagonal metrics of arbitrary signature.

F.A. Dall' Acqua (1908,1912)
P. Burgatti (1911) Necessary and sufficient conditions for metrics with nonzero diagonal elements.

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Separation of variables and Hamiltonian formulation

$$p_\alpha - W'_\alpha(x^\alpha, c) = 0 \quad \Rightarrow \quad c_a = F_a(x, p)$$

Proposition. Functions F_a are first integrals of the Hamiltonian equations.
These integrals are in involution $[F_a, F_b] = 0$.

In other words. There are n independent involutive conservation laws, and the Hamiltonian system is Liouville integrable. Therefore solution for geodesics can be written in quadrature and analyzed.

A general case

Strategy:
1) List all possible classes of separable metrics.
2) Define the set of independent parameters and separating functions.
3) Solve the functional Hamilton-Jacobi equation.
4) Choose the canonical separable metric in each equivalence class.
5) Prove that any additional conservation law is functionally dependent.

Definition. A coordinate line x^φ is called coisotropic if $g^{\varphi\varphi} \equiv 0$

Let there be N commuting Killing vector fields, M quadratic conservation laws, and $n-N-M$ coisotropic coordinates.

Three groups of coordinates: $(x^\alpha, y^\mu, z^\varphi) \in \mathbb{M}$

$\alpha, \beta, \ldots = 1, \ldots, N$ - Killing vector fields

$\mu, \nu, \ldots = N+1, \ldots, N+M$ - quadratic conservation laws, $g^{\mu\mu} \neq 0$

$\varphi, \chi, \ldots = N+M+1, \ldots, n$ - coisotropic coordinates, $g^{\varphi\varphi} \equiv 0$

We assume that all cyclic coordinates for Killing vector fields are separated:

$$W'_\alpha = c_\alpha, \qquad p_\alpha = c_\alpha$$

A general case (continued)

Definition of parameters: $d_{ii} := W'^2_\mu\big|_{p,\mu=i}, \quad i = N+1,\dots,N+M$

$a_r := W'_\varphi\big|_{p,\varphi=r}, \quad r = N+M+1,\dots,n$

$p \in \mathbb{M}$

$$(c_\alpha, d_{ij}, a_r;\; a_n := 2E) \qquad \text{(Case 1)}$$

$$(c_\alpha, d_{ij}, a_r;\; d_{N+M\;N+M} := 2E) \qquad \text{(Case 2)} \qquad (1)$$

Separating functions:

$$W'^2_\mu := b_{\mu\mu}{}^{ij}(y^\mu, c)d_{ij} + b_{\mu\mu}{}^{r}(y^\mu, c)a_r + k_{\mu\mu}(y^\mu, c)$$
$$W'_\varphi := b_\varphi{}^{ij}(y^\mu, c)d_{ij} + b_\varphi{}^{r}(y^\mu, c)a_r + l_\varphi(y^\mu, c) \qquad (2)$$

Lemma. If independent parameters and separating functions are chosen as in equations (1) and (2) then separable metric has block form

$$g^{**} = \begin{pmatrix} g^{\alpha\beta}(y,z) & 0 & g^{\alpha\phi}(y,z) \\ 0 & g^{\mu\nu}(y,z) & 0 \\ g^{\varphi\beta}(y,z) & 0 & 0 \end{pmatrix}$$

$g^{\mu\nu} \equiv 0, \quad \mu \neq \nu$

$* = (\alpha, \mu, \varphi)$

Coisotropic coordinates

Coordinates and parameters: $x^{\alpha}, \quad \alpha = 1, \dots, N \qquad z^{\varphi}, \ \varphi = N+1, \dots, n$

$$c_a, \quad a = 1, \dots, N \qquad a_r, \ \ r = N+1, \dots, n$$

$$a_n := 2E \qquad n/2 \le N < n$$

Invertible matrix
$$\phi_{\varphi}{}^{r} = \begin{pmatrix} 1 & \phi_{N+1}{}^{N+2} & \cdots & \phi_{N+1}{}^{n} \\ \phi_{N+2}{}^{N+1} & 1 & \cdots & \phi_{N+2}{}^{n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n}{}^{N+1} & \phi_{n}{}^{N+1} & \cdots & 1 \end{pmatrix}$$

Invertible matrix $b_{\varphi}{}^{r}(z^{\varphi}, c) := \dfrac{\phi_{\varphi}{}^{r}}{h_{\varphi}^{\alpha} c_{\alpha}}$

$b_r{}^{\varphi}$ - the inverse matrix

$b_n{}^{\varphi} = \phi_n{}^{\varphi} h_{\varphi}^{\alpha} c_{\alpha}$ - the last row

$\phi_{\varphi}{}^{r}(z^{\varphi}), \ h_{\varphi}^{\alpha}(z^{\varphi})$ - arbitrary functions

Coisotropic coordinates (continued)

Theorem. If separable metric has exactly N and not more commuting Killing vector fields, and all other coordinates are coisotropic, then there is such set of independent parameters a_r and separating functions that the Hamilton-Jacobi equation is separable if and only if (i) $n = 2N$, (ii) all conservations laws are linear, and (iii) canonical separable metric has block form

$$g^{**} = \frac{1}{2}\begin{pmatrix} 0 & b_n{}^{\alpha\varphi} \\ b_n{}^{\alpha\varphi} & 0 \end{pmatrix}, \qquad b_n{}^{\alpha\varphi}(z) := \phi_n{}^{\varphi}(z) h_{\varphi}^{\alpha}(z^{\varphi})$$

The Hamilton-Jacobi equation is $\quad b_n{}^{\alpha\varphi} W'_\alpha W'_\varphi = 2E$

Variables are separated by $\quad W'_\alpha = c_\alpha, \qquad W'_\varphi = \dfrac{\phi_\varphi{}^{r}}{h_\varphi^\alpha c_\alpha} a_r$

Conservation laws are $\quad p_\alpha = c_\alpha$

$$\phi_r{}^{\varphi} h_\varphi^\alpha c_\alpha p_\varphi = a_r \qquad \text{- linear}$$

A general case

Energy enters either as $a_n = 2E$ (case 1) or $d_{N+M\ N+M} = 2E$ (case 2)

Every separable metric belongs to one of the classes: $[N, M, n-N-M]_1$

$[N, M, n-N-M]_2$

Theorem. There exists such set of independent parameters and separating functions, that canonical separable metric belongs to class $[N, M, n-N-M]_2$ if and only if metric has block form

$$g^{**} = \begin{pmatrix} -b_{N+M\ N+M}{}^{\mu\nu} k^{\alpha\beta}_{\mu\nu} & 0 & b_{N+M\ N+M}{}^{\alpha\varphi}/2 \\ 0 & b_{N+M\ N+M}{}^{\mu\mu} & 0 \\ b_{N+M\ N+M}{}^{\alpha\varphi}/2 & 0 & 0 \end{pmatrix}$$

where $b_{N+M\ N+M}{}^{\varphi}(y,z,c) = b_{N+M\ N+M}{}^{\alpha\varphi}(y,z)c_\alpha$

Conservation laws: $p_\alpha = c_\alpha$

$$b_{ii}{}^{\mu\nu}(p_\mu p_\nu - k^{\alpha\beta}_{\mu\nu} c_\alpha c_\beta) + b_{ii}{}^{\varphi} p_\varphi = d_{ii}$$

$$b_r{}^{\mu\nu}(p_\mu p_\nu - k^{\alpha\beta}_{\mu\nu} c_\alpha c_\beta) + b_r{}^{\varphi} p_\varphi = a_r$$

A general case (continued)

Case 1: $a_n = 2E$

Theorem. There exists such set of independent parameters and separating functions, that canonical separable metric belongs to class $[N, M, n-N-M]_1$ if and only metric has block form

$$g^{**} = \begin{pmatrix} -b_n{}^{\mu\nu} k^{\alpha\beta}_{\mu\nu} & 0 & b_n{}^{\alpha\varphi}/2 \\ 0 & b_n{}^{\mu\mu} & 0 \\ b_n{}^{\alpha\varphi}/2 & 0 & 0 \end{pmatrix}$$

where $b_n{}^{\varphi}(y,z,c) = b_n{}^{\alpha\varphi}(y,z)c_\alpha$

Conservation laws are $p_\alpha = c_\alpha$

$$b_{ii}{}^{\mu\nu}(p_\mu p_\nu - k^{\alpha\beta}_{\mu\nu} c_\alpha c_\beta) + b_{ii}{}^{\varphi} p_\varphi = d_{ii}$$

$$b_r{}^{\mu\nu}(p_\mu p_\nu - k^{\alpha\beta}_{\mu\nu} c_\alpha c_\beta) + b_r{}^{\varphi} p_\varphi = a_r$$

Theorem. Any additional conservation law is some function of n conservation laws described in previous theorems.

Four dimensions (ten classes)

1) Class $[4,0,0]$

Coordinates and parameters: $(x^\alpha, y^\mu, z^\varphi) \mapsto (x^1, x^2, x^3, x^4)$

$(c_\alpha, d_{ij}, a_r) \mapsto (c_1, c_2, c_3, c_4)$

Canonical separable metric $g^{\alpha\beta} = \eta^{\alpha\beta} \qquad \alpha, \beta = 1, 2, 3, 4$

The Hamilton-Jacobi equation $\eta^{\alpha\beta} W'_\alpha W'_\beta = 2E$

Variables are separated by $W'_\alpha = c_\alpha$

Conservation laws: $p_\alpha = c_\alpha$

Classes 2-9: $[3,1,0]_2\ [2,2,0]_2\ [2,0,2]_1\ [2,1,1]_1\ [2,1,1]_2\ [1,3,0]_2\ [0,4,0]_2$

$[3,1,0]_2$ The Kasner solution

$[2,2,0]_2$ The Schwarzschild, Reisner-Nordstrom, Kerr solutions

$[1,3,0]_2$ The Friedman solution

Four dimensions (ten classes)

10) Class $[1,2,1]_2$

Coordinates and parameters: $(x^\alpha, y^\mu, z^\varphi) \mapsto (x, y^2, y^3, z)$

$(c_\alpha, d_{ij}, a_r) \mapsto (c, d, 2E, a)$

Stackel matrix $$(\phi_{\mu\mu}{}^{ii}) = \begin{pmatrix} 1 & \phi_{23}(y^2) & \phi_{24}(y^2) \\ \phi_{32}(y^3) & 1 & \phi_{34}(y^3) \\ \phi_{42}(y^4)/c & \phi_{43}(y^4)/c & 1/c \end{pmatrix}$$

$$(\phi_{ii}{}^{\mu\mu}) = \frac{1}{\Delta} \begin{pmatrix} \Delta_{22} & \Delta_{32} & \Delta_{42} \\ \Delta_{23} & \Delta_{33} & \Delta_{43} \\ \Delta_{24} & \Delta_{34} & \Delta_{44} \end{pmatrix}$$

where $\Delta := \det(\phi_{\mu\mu}{}^{ii})$

$\Delta_{\mu i}$ denotes the cofactor of element $\phi_{\mu\mu}{}^{ii}$

Six arbitrary functions

Four dimensions (ten classes)

10) Class

$[1,2,1]_2$

Canonical separable metric

$$g^{**}=\frac{1}{\Delta}\begin{pmatrix}-\Delta_{24}k_2-\Delta_{34}k_3 & 0 & 0 & \Delta_{43}/2c\\ 0 & \Delta_{24} & 0 & 0\\ 0 & 0 & \Delta_{34} & 0\\ \Delta_{43}/2c & 0 & 0 & 0\end{pmatrix}$$

The Hamilton-Jacobi equation

$$\frac{1}{\Delta}\left[(-\Delta_{23}k_2-\Delta_{33}k_3)c^2+\Delta_{23}W_2'^2+\Delta_{33}W_3'^2+\Delta_{43}W_4'\right]=2E$$

Variables are separated by

$$W_2'^2=d+2\phi_{23}E+\phi_{24}a+k_2c^2$$

$$W_3'^2=b_{32}d+2E+2b_{34}a+k_3c^2$$

$$W_4'=(\phi_{43}d+2\phi_{43}E+a)/c$$

Conservation laws:

$$\frac{1}{\Delta}\left[\Delta_{22}(p_2^2-k_2c^2)+\Delta_{32}(p_3^2-k_3c^2)+\Delta_{42}p_4\right]=d$$

$$\frac{1}{\Delta}\left[\Delta_{23}(p_2^2-k_2c^2)+\Delta_{33}(p_3^2-k_3c^2)+\Delta_{43}p_4\right]=2E$$

$$\frac{1}{\Delta}\left[\Delta_{24}(p_2^2-k_2c^2)+\Delta_{34}(p_3^2-k_3c^2)+\Delta_{44}p_4\right]=a$$

Conclusion

The Stackel problem (1891) is completely solved:
All separable metrics of arbitrary signature in any number n of dimensions are explicitly described. All conservation laws are found.

Separable metrics are divided into equivalence classes denoted by

$$[N, M, n-N-M]_{1,2}$$

where N is the number of commuting Killing vector fields, M is the number of indecomposable quadratic conservation laws, and $n-N-M$ is the number of coisotropic coordinates. Indices $1, 2$ denote position of energy. Within each equivalence class, separable metrics are related by canonical transformations and arbitrary nondegenerate transformation of parameters.

All separable metrics in two (three classes), three (six classes), and four (ten classes) dimensions are constructed, as examples.