Effective Function Theory on Riemann Surfaces

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Constructive Methods for Riemann Surfaces and Applications
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Crash course on Riemann Surfaces

Surfaces: topology

(Bordered) surfaces are spaces with the same local structure as a (half-) plane.

Theorem

(Moebius). Two compact surfaces are topologicaly equivalent iff they have the same set of invariants:

- k – number of boundary components;
- $\mathcal{O}$ – orientability;
- $\chi = V - E + F$ – Euler characteristic.

Full collection of orientable $\mathcal{O} = +1$ surfaces: $\oplus g\mathbb{T} \ominus k\mathbb{D}$, $g, k = 0, 1, \ldots$. $\chi = 2 - 2g - k$. Nicknames: $(g, k) := (0, 0)$ 'Sphere', $(1, 0)$ 'Torus', $(3, 0)$ 'Brezel', $(0, 1)$ 'Disc', $(0, 2)$ 'Annulus', $(0, 3)$ 'Pants', $(1, 1)$ 'Handle' ...

Full collection of nonorientable $\mathcal{O} = -1$ surfaces: $\oplus h\mathbb{RP}^2 \ominus k\mathbb{D}$, $h - 1, k = 0, 1, \ldots$. $\chi = 2 - h - k$. Nicknames: $(h, k) := (1, 0)$ 'Projective Plane', $(2, 0)$ 'Kleinian bottle', $(1, 1)$ 'Moebius Strip', $(2, k)$ 'Kleinian saxophone'.

Kleinian saxophone

Surfaces: conformal (complex) structure

Several equivalent ways to determine the complex structure:

1. Patchwork with conformal (holomorphic) glueings
2. Conformal class of metrics
3. Field of operators J, $J^2 = -I$, in tangent space (emulation of rotation by $\pi/2$)

Keywords: Beltrami differential, Beltrami equation, Liouville equation, hyperbolic geometry

Riemann's computation: space of complex structures has FINITE real dimension $6g - 6 + 3k$.

Function theory on surfaces

Several objects which locally may be presented as

$\phi(z)$	meromorphic function
$\phi(z)dz$	meromorphic differential
$\phi(z)(dz)^2$	meromorphic quadratic differential
$\phi(z)\sqrt{dz}$	meromorphic spinor (half-differential)

Latter depends on 'spin-structure', the choice of signs in the gluing maps $\pm\sqrt{dz/dw}$. All above may be considered as (meromorphic) sections of certain line bundles over the surface.

Also: kernel functions like Bergman, Szegö, Schottky-Klein prime form $E(z,w)/\sqrt{dz \cdot dw}$..

Two major computational approaches for RS:
Riemann theta functions and Poincare seria.

Cycles, Period matrix

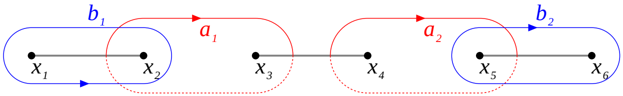


Figure: Two sheeted model of the genus 2 curve $y^2 = \prod_{s=1}^{6}(x - x_s)$ and symplectic homology basis

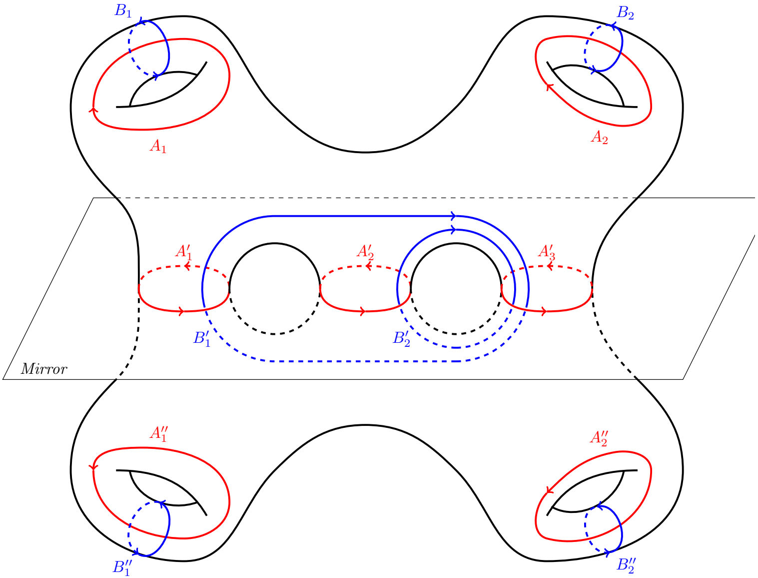
A reflection $\bar{J}(x, y) := (\bar{x}, \bar{y})$ acts on the curve and in particular, on just introduces cycles

$$\bar{J}a_s = a_s, \quad \bar{J}b_s = -b_s, \qquad s = 1, 2. \tag{1}$$

We consider the dual basis of holomorphic differentials, which are "real" and generate purely imaginary period matrix.

$$\int_{a_s} du_j := \delta_{sj}; \quad \Pi_{sj} := \int_{b_s} du_j; \qquad s, j = 1, 2,$$

Another surface with reflection: cycles



Jacobian and Abel-Jacobi mapping

Period matrix Π in turn generates a full rank lattice in $\mathbb{C}^g$:

$$L(\Pi) = \Pi\mathbb{Z}^g + \mathbb{Z}^g = \int_{H_1(X,\mathbb{Z})} d\boldsymbol{u}, \qquad d\boldsymbol{u} := (du_1, du_2, \ldots, du_g)^t,$$

its factor is a 2g-torus $Jac(X) := \mathbb{C}^g / L(\Pi)$ a.k.a. the Jacobian of the curve X.

The natural Abel-Jacobi embedding of the curve to its Jacobian

$$\boldsymbol{u}(p) := \int_{p_*}^{p} d\boldsymbol{u} \mod L(\Pi), \quad d\boldsymbol{u} := (du_1, du_2, \ldots, du_g)^t,$$

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Riemann thetas

Principal function theoretic objects can be expressed in terms of theta function of jacobian coordinates

$$\theta(\boldsymbol{u},\Pi) := \sum_{\boldsymbol{m}\in\mathbb{Z}^g} \exp(2\pi i \boldsymbol{m}^t \boldsymbol{u} + \pi i \boldsymbol{m}^t \Pi \boldsymbol{m}).$$

here $\boldsymbol{u} \in \mathbb{C}^g$ and $\Pi \in \mathbb{C}^{g\times g}$ is a Riemann matrix, i.e. $\Pi = \Pi^t$ and $Im\ \Pi > 0$.

In terms of this function it is possible to

- Localize the curve inside its Jacobian;
- Represent any meromorphic function ($=$ many- sheeted branched cover of the surface to the sphere);
- Compute all abelian integrals

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Riemann thetas on a surface

Many constructive formulas follow from the Riemann theorem about zeros of theta function

1 The AJ image of the curve X is shifted θ divisor (Riemann) $\theta[35](\boldsymbol{u}) = 0$ when $g = 2$

2 The 2:1 projection of X to the sphere $(x, y) \to x$ with normalization $x(p_s) := 0; x(p_l) := \infty$ takes the appearance

$$x(\boldsymbol{u}) := Const \frac{\theta^2[35sk](\boldsymbol{u})}{\theta^2[35lk](\boldsymbol{u})}, \qquad \boldsymbol{u}_* = \boldsymbol{u}(p_*), k \neq s, l.$$

3 Normalized 3rd kind integral (a building block for the whole function theory) is represented by an explicit formula

$$\eta_{pq}(\boldsymbol{u}) = \log \frac{\theta[..](\boldsymbol{u}(t) - \boldsymbol{u}(p))}{\theta[..](\boldsymbol{u}(t) - \boldsymbol{u}(q))}$$

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Computation of Riemann theta

Truncation principle for infinite series: sum up large elements first. Fix a small threshold ϵ for the absolute value of the theta series terms. Then you get a condition: the lattice $\mathbb{Z}^g$ elements lie inside an ellipsoid with semiaxes defined by $Im\ \Pi$ and ϵ. An apriori error estimate for this summation rule is available.
Variations of moduli: differentiate with respect to period matrix entries, which serve as the local coordinates in the moduli space (Rauch formulas).
Disadvantage: sophisticated moduli description, notorious Schottky problem.
Benefit: excellent convergence (even in the worst case!); excellent parallelizability
Degeneration of period matrix: use modular transformations.
Realization: e.g. MAPLE, Wolfram Math.

Schottky model of surface

Consider g pairs of disjoint discs D_s, D'_s in the complex plane. Fix Möbius transformations $S_j : Int\ D_j \to Ext\ D'_j$. Classical Schottky group $\mathfrak{S}$ is a free group with generators S_j.

$$\frac{Su - \alpha}{Su - \beta} = \mu \frac{u - \alpha}{u - \beta}$$

where α and β the attractive and the repulsive fixed point, $|\mu| < 1$ is a dilation coefficient. The exterior of all discs is the fundamental domain $\mathcal{R}$ of the group.

Schottky model of surface, 2

Mirror symmetry may be taken into account:

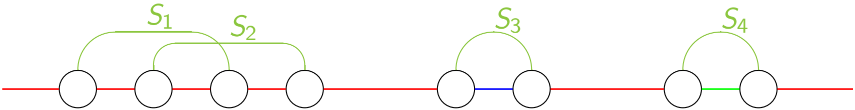


Figure: Schottky group $\mathfrak{S}$ with oriented factor of genus 1 and 3 ovals

Bicolored chord diagram with weights from $PGL_2(\mathbb{R})$: orientability $=$ generators are in $PSL_2(\mathbb{R})$; Euler characteristics $= 2 - 2rk\mathfrak{S}$, number of ovals $=$ combinatorics of the diagram.
Standard blocks for a handle and a hole

Function theory: Differentials

Abelian differential with two simple poles z, z' (H.Poincare, 1882).

$$d\eta_{zz'}(u) := \sum_{S\in\mathfrak{S}} \left\{ \frac{1}{Su - z} - \frac{1}{Su - z'} \right\} dS(u) =$$

$$\sum_{S\in\mathfrak{S}} \left\{ \frac{1}{u - Sz} - \frac{1}{u - Sz'} \right\} du. \qquad z, z' \in \mathcal{R}.$$

Holomorphic differential: place poles z, z' in the previous formula into the same orbit and consider the 'telescopic' sum:

$$d\zeta_j(u) := d\eta_{S_jz\ z} = \sum_{S\in\mathfrak{S}} \left\{ (u - SS_jz)^{-1} - (u - Sz)^{-1} \right\} du =$$

$$\cdots = \sum_{S\in\mathfrak{S}|\langle S_j\rangle} \left\{ (u - S\alpha_j)^{-1} - (u - S\beta_j)^{-1} \right\} du =$$

$$\sum_{S\in\langle S_j\rangle|\mathfrak{S}} \left\{ (Su - \alpha_j)^{-1} - (Su - \beta_j)^{-1} \right\} dS(u), \quad j = 1, \dots, g.$$

α_j and β_j are the attractive/ repulsive fixed points of $S_j(u)$.

Differentials II

Differential of the 2nd kind (with poles of higher orders) may be obtained by taking derivatives with respect to the location of the simple pole:

$$d\omega_{mz}(u) := D_z^m d\eta_{zy}(u) = m! \sum_{S \in \mathfrak{S}} (Su - z)^{-m-1} dS(u), \qquad m = 1, 2, ...$$

Thus obtained differentials will automatically have the normalization with respect to the boundary circles of the fundamental domain:

$$\int_{C_s} d\eta_{zz'} = 0; \quad \int_{C_s} d\omega_{mz} = 0; \quad \int_{C_s} d\zeta_j = 2\pi i \delta_{sj}; \qquad s, j = 1, \ldots, g.$$

provided the poles z, z' are contained in the fundamental domain.

Constructive function theory

The following two functions with their arguments belonging to the discontinuity domain of the Schottky group were introduced by F.Schottky in 1887. Later W.Burnside in 1892 related those functions to Poincare series.

$$(u, u'; z, z') := \exp \int_{u'}^{u} d\eta_{zz'} = \prod_{S \in \mathfrak{S}} \frac{u - Sz}{u - Sz'} : \frac{u' - Sz}{u' - Sz'}, \qquad (2)$$

$$E_j(u) := \exp \int_{\infty}^{u} d\zeta_j = \prod_{S \in \mathfrak{S} | \langle S_j \rangle} \frac{u - S\alpha_j}{u - S\beta_j}, \quad j = 1, \ldots, g.$$

Schottky functions are subjected to the obvious transformations under the permutations of their arguments and the substitutions from the group $\mathfrak{S}$.

Transformations of Schottky functions

$$(u, u'; z, z') = (z, z'; u, u') = 1/(u', u; z, z') = 1/(z', z; u, u')$$

$$(S_j u, u'; z, z') = (u, u'; z, z') E_j(z)/E_j(z'), \quad j = 1, \dots, g,$$
$$E_l(S_j u) = E_l(u) E_{lj}, \quad l, j = 1, \dots, g,$$

where the constants $E_{lj} := \exp(\int_u^{S_j u} d\zeta_l)$ (exp of the periods) admit the following representation

$$E_{lj} = E_{jl} = \prod_{S \in \langle S_l \rangle | \mathfrak{S} | \langle S_j \rangle} \frac{S\alpha_j - \alpha_l}{S\alpha_j - \beta_l} : \frac{S\beta_j - \alpha_l}{S\beta_j - \beta_l}, \quad l, j = 1, \dots, g.$$

Here the product is taken over the two-sided cosets of the group $\mathfrak{S}$ and the factor $0/\infty$ corresponding to $S = 1$ when $j = l$ is replaced by the dilatation coefficient $\lambda_l := S_l'(\alpha_l)$.

Representation of automorphic functions

Let $F(u) = F(S(u))$, $S \in \mathfrak{S}$. Then

$$dF/F = \sum_{k=1}^{deg\ F} d\eta_{z_k p_k} + \sum_{j=1}^{g} m_j d\zeta_j$$

where z_k, p_k represent zeros, poles in fundamental domain $\mathcal{R}_{\mathfrak{S}}$ and integer $m_j := \log(F)|_{C_j}/(2\pi i)$. Hence,

$$F(u) = const \prod_{k=1}^{deg\ F} (*, u; z_k, p_k) \prod_{j=1}^{g} E_j(u)^{m_j}$$

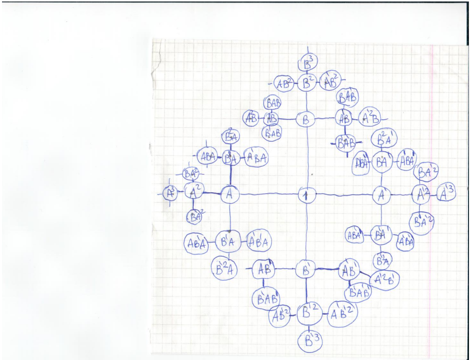
and its automorphicity is equivalent to Abel's criterion for the divisor (F).

Logistics of summation over Schottky group

Example. Computation of 3rd kind abelian differential:

$$d\eta_{zz'}(u) := \sum_{S\in\mathfrak{S}} \left\{ \frac{1}{u - Sz} - \frac{1}{u - Sz'} \right\} du.$$

Summation over infinite vocabulary. Keep the values Sz, Sz' in each node S of Caley graph to use them for the computation of the term in all immediate descendants $S_j^{\pm 1} S$.



Logistics of summation over Schottky group

Use the same principle: sum up large terms first. Cut off threshold is the value of $|Sz - Sz'|$. A posteriori estimate of the error is available. Lexicographic go-round of a tree.

Disadvantage: (sometimes) poor convergence;

Benefit: Comprehensive moduli description; excellent parallelizability

Variational formulae

Let $\mathfrak{S}$ be the Schottky group with generators $S_1, S_2, \ldots, S_g$ represented in the matrix form:

$$S_j(u) := \frac{c_{11}u + c_{12}}{c_{21}u + c_{22}} \longrightarrow \hat{S}_j := \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

Theorem (A.B. 1997) *The following variational formulae are valid for the definite abelian integrals:*

$$\delta \int_v^{v'} d\eta = (2\pi i)^{-1} \sum_{l=1}^{g} \int_{C_l} d\eta(u) d\eta_{vv'}(u) \mathrm{tr}[\mathcal{M}(u) \cdot \delta \hat{S}_l \cdot \hat{S}_l^{-1}]/du + o,$$

Here $d\eta(u)$ is any of the differentials $d\eta_{zz'}$, $d\zeta_s$, $d\omega_{mz}$; the endpoints $v, v' \in \mathcal{R}_{\mathfrak{S}}$; and $\mathcal{M}(u) := \left\| \begin{matrix} -u & u^2 \\ -1 & u \end{matrix} \right\|$ is the Hejhal matrix; $o := O(\sum_{sj} (\delta c_{sj})^2)$.

Similar variational formulae exist for the *periods of abelian differentials* and *Schottky-Klein prime factor*

Effective usage: Hejhal formulae

Example The following relative quadratic Poincare seria

$$\begin{aligned} \Theta_2^j(du)^2 &:= \sum_{S\in\langle S_j\rangle|\mathfrak{G}} R(Su)(dS(u))^2, \\ R(u) &:= (u-\alpha_j)^{-2}(u-\beta_j)^{-2}; \end{aligned} \qquad j=1,\dots,g, \tag{3}$$

represent g holomorphic quadratic differentials on the surface M. All integrals from right hand sides of the variational formulae may be computed explicitly for those seria:

$$\int_{C_s} \Theta_2^j(u)\mathcal{M}(u)du = \frac{i\pi}{2\alpha_j^2}\begin{pmatrix} 0 & \alpha_j \\ \alpha_j^{-1} & 0 \end{pmatrix}\delta_{js}. \tag{4}$$

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Thank you for the patience