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Traces of first-order Sobolev spaces to lower content regular subsets of metric measure spaces

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Roots of the Problem

The classical trace problem. *Let $S \subset \mathbb{R}^n$ be a closed nonempty set and $p \in (1, \infty)$. Given a Borel function $f : S \to \mathbb{R}$, how can we decide whether f extends to a $W_p^1(\mathbb{R}^n)$-function?*

The famous result of Gagliardo (1957) reads as follows:

A Borel function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ extends to $W_p^1(\mathbb{R}^n)$-function if and only if $f \in B_{p,p}^{1-1/p}(\mathbb{R}^{n-1})$, i.e.,

$$\|f|B_{p,p}^{1-\frac{1}{p}}(\mathbb{R}^{n-1})\| := \|f|L_p(\mathbb{R}^{n-1})\| + \Big(\int\limits_{\mathbb{R}^{n-1}} \int\limits_{B_1(0)} \frac{|f(x) - f(x+h)|^p}{|h|^{n+p-1}} \, dh \, dx\Big)^{\frac{1}{p}} < +\infty.$$

For general subsets $S \subset \mathbb{R}^n$ the most powerful results in the case $p > n$ were given by P. Shvartsman (2010) and in the case $1 < p \leqslant n$ by T. and S. K. Vodop'yanov (2020).

Metric Measure Spaces

A metric measure space (m.m.s.) is a triple $\mathsf{X} = (\mathsf{X}, \mathsf{d}, \mu)$, where (X, d) is *a complete separable metric space* and μ is *a Borel locally finite positive measure*. Given $q \in (1, \infty)$, we say that X is q-admissible if:

A) the measure μ has *the uniformly locally doubling property*, i.e.,

$$C(R) := \sup_{r \in (0,R]} \sup_{x \in \mathsf{X}} \frac{\mu(B_{2r}(x))}{\mu(B_r(x))} < +\infty \quad \forall R > 0;$$

B) X supports a local $(1, q)$-Poincaré inequality, i.e., $\forall R > 0$ $\exists C > 0$, $\exists \lambda \geqslant 1$ s. t. $\forall f \in \mathrm{LIP}(X)$

$$\fint_{B_r(x)} \left| f(y) - \fint_{B_r(x)} f(z)\, d\mu(z) \right| d\mu(y) \leqslant C r^q \fint_{B_{\lambda r}(x)} (\mathrm{lip}[f](y))^q \, d\mu(y) \quad \forall r \in (0, R),$$

where $\mathrm{lip}[f]$ is the local Lipschitz constant of f, i.e.,

$$\mathrm{lip}[f](x) := \begin{cases} \overline{\lim\limits_{z \to x}} \frac{|f(z) - f(x)|}{\mathrm{d}(z,x)}, & x \text{ is a limit point;} \\ 0, & x \text{ is a isolated point.} \end{cases} \tag{1}$$

Sobolev spaces, approach of J. Cheeger

Following J. Cheeger (1999) for each $p \in (1, \infty)$ we define *the Cheeger-Sobolev space* $W_p^1(\mathrm{X})$ by letting

$$W_p^1(\mathrm{X}) := \{F \in L_p(\mathrm{X}) : \mathrm{Ch}_p[F] < +\infty\},$$

where $\mathrm{Ch}_p[f]$ is *a Cheeger energy* defined as

$$\mathrm{Ch}_p[F] := \inf\{\varliminf_{n\to\infty} \int\limits_{\mathrm{X}} (\mathrm{lip}[F_n])^p \, d\mu : \{F_n\} \subset \mathrm{LIP}(\mathrm{X}), F_n \to F \text{ in } L_p(\mathrm{X})\}.$$

The space $W_p^1(\mathrm{X})$ is normed by

$$\|F|W_p^1(\mathrm{X})\| := \|F|L_p(\mathrm{X})\| + \left(\mathrm{Ch}_p[F]\right)^{\frac{1}{p}}.$$

Capacity C_p

Let $K \subset \mathrm{X}$ be a compact set. For each $p \in (1, \infty)$ we set

$$C_p(K) := \inf \|\varphi | W_p^1(\mathrm{X})\|^p,$$

where the "inf"is taken over all $\varphi \in \mathrm{LIP}_c(\mathrm{X})$, $\varphi \geqslant 1$ on K.

If $\Omega \subset \mathrm{X}$ is open

$$C_p(\Omega) := \sup_{K \subset \Omega} C_p(K).$$

For any Borel set $E \subset \mathrm{X}$ we put

$$C_p(E) := \inf_{E \subset \Omega} C_p(\Omega).$$

Definition of the sharp trace

It is well known that $\forall F \in W_p^1(\mathrm{X})$ there is a set $E_F \subset \mathrm{X}$ with $C_p(E_F) = 0$ and a representative $\overline{F}$ s.t.

$$\lim_{r \to 0} \fint_{B_r(x)} |\overline{F}(x) - F(y)| \, d\mu(y) = 0, \quad \forall x \in \mathrm{X} \setminus E_F.$$

Hence, if $C_p(S) > 0$ for each $F \in W_p^1(\mathrm{X})$ we define the sharp trace $F|_S := \overline{F}|_S$. The sharp trace is well defined up to a set of C_p-capacity zero.

The sharp trace space

Given $S \subset \mathrm{X}$ with $C_p(S) > 0$, by $W_p^1(\mathrm{X})|_S$ we denote the sharp trace space on S of the space $W_p^1(\mathrm{X})$, i.e.,

$$W_p^1(\mathrm{X})|_S := \{f : S \to \mathbb{R} | \exists F \in W_p^1(\mathrm{X}) \text{ s.t. } F|_S = f\}$$

with a quotient-space norm

$$\|f|W_p^1(\mathrm{X})|_S\| = \inf_{F|_S = f} \|F|W_p^1(\mathrm{X})\|.$$

$\mathrm{Tr}\,|_S : W_p^1(\mathrm{X}) \to W_p^1(\mathrm{X})|_S$ – the sharp trace operator.

The sharp trace problem

The sharp trace problem. *Let $p \in (1,\infty)$ and let $S \subset \mathrm{X}$ be a closed nonempty set with $C_p(S) > 0$.*

$(\mathbf{Q}1)$ *Given a Borel function $f : S \to \mathbb{R}$, find necessary and sufficient conditions for the existence of a Sobolev extension F of f, i.e., $F \in W_p^1(\mathrm{X})$ and $F|_S = f$.*

$(\mathbf{Q}2)$ *Using only geometry of the set S and values of the function f compute the trace norm $\|f|W_p^1(\mathrm{X})|_S\|$ up to some universal constants.*

$(\mathbf{Q}3)$ *Does there exist a bounded linear operator $\mathrm{Ext}_{S,p} : W_p^1(\mathrm{X})|_S \to W_p^1(\mathrm{X})$ such that $\mathrm{Tr}\,|_S \circ \mathrm{Ext}_{S,p} = \mathrm{Id}$ on $W_p^1(\mathrm{X})|_S$?*

The $\mathfrak{m}$-trace

Given $p \in (1, \infty)$, let $S \subset \mathrm{X}$ be a closed set with $C_p(S) > 0$. Let $\mathfrak{m}$ be a Borel measure with $\operatorname{supp} \mathfrak{m} = S$ that is absolutely continuous w.r.t. C_p. We say that f is an $\mathfrak{m}$-trace of a function $F \in L_1^{loc}(\mathrm{X}, \mu)$ and write $f = F|_S^{\mathfrak{m}}$ if

$$\lim_{r \to 0} \fint_{B_r(x)} |f(x) - F(y)| \, d\mu(y) = 0 \text{ for } \mathfrak{m} - \text{a.e. } x \in S.$$

The $\mathfrak{m}$-trace operator $\operatorname{Tr}|_S^{\mathfrak{m}} : W_p^1(\mathrm{X}) \to L_0(S, \mathfrak{m})$ takes $F \in W_p^1(\mathrm{X})$ and gives back $F|_S^{\mathfrak{m}} \in L_0(S, \mathfrak{m})$. We set

$$W_p^1(\mathrm{X})|_S^{\mathfrak{m}} := \{f : S \to \mathbb{R} | \exists F \in W_p^1(\mathrm{X}) \text{ s.t. } F|_S^{\mathfrak{m}} = f\}$$

and equip it with a quotient-space norm.

The $\mathfrak{m}$-trace problem

The $\mathfrak{m}$-trace problem. *Let $p \in (1, \infty)$, $S \subset \mathrm{X}$ be a closed set with $C_p(S) > 0$. Let $\mathfrak{m}$ be an absolutely continuous w.r.t. C_p measure such that* $\operatorname{supp} \mathfrak{m} = S$.

$(\mathbf{MQ}1)$ *Given $f : S \to \mathbb{R}$, find necessary and sufficient conditions for the existence of a Sobolev extension F of f, i.e., $F \in W_p^1(\mathrm{X})$ and $F|_S^{\mathfrak{m}} = f$.*

$(\mathbf{MQ}2)$ *Using only geometry of the set S and values of the function f compute the trace norm $\|f|W_p^1(\mathrm{X})|_S^{\mathfrak{m}}\|$ up to some universal constants.*

$(\mathbf{MQ}3)$ *Does there exist a bounded linear operator* $\operatorname{Ext}_{\mathfrak{m},p} : W_p^1(\mathrm{X})|_S^{\mathfrak{m}} \to W_p^1(\mathrm{X})$ *such that* $\operatorname{Tr}|_S^{\mathfrak{m}} \circ \operatorname{Ext}_{\mathfrak{m},p} = \operatorname{Id}$ *on* $W_p^1(\mathrm{X})|_S^{\mathfrak{m}}$*?*

Regular sets

A closed set $S \subset \mathrm{X}$ is regular if $\exists \lambda \in (0,1)$ s.t.

$$\mu(B_r(x) \cap S) \geqslant \lambda \mu(B_r(x)), \quad \forall x \in S, \quad \forall r \in (0,1].$$

We set

$$f^{\sharp}_{S,\mu}(x) := \sup_{r \in (0,1]} \frac{1}{r} \inf_{c \in \mathbb{R}} \fint_{B_r(x) \cap S} |f(y) - c| \, d\mu(y), \quad x \in S.$$

Shvartsman's criterion (2007). *A function $f : S \to \mathbb{R}$ belongs to $W^1_p(\mathrm{X})|^{\mu}_S$ if and only if $f \in L_p(S,\mu)$ and $f^{\sharp}_{S,\mu} \in L_p(S,\mu)$. Furthermore,*

$$\|f|W^1_p(\mathrm{X})|^{\mu}_S\| \approx \|f|L_p(S,\mu)\| + \|f^{\sharp}_{S,\mu}|L_p(S,\mu)\|$$

and there exists a bounded linear extension operator $\mathrm{Ext}_{S,\mu}$ *s.t.* $\mathrm{Tr}\,|^{\mu}_S \circ \mathrm{Ext}_{S,\mu} = \mathrm{Id}$ *on* $W^1_p(\mathrm{X})|^{\mu}_S$.

The codimension θ Hausdorff content and the codimension θ Hausdorff measure

Let $\theta \geqslant 0$, $S \subset X$. For each $\delta \in (0, \infty]$ we set

$$\mathcal{H}_{\theta,\delta}(S) = \inf \sum_{j} \frac{\mu(B_{r_j}(x_j))}{r_j^{\theta}},$$

where the "infimum"is taken over all coverings of S by sequences of balls $\{B_{r_j}(x_j)\}$ with $r_j \in (0, \delta)$.

$\mathcal{H}_{\theta,\infty}(S)$ – the codimension θ Hausdorff content of S.

The codimension θ Hausdorff measure of S is defined as

$$\mathcal{H}_{\theta}(S) := \lim_{\delta \to 0} \mathcal{H}_{\theta,\delta}(S).$$

The Ahlfors-David codimension θ regular sets

Given $\theta \geqslant 0$, a set $S \subset \mathrm{X}$ is Ahlfors-David codimension θ regular if

$$\mathcal{H}_\theta(S \cap B_r(x)) \approx \frac{\mu(B_r(x))}{r^\theta}, \quad \forall x \in S, \quad \forall r \in (0,1].$$

Maly, Saksman and Soto criterion. *Let $\theta \geqslant 0$ and $p \in [0,\theta)$. A function $f : S \to \mathbb{R}$ belongs to $W_p^1(\mathrm{X})|_S^{\mathcal{H}_\theta}$ if and only if $f \in B_{p,p}^{1-\theta/p}(S)$ Furthermore,*

$$\|f|W_p^1(\mathrm{X})|_S^{\mathcal{H}_\theta}\| \approx \|f|B_{p,p}^{1-\theta/p}(S)\| := \|f|L_p(S,\mathcal{H}_\theta)\|$$
$$+ \Big(\sum_{k=1}^{\infty} \epsilon^{k(\theta-p)} \int_S \Big(\fint_{B_k(x)} \fint_{B_k(x)} |f(y)-f(z)|\, d\mathcal{H}_\theta(y)\, d\mathcal{H}_\theta(z)\Big)^p d\mathcal{H}_\theta(x)\Big)^{\frac{1}{p}}$$

and there exists a bounded linear extension operator $\mathrm{Ext} : W_p^1(\mathrm{X})|_S^{\mathcal{H}_\theta} \to W_p^1(\mathrm{X})$.

Codimension θ lower content regular sets

Given $\theta \geqslant 0$, a set $S \subset \mathrm{X}$ is said to be θ-thick, or equivalently codimension θ lower content regular if $\exists \lambda_S > 0$ s.t.

$$\mathcal{H}_{\theta,\infty}(B_r(x) \cap S) \geqslant \lambda_S \frac{\mu(B_r(x))}{r^\theta}, \quad \forall x \in S, \quad \forall r \in (0,1].$$

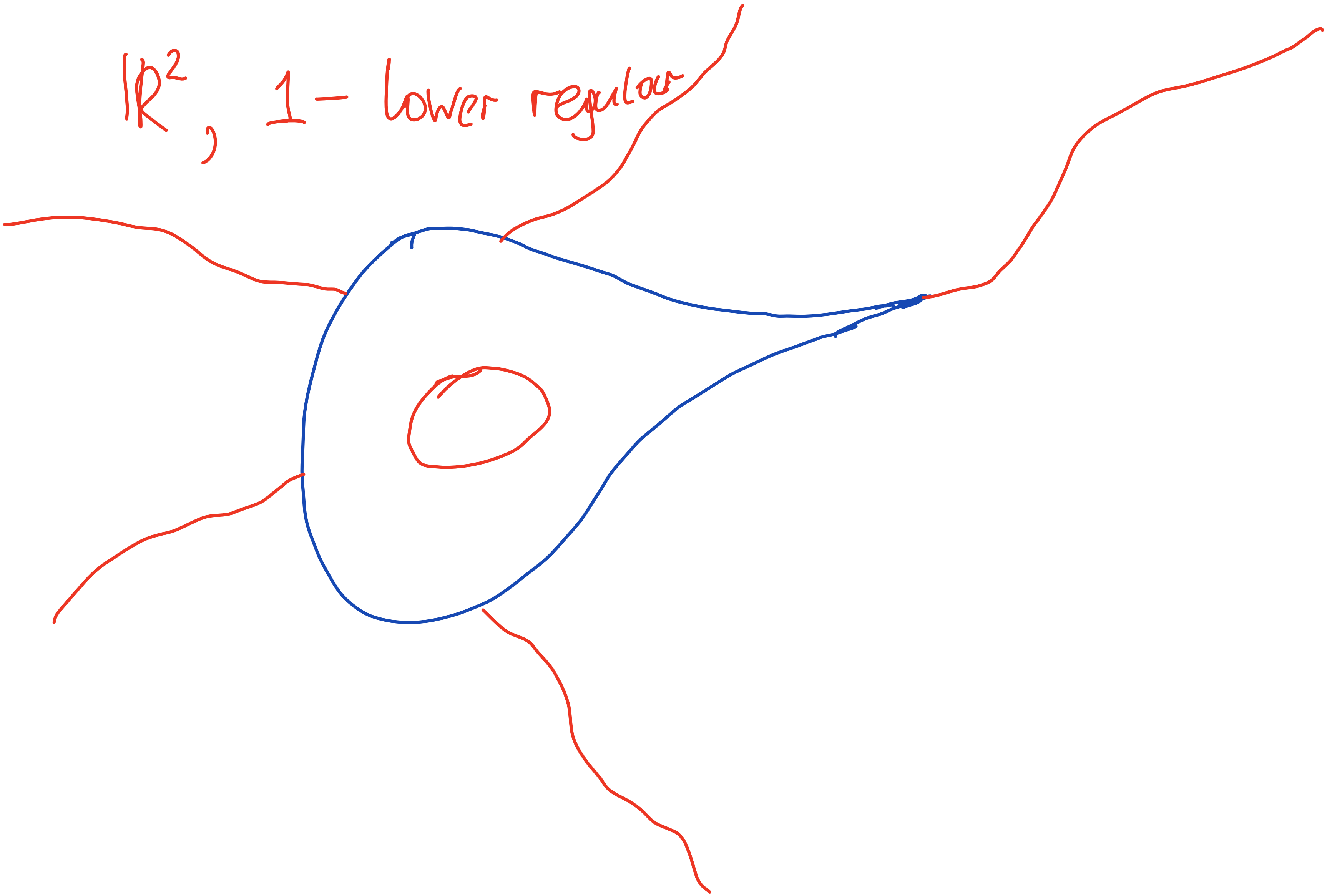
By $\mathcal{LCR}_\theta(\mathrm{X})$ we denote the class of all codimension θ lower content regular sets.

Examples:

(1) Any path-connected set $S \subset \mathbb{R}^n$ s.t. card $S > 1$ is $n-1$-thick;

(2) $S \subset \mathbb{R}^n$ is Ahlfors-David θ-regular $\Longrightarrow$ θ-thick. The converse is false (ball on a rope).

Codimension θ lower content regular sets



Regular sequences of measures

Given $\theta \geqslant 0$, a set $S \in \mathcal{LCR}_\theta(\mathrm{X}) \Leftrightarrow \mathfrak{M}_\theta(\mathrm{X}) \neq \emptyset$. A sequence of measures $\{\mathfrak{m}_k\}_{k\in\mathbb{N}_0} \in \mathfrak{M}_\theta(\mathrm{X})$ if $\operatorname{supp}\mathfrak{m}_k = S$ for all $k \in \mathbb{N}$ and $\exists\epsilon \in (0,1)$:

(M1) $\exists C^1 > 0$ s. t. $\forall k \in \mathbb{N}_0$

$$\mathfrak{m}_k(B_r(x)) \leqslant C^1 \frac{\mu(B_r(x))}{r^\theta} \quad \forall x \in \mathrm{X} \quad \text{and} \quad \forall r \in (0, \epsilon^k];$$

(M2) $\exists C^2 > 0$ s. t. $\forall k \in \mathbb{N}_0$

$$\mathfrak{m}_k(B_r(x) \cap S) \geqslant C^2 \frac{\mu(B_r(x))}{r^\theta} \quad \forall r \in (\epsilon^k, 1];$$

(M3) $\mathfrak{m}_k = w_k \mathfrak{m}_0$ with $w_k \in L_\infty(S, \mathfrak{m}_0)$ $\forall k \in \mathbb{N}_0$ and $\exists C^3 > 0$ s. t.

$$\frac{1}{C^3} w_{k+1}(x) \leqslant w_k(x) \leqslant C^3 w_{k+1}(x) \quad \text{for} \quad \mathfrak{m}_0 - \text{a.e.} \quad x \in S;$$

(M4) for any Borel set $E \subset S$

$$\overline{D}_E^{\{\mathfrak{m}_k\}} := \varlimsup_{k\to\infty} \frac{\mathfrak{m}_k(E \cap B_{\epsilon^k}(x))}{\mathfrak{m}_k(B_{\epsilon^k}(x))} > 0 \quad \text{for} \quad \mathfrak{m}_0 - \text{a.e. } x \in E.$$

New Calderón-type maximal functions

Let $\theta \geqslant 0$ and $S \in \mathcal{LCR}_\theta(\mathrm{X})$. Let $\{\mathfrak{m}_k\}$ be a θ-regular on S sequence of measures. Given $f \in L_1^{loc}(\{\mathfrak{m}_k\})$, we set

$$f^\sharp_{\{\mathfrak{m}_k\}}(x) := \sup_{k \in \mathbb{N}_0} \frac{1}{\epsilon^k} \mathcal{E}_{\mathfrak{m}_k}\Big(f, B_{\epsilon^k}(x)\Big) \quad x \in \mathrm{X},$$

where

$$\mathcal{E}_{\mathfrak{m}_k}\Big(f, B_{\epsilon^k}(x)\Big) := \begin{cases} \inf\limits_{c \in \mathbb{R}} \fint\limits_{2B_{\epsilon^k}(x)} |f(y) - c| \, d\mathfrak{m}_k(y), & B_{\epsilon^k}(x) \cap S \neq \emptyset; \\ 0, \quad B_{\epsilon^k}(x) \cap S = \emptyset. \end{cases}$$

If S is a regular set and $\mathfrak{m}_k = \mu$ for all $k \in \mathbb{N}_0$, then we get $f^\sharp_{S,\mu}$.

Calderón-style characterization

Theorem 1. *Let* $p \in (1, \infty)$ *and let* X *be a p-admissible space. Let* $\theta \in [0, p)$ *and* $S \in \mathcal{LCR}_{\theta}(\mathrm{X})$*. A function* $f \in W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0} \iff f \in L_p(S, \mathfrak{m}_0)$ *and* $f^{\sharp}_{\{\mathfrak{m}_k\}} \in L_p(\mathrm{X}, \mu)$*. Furthermore,*

$$\|f|W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}\| \approx \|f|L_p(S, \mathfrak{m}_0)\| + \|f^{\sharp}_{\{\mathfrak{m}_k\}}|L_p(\mathrm{X}, \mu)\|$$

and there exists a bounded linear extension operator $\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}$ *s.t.* $\mathrm{Tr}\,|_S^{\mathfrak{m}_0} \circ \mathrm{Ext}_{S,\{\mathfrak{m}_k\}} = \mathrm{Id}$ *on* $W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}$*. The constants of equivalence depend only on* θ, p, C^1, C^2, C^3*.*

Brudny-Shvartsman-style characterization

Theorem 2. *Let $p \in (1, \infty)$ and let X be a p-admissible space. Let $\theta \in [0, p)$ and $S \in \mathcal{LCR}_\theta(\mathrm{X})$. A function $f \in W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0} \iff f \in L_p(S, \mathfrak{m}_0)$ and for some $c \geqslant 10$*

$$\mathcal{BSN}_{p,\{\mathfrak{m}_k\},c}(f) := \sup\Big(\sum_{i=1}^{N} \frac{\mu(B_{r_i}(x_i))}{r_i^p}\Big(\mathcal{E}_{\mathfrak{m}_{k(r_i)}}(f, B_{(c+3)r_i}(x_i))\Big)^p\Big)^{\frac{1}{p}} < +\infty, \quad (2)$$

where the supremum is taken over all families $\{B_{r_i}(x_i)\}_{i=1}^N$ s.t.:

(F1) $B_{r_i}(x_i) \cap B_{r_j}(x_j)$ *if* $i \neq j$;

(F2) $\max\{r_i : i = 1, ..., N\} \leqslant 1$;

(F3) $B_{cr_i}(x_i) \cap S \neq \emptyset$ *for all* $i \in \{1, ..., N\}$.

Furthermore,

$$\|f|W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0}\| \approx \|f|L_p(S, \mathfrak{m}_0)\| + \mathcal{BSN}_{p,\{\mathfrak{m}_k\},c}(f).$$

Besov-style characterization

Theorem 3. *Let* $p \in (1, \infty)$ *and let* X *be a* p*-admissible space. Let* $\theta \in [0, p)$, $S \in \mathcal{LCR}_\theta(\mathsf{X})$ *and* $\{\mathfrak{m}_k\} \in \mathfrak{M}_\theta(\mathsf{X})$*. Then* $f \in W^1_p(\mathsf{X})|^{\mathfrak{m}_0}_S$ $\Longleftrightarrow$ $f \in L_p(S, \mathfrak{m}_0)$ *and*

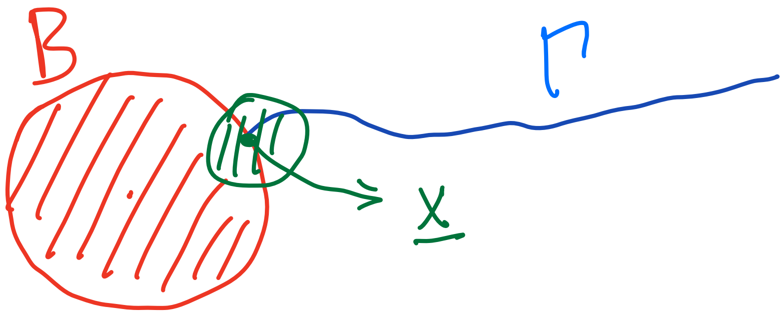
$$\mathcal{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f) := \Big(\sum_{k=1}^{\infty} \epsilon^{k(\theta-p)} \int\limits_{S_k(\sigma)} \Big(\mathcal{E}_{\mathfrak{m}_k}(f, B_k(x))\Big)^p \, d\mathfrak{m}_k(x)\Big)^{\frac{1}{p}} < +\infty. \qquad (3)$$

Furthermore,

$$\|f|W^1_p(\mathsf{X})|^{\mathfrak{m}_0}_S\| \approx \|f|L_p(S, \mathfrak{m}_0)\| + \mathcal{BN}_{p,\{\mathfrak{m}_k\},\sigma}(f).$$

Example

Let X be an Ahlfors-David Q-regular space,i.e., $\mu(B_r(Q)) \approx r^Q$ for some $Q > 1$.



One can take $\mathfrak{m}_k = 2^{k(Q-1)}\mu\lfloor_B + \mathcal{H}_{Q-1}\lfloor_\Gamma$, $k \in \mathbb{N}_0$.

Trace criterion. *Let* $p \in (\max\{1, Q-1\}, \infty)$. *Then*

$f \in W_p^1(\mathrm{X})|_S^{\mathfrak{m}_0} \iff$

1) $f \in W_p^1(B) \cap B_{p,p}^{1-\frac{Q-1}{p}}(\Gamma)$;

2) the gluing condition holds

$$\sum_{k=1}^{\infty} 2^{k(Q-p)} \Big(\fint_{B_k(\underline{x}) \cap B} \fint_{B_k(\underline{x}) \cap \Gamma} |f(y) - f(z)| \, d\mu(y) \, d\mathcal{H}_{Q-1}(z) \Big)^p < +\infty.$$

THANK YOU FOR YOUR ATTENTION!

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