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Attainability of infimum in the definition of the relative entropy of entanglement and its corollaries

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The relative entropy of entanglement (REoE)

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be separable Hilbert spaces, $\mathcal{H}_{AB} \doteq \mathcal{H}_A \otimes \mathcal{H}_B$.

$\mathfrak{T}(\mathcal{H}_X)$ – the Banach space of all trace-class operators on $\mathcal{H}_X$, $X = A, B, AB$, $\mathfrak{S}(\mathcal{H}_X) = \{\rho \in \mathfrak{T}(\mathcal{H}_X)\,|\,\rho \geq 0, \mathrm{Tr}\rho = 1\}$ – the set of quantum states of X, $\mathfrak{S}_{\mathrm{sep}}$ – the set of separable states – the convex closure of the set $\{\rho \otimes \sigma\,|\,\rho \in \mathfrak{S}(\mathcal{H}_A), \sigma \in \mathfrak{S}(\mathcal{H}_B)\}$.

$$E_R(\rho) = \inf_{\sigma \in \mathfrak{S}_{\mathrm{sep}}} D(\rho\,\|\,\sigma),$$

where

$D(\rho\|\sigma) = \sum_i \langle\varphi_i|\,\rho\ln\rho - \rho\ln\sigma\,|\varphi_i\rangle$ –– the quantum relative entropy

($\{\varphi_i\}$ is the basis of eigenvectors of ρ and it is assumed that $D(\rho\,\|\sigma) = +\infty$ if $\mathrm{supp}\rho$ is not contained in $\mathrm{supp}\sigma$)

REoE is a partial case of the relative entropy of resourse

If $\mathfrak{F}$ is a set of **free states** in $\mathfrak{S}(\mathcal{H})$ then

$$D_{\mathfrak{F}}(\rho) = \inf_{\sigma\in\mathfrak{F}} D(\rho\,\|\,\sigma)$$

Examples:

- Relative entropy of NPT-entanglement in bipartite systems;
- Relative entropy of π-entanglement in multipartite systems;
- Relative Entropy of non-gaussianity;
- Relative entropy of non-classicality;
- Relative entropy of Wigner non-positivity.

Attainability and lower semicontinuity

If $\mathfrak{F}$ is a compact subset of $\mathfrak{S}(\mathcal{H})$ then

- the for any $\rho\in\mathfrak{S}(\mathcal{H})$ there is $\sigma_\rho\in\mathfrak{F}$ s.t. $D_{\mathfrak{F}}(\rho)=D(\rho\,\|\,\sigma_\rho)$

- the function $D_{\mathfrak{F}}(\rho)$ is lower semicontinuous on $\mathfrak{S}(\mathcal{H})$.

These claims directly follow from the lower semicontinuity of the function $(\rho,\sigma)\mapsto D(\rho\,\|\,\sigma)$.

if $\rho_n\to\rho_0$ then the sequence $\{\sigma_{\rho_n}\}$ has a limit point $\sigma_*\in\mathfrak{F}$ and

$$\liminf_{n\to+\infty} D_{\mathfrak{F}}(\rho)=\liminf_{n\to+\infty} D(\rho_n\,\|\,\sigma_{\rho_n})\geq D(\rho_0\,\|\,\sigma_*)\geq D_{\mathfrak{F}}(\rho_0)$$

Problem: in all the interesting cases the set $\mathfrak{F}$ is closed but not compact (alas!)

Basic idea: 1) to find a weaker topology τ in which $\mathfrak{F}$ is relatively compact and to apply the above arguments to a τ -compact set $\widehat{\mathfrak{F}}$ containing $\mathfrak{F}$ by using appropriate extension $\widehat{D}(\cdot\,\|\,\cdot)$ of $D(\cdot\,\|\,\cdot)$;

2) to prove the lower semicontinuity of $(\rho,\sigma)\mapsto\widehat{D}(\rho\,\|\,\sigma)$ w.r.t. τ;

3) to prove that $\sigma_\rho\in\mathfrak{F}$ for any ρ in $\mathfrak{S}(\mathcal{H})$.

<u>Realization</u>

1) Let τ be the weak-* topology on $\mathfrak{T}(\mathcal{H})$ induced by the duality $\mathfrak{T}(\mathcal{H}) = [\mathfrak{C}(\mathcal{H})]^*$, where $\mathfrak{C}(\mathcal{H})$ is the Banach space of compact operators on $\mathcal{H}$ with the operator norm.

By the Banach–Alaoglu theorem any subset $\mathfrak{F}$ of $\mathfrak{S}(\mathcal{H})$ is τ-relatively compact!

2) Let $\widehat{\mathfrak{F}}$ be the intersection of the unit ball of $\mathfrak{T}(\mathcal{H})$ with the cone generated by $\mathfrak{F}$.

3) Let $\widehat{D}(\cdot\,\|\,\cdot)$ be Lindblad's extension of the q.r.e. defined as

$$\widehat{D}(\rho\|\sigma) = \sum_i \langle\varphi_i|\,\rho\ln\rho - \rho\ln\sigma + \sigma - \rho\,|\varphi_i\rangle, \qquad \rho,\sigma\in\mathfrak{T}_+(\mathcal{H}),$$

($\{\varphi_i\}$ is the basis of eigenvectors of ρ and it is assumed that $\widehat{D}(\rho\,\|\sigma) = +\infty$ if suppρ is not contained in suppσ)

To prove weak-* compactness of $\widehat{\mathfrak{F}}$ it suffices to prove that the cone $\mathrm{cone}(\mathfrak{F})$ generated by $\mathfrak{F}$ is weak-* closed.

Theorem 1 *Let $\mathfrak{F}$ be a norm closed and convex subset of $\mathfrak{S}(\mathcal{H})$.*

The set $\mathrm{cone}(\mathfrak{F})$ is weak closed if there is a sequence $\{M_n\}$ of compact operators strongly converging to the unit operator $I_{\mathcal{H}}$ such that*

$$M_n\,\mathfrak{F}\,M_n^* \subseteq \mathrm{cone}(\mathfrak{F}) \quad \forall n \in \mathbb{N} \tag{1}$$

Corollary 1 $\mathrm{cone}(\mathfrak{S}_{\mathrm{sep}}(\mathcal{H}_{AB}))$ *is a weak* closed subset of $\mathfrak{T}(\mathcal{H}_{AB})$.*

Let $M_n = P_n \otimes Q_n$, where $\{P_n\} \subset \mathfrak{B}(\mathcal{H}_A)$ and $\{Q_n\} \subset \mathfrak{B}(\mathcal{H}_B)$ are sequences of finite-rank projectors strongly converging to the unit operators I_A and I_B. Then (1) holds with $\mathfrak{F} = \mathfrak{S}_{\mathrm{sep}}(\mathcal{H}_{AB})$.

Theorem 2 *The REoE* E_R *in an infinite-dimensional bipartite system* AB *is a lower semicontinuous function on* $\mathfrak{S}(\mathcal{H}_{AB})$ *and for any state* ρ *there exists a separable state* σ_ρ *such that*

$$E_R(\rho) = D(\rho \,\|\, \sigma_\rho).$$

If ρ *is a faithful state then the state* σ_ρ *is unique.*

The same claims are valid for the relative entropy of resource $D_{\mathfrak{F}}$ provided that cone($\mathfrak{F}$) is a weak* closed subset of $\mathfrak{T}(\mathcal{H}_{AB})$.

$I(A\!:\!B)_\rho \doteq D(\rho\,\|\,\rho_A\otimes\rho_B) = H(\rho_A)+H(\rho_B)-H(\rho)$ – the quantum mutual information of $\rho\in\mathfrak{S}(\mathcal{H}_{AB})$ $\;(\rho_A = \mathrm{Tr}_B\rho,\ \rho_B = \mathrm{Tr}_A\rho)$.

Corollary 2 *Let $\{\rho_n\}\subset\mathfrak{S}(\mathcal{H}_{AB})$ be a sequence of states converging to a state $\rho_0\in\mathfrak{S}(\mathcal{H}_{AB})$. If*

$$\lim_{n\to+\infty} I(A\!:\!B)_{\rho_n} = I(A\!:\!B)_{\rho_0} < +\infty$$

then

$$\lim_{n\to+\infty} E_R(\rho_n) = E_R(\rho_0) < +\infty.$$

L.Lami, M.E.Shirokov "Continuity of the relative entropy of resource", arXiv:2308.00696

THANK YOU FOR YOUR ATTENTION!