A free boundary problem for a parabolic equation with power nonlinearities

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Introduction

The heat conduction equation is a non-linear equation when the temperature dependence of thermal parameters is taken into account. It is proved that the mathematical condition for reducing the one-dimensional nonlinear heat equation to a linear form is the constancy of the Storm condition for heat capacity and heat conduction. This discovery was the reason for studying the relationship between the thermal parameters of simple metals on the basis of solid state theory and available experimental data. The nonlinear heat equation for an isotropic solid has the form

$$\nabla(b(u)\nabla u) = a(u)u_t.$$

Here u is the temperature of the solid, $b(u)$ is the thermal conductivity, $a(u) = \rho C_\rho$, where ρ is the density, C_ρ is the specific heat at constant pressure. The two quantities $b(u)$ and $a(u)$ are called "thermal parameters". It is assumed that the metal has nonlinear thermal characteristics, so that the heat capacity $a(u)$ and thermal conductivity $b(u)$ satisfy the Storm condition [[1]],

[1] Storm, M.L. *Heat conduction in simple metals.*, //Nonlinear Analysis: Real World Applications. 10 J. Appl. Phys. 1951, 22, 940951.

$$\frac{d}{du}\frac{\sqrt{\frac{a(u)}{b(u)}}}{a(u)} = \lambda = const > 0. \quad (*)$$

Condition (*) was originally obtained in [1] in the study of thermal conductivity in simple monatomic metals. In this work, it was shown that if this condition is met, the heat equation can be transformed to a linear form. There the condition (*) is checked for aluminium, silver, sodium, cadmium, zinc, copper and lead. The usual assumption of constant thermal parameters is unacceptable for thermal conduction problems in devices such as jet engines and rockets, where large temperature ranges and high heating rates are encountered.Previous researchers, who took into account the change in thermal parameters, had more success in solving the stationary heat equation than in solving the problem of non-stationary heat conduction. In the latter case, the least restrictive solutions were obtained in problems with a small change in thermal parameters, which made it possible to obtain approximate solutions.

Statement of the problem. Preliminary results

It is required to find a pair of functions $(s(t), u(t,x))$ such that the function $s(t)$ is continuously differentiable on the interval $0 < t \leqslant T$, $s(0) = 0$, $s(t) > 0$ and the function $u(t,x)$ in $D_T = \{(t,x) : 0 < t \leqslant T, 0 < x < s(t)\}$ satisfies the equation

$$u_t = (u^{-2}u_x)_x - au^{-3}, \quad (t,x) \in D_T \tag{1}$$

is continuous in D_T together with the derivative $u_x(t,x)$ and satisfies the conditions

$$u_x(t,0) = 0, \quad 0 < t \leq T, \tag{2}$$

$$u_x(t,s(t)) = 0, \quad 0 < t \leq T, \tag{3}$$

$$u(t,s(t)) = g(s(t)), \quad 0 < t \leq T. \tag{4}$$

Here $g(x) > 0$ is defined and continuous in the interval $0 \leq x \leq x_0$, $0 < s(t) < x_0$.

The study is carried out according to the following scheme. First, with the help of some transformations (hodographs), the problem is reduced to a problem with a free boundary for a new function $v(t, y)$ in some non-standard domain for a parabolic equation with the lowest term , without an initial condition with a homogeneous boundary condition of the second kind. Some initial a priori estimates for $v(t, y)$ are established and the uniqueness theorem for the solution is proved. It can be seen that problem (1)-(4) is a Florin-type problem, which is characterized by a number of features: the free boundary begins at the solid wall $x = 0$, a free boundary condition is implicitly specified for this boundary; the behavior of the free boundary is unknown. Therefore, below we consider a problem with an initial condition, which reduces to a Stefan-type problem. Their equivalence is proved. To solve the Stefan-type problem, a priori Schauder-type estimates are established and, on their basis, an existence theorem is proved. At the same time, for the unknown boundary, two-sided estimates are established from known curves that determine the behavior of the unknown boundary at $t > 0$.

t the end of the article, it was proved that with an unlimited increase in time, the free boundary tends to some constant x_0.

Let us now reduce the problems to a problem for a linear equation. Let us introduce a new desired function $v(t,y)$, $y = y(t,x)$ as follows [4, 5]:

$u(t,x) = \frac{1}{v(t,y)}$, $\quad y_x(t,x) = \frac{1}{v(t,y)} = u(t,x)$,

$y_t(t,x) = u_x^2(t,x)/u^2(t,x)$, $y_{xt} = y_{tx}$. In this case, the boundary $x = 0$ passes into

$$y = y(t,0) = \int_0^t u_x(\eta,0) = 0,$$

and the free boundary $x = s(t)$ - on

$$y = y(t,s(t)) = \int_0^{s(t)} g(\eta)d\eta = h(t).$$

The domain D_T passes into the domain

$\Omega_T = \{(t,y) : 0 < t \leq T, 0 < y < h(t)\}$.

For the new function $v(t, y)$, we obtain the problem

$$v_t = v_{yy} + av_y(t, y) \quad in \quad \Omega_T = (t, y) : 0 < t \leq T, 0 < y < h(t), \tag{5}$$

$$v_y(t, 0) = 0, \tag{6}$$

$$v_y(t, h(t)) = 0, \quad 0 < t \leq T, \tag{7}$$

$$v(t, h(t)) = \frac{1}{g(s(t))} = b + \int_0^t q(h(\eta))d\eta, \quad 0 < t \leq T. \tag{8}$$

Let us establish some initial a priori estimates used in the proof of the uniqueness theorem.

Lemma 1.

Let $q(x) > 0$, $q_x(x) < 0$. Then the continuous function $v(t, y)$ in Ω_T satisfies the estimates

$$0 \leq v(t, y) \leq M_1, \quad 0 \leq v_y(t, y), \quad (t, y) \in \Omega_T.$$

Theorem 1.

Let the conditions of Lemma 1 be satisfied. Then the solution of problem (5)-(8) ((1)-(4)) is unique.

Existence of the Solution

To do this, we consider a problem with an initial condition, and this problem is reduced to a Stefan-type problem. Their equivalence is proved. To solve the Stefan-type problem, a priori Schauder-type estimates ar established and, on their basis, an existence theorem is proved. At the same time, for the unknown boundary, two-sided estimates are established using known curves that determine the behavior of the unknown boundary at $t \to 0$.

Theorem 2.

Suppose that the function $q(y)$ and its derivatives q_y and q_{yy} are continuous on the segment $0 < y < y_0$ and $q(y) > 0$, $q_y(y) < 0$. Then there is a $T > 0$ such that problem (5)-(8) admits a unique solution $(v, h) \in C^{(1+\alpha)/2, 1+\alpha}(\Omega_T) \times C^{1+\alpha/2}([0, T])$; moreover,

$$\| v \|_{C^{(1+\alpha)/2,(1+\alpha)}} (\Omega_t) + \| h \|_{C^{1+\alpha/2}} ([0, T]) \leq C$$

where $\Omega_t = \{(t, x) \in R^2 : x \in [0, h(t)], t \in [0, T]\}$.

Lemma 2.

Suppose that a couple of functions $(h_l(t), U^l(t,y))$ is a solution of problem (15)-(19). Then the couple $(h_l(t), v^l(t,y))$ where

$$v(t,y) = \varphi_l(0) + a\varphi_l(0)y - a\int\limits_0^y \varphi_l(\xi)d\xi + \int\limits_0^t U^l(\eta)d\eta + ay\int\limits_0^t U(\eta,0)d\eta -$$

$$- a\int\limits_0^y d\xi \int\limits_0^t U^l(\eta,\xi)d\eta + \int\limits_0^y d\xi \int\limits_0^\xi U(t,x) + \delta y \qquad (9)$$

is a solution of problem (10)-(14).

Lemma 3.

Let $q''(y) + aq'(y) \leq 0$ and there exists constants $k_1 > 0, m_1 > 0$ such that $k_1 y_0 \leq m_1$, $\delta(l) \leq kl^2$,

$$my_0 + \int_0^{y_0} q(\xi)d\xi + \delta l \geq kl^2.$$

If $h_l(t)$ is a solution of the equation

$$-\delta(l) + kl^2 = m_1 h_l^-(t) + \int_0^{h_l^-(t)} q(\xi)d\xi \tag{10}$$

then

$$h_l^-(t) \leq h_l(t), \quad 0 \leq t \leq T_l$$

Remark 1

The solvability of equation (10) for $h_l^-(t)$ is proved as follows: In the segment $0 \le y \le y_0$, we consider a function

$$B(t,y) = m_1 y + \int_0^y q(\xi)d\xi + \delta(l) - k_1 l^2$$

we check the property of continuous functions

$$B(t,0) = \delta(l) - k_1 l^2 < 0, \quad B(t,y_0) = m_1 y_0 + \int_0^{y_0} q(s)ds + \delta(l) - k_1 l^2 > 0$$

$$B_y'(t,y) = m_1 + q(y) > 0$$

Therefore, there is a root of the function $B(t,y)$ by y.

Lemma 4.

Let the inequalities hold for constants $m_2 > 0, k_2 > 0,$
$-q''(y) - aq'(y) - ak_2 \leq 0, k_2 l \leq k_2 y_0 \leq m_2$ and a function $h_2^+(t)$ is a solution of the equation

$$aq(0)(T+t) - \delta(l) = \int\limits_0^{h_l^+(t)} q(\xi)d\xi + (k_2 l - m_2)h_l^+(t). \qquad (11)$$

Then $h_l(t) \leq h_l^+(t), \quad 0 \leq t.$

Remark 2.

To prove the solvability of (11) with respect $h_l^+(t)$ we consider a function

$$R(t,y) = aq(0)(T+t) - \delta(l) - \int\limits_0^y q(\xi)d\xi - M_3 y, \quad M_3 = k_2 l - m_2, 0 \le y \le y_0.$$

$$R(t,0) = aq(0)(T+t) - \delta(l) > 0,$$

$$R(t,y_0) = aq(0)(T+t) - \delta(l) - \int\limits_0^{y_0} q(\xi)d\xi - M_3 y_0 < 0$$

$$R_y(t,y) = -q(y) - M_3 < 0$$

The inequalities are satisfied by the choice of M_3.

Remark 3.

In the limit as $l \to +0$, the function $h_l^+(t)$ monotonically converges to the function $h^+(t)$, which is a solution of the equation

$$m_2 h^+(t) + (t + T)q(0)a = \int_0^{h^+(t)} q(\xi) d\xi$$

Lemma 5.

The estimate $0 < h_l'(t) \leq N$ holds uniformly in $1 \leq l_0 \leq y_0$, $N = const > 0, 0 \leq t \leq T_l$.

Thus, it has been established that problems (1)-(4) and (15)-(19) are solvable on any time interval $[0, T]$.

Applying the estimates in the space $C^{1+\alpha}$ from [2], we obtain uniform (in $l \leq 1_0$) estimates for the Hölder norms for the functions $U^l(t, y)$: From condition (9) it follows that the derivatives $h_l(t)$ uniformly in $l \leq 1_0$ satisfy the Hölder condition on the interval $[0, T]$ and, therefore, on the basis of well-known Schauder-type estimates [2] for solutions of parabolic equations, we can assert that, uniformly in l, the Hölder norms $U^l_t(t, y), U^l_{yy}(t, y)$ are bounded on any set

$$\{(t, y) : 0 < \delta \leq t \leq T, \quad 0 < s \leq y \leq h_l(t)\}.$$

[2],

[2] Ladyzhenskaja O. A., Solonnikov V. A. and Uralceva N. N. *Linear and Quasi-linear Equations of Parabolic Type, American Mathematical Society: Translations of Mathematical Monographs*, //Nonlinear Analysis: Real World Applications. 10 Vol. 23, Providence, RI, 1968.

Taking into account the connection established in lemma 2 between the functions $v^l(t,y)$ and $U^l(t,y)$, as well as the estimates proved in Lemmas 3-5, and using the Arzela theorem, we obtain the existence of such a sequence $l = l_k \to +0$ as $k \to +\infty$ that:
1) the functions $h_{l_k}(t) \to h(t)$ uniformly on $[0, T]$ and $h(t) > 0, h'(t) > 0$ on $[0, T]$ near the point $t = 0$, the estimates $h^-(t) \leq h_{l_k}(t) \leq h^+(t)(h^-(0) = h^+(0) = 0$ are essentially used;
2) the function $U^{l_k}(t,y)$ in the domain D_τ converges to a solution $U(t,y)$ of (15) continuous in D_τ, having on any set $\{(t,y) : 0 < \delta \leq t \leq T, 0 < x < h(t)\}$ continuous derivatives $U_y(t,y)$, $U_t(t,y)$, $U_{yy}(t,y)$, and

$$U(t,h(t)) = q(h(t)), U_x(t,h(t)) = -q(h(t))\dot{h}_-(t)$$

3) at each point D_τ the function $v^{l_k}(t,y)$ converges to the solution of problem (5)-(8).

On the asymptotics of the free boundary

By virtue of the results obtained in Lemmas 2-5, the function $h_l(t)$ is strictly monotonic and, therefore, there exists $h_\infty \in [0, \infty)$ such that $\lim h_l(t) = h_\infty$.

Lemma 6.

The equality $\lim\limits_{t\to+\infty} h_l(t) = h_l(\infty) = y_0$ is true.

END!!!!