

Three lectures on dilators & Π^1_2 Proof Theory

Logic Online Summer

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November 6th

December 4th

December 11th



Added in 2024:

These are the notes from the three lectures given at the logic online summer at the Steklov Institute (in spirit) and de facto in Vienna on December 6th, December 4th, and December 11th.

Topics:

- introduction to dilators
- proof-theoretic norms
- a taste of Π_2 -analysis.

Results are mostly either due to Girard (late 70s - early 80s) or joint by Beklemishev and the speaker.

Thanks to Leo Beklemishev for the invitation and to the participants for participating.

Let T be a theory extending ACA_0 and suppose T is recursively enumerable.

Define $|T|_{\Pi_1^1} = \sup \left\{ \alpha \in \text{Ord} : \begin{array}{l} \exists A^1 \text{ linear ordering of } \mathbb{N} \\ A \cong \alpha \text{ and } \underbrace{T \vdash A \in WO}_{\text{so}} \end{array} \right\} \stackrel{\text{recursive}}{=} \Sigma_1^1$

Fact: suppose T is Π_1^1 -sound. Then $|T|_{\Pi_1^1} < \omega_1^{ck}$.

Theorem (Spector) Suppose $S \subseteq WO$ is Σ_1^1 . Then there is $\alpha < \omega_1^{ck}$ such that $|A| < \alpha$ for all $A \in S$.

Theorem (Kleene) Suppose R is Π_1^1 . Then there is a recursive $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall \alpha \in R \leftrightarrow f(\alpha)$ is the index of a recursive wellorder.

Question. Let T be given explicitly. What is $|T|_{\Pi_1^1}$?

Example. (Gentzen) $|ACA_0|_{\Pi_1^1} = \epsilon_0 = \sup \{ \omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \dots \}$.

Short-term goal: what is a good definition for $|T|_{\Pi_1^1}$.

Decree: from now on, T is a recursively enumerable extension of ACA_0 .

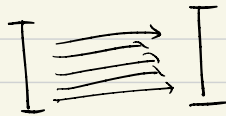
$|T|_{\Pi_1^1} = \sup \left\{ \alpha : \underbrace{\exists A \cong \alpha \quad T \vdash A \in WO}_{\text{wellordered if } T \text{ is } \Pi_1^1\text{-sound}} \right\}$ taken along the index of the ord.

Fact: $ACA_0 \vdash |T|_{\Pi_1^1}$ is wellordered iff T proves no false Π_1^1 sentences.

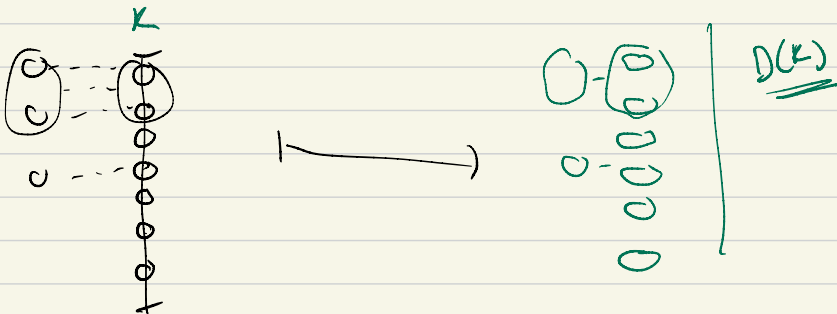
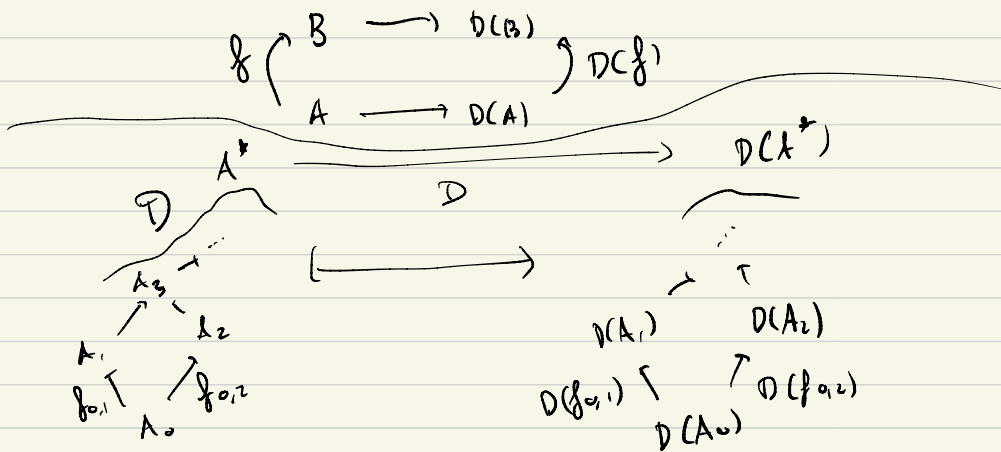
Abunden ordinals.

Dilator (Girard late 1970s)

wo category: objects are ordinals
morphisms are order-preserving embeddings



Dilators are particular types of functors on wo.



A dilator is countable if $D(n)$ is countable for every $n \in \mathbb{N}$. Then the set

$$\{D(n) : n \in \mathbb{N}\} \cup \{D(f) : f: \mathbb{N} \rightarrow \mathbb{N}\}$$

is countable and codes D .

Suppose we are given data as in $(*)$.
Does it come from a dilator?

$$\{L_n : n \in \mathbb{N}\} \cup \{F_{n,m} : n \leq m \in \mathbb{N}\} \} \Leftarrow$$

The data can be uniquely extended to a **pre-dilator**, a functor $\omega_0 \rightarrow \omega_0$ "like a dilator" but might have illfounded ordinals in its range.

Theorem (Girard) The collection of sets of natural numbers which code dilators is Π_2^1 -complete.

Why is it Π_2^1 ? $D \in \Pi_1^1$ if D is a pre-dilator and $\forall A (\lambda e \omega_0 \rightarrow D(A) e \omega_0) \} \Pi_2^1$.

Fundamental theorem of dilators (Girard)

Every dilator is isomorphic to a denotational system on ordinals.

CNF. $\omega^{\alpha_0} \cdot n_0 + \omega^{\alpha_1} \cdot n_1 + \dots + \omega^{\alpha_k} \cdot n_k$

parameters $\alpha_0 > \alpha_1 > \dots > \alpha_k$ (INPUT)

formal data $\omega \cdot n_0 \quad \omega \cdot n_1 \quad \omega \cdot n_k$ (GIVEN)

Fix set of parameters $\{\beta : \alpha \geq \beta\} \leftarrow$

Example $\alpha = \omega \cdot 2$

$$\underbrace{\underbrace{\omega}_{\varepsilon_2} \cdot 2}_{\varepsilon_3} + \underbrace{\underbrace{7}_{\varepsilon_1} \cdot 3}_{\varepsilon_1} = \underbrace{\underbrace{2}_{\varepsilon_0} \cdot 1}_{\varepsilon_0}$$

} names an ordinal

$$\underbrace{\underbrace{\omega+1}_{\varepsilon_3} \cdot 11}_{\varepsilon_3} + \underbrace{\underbrace{2}_{\varepsilon_0} \cdot 8}_{\varepsilon_0}$$

} names an ordinal

$$\varepsilon_3 > \varepsilon_2 > \varepsilon_1 > \varepsilon_0$$

$$\omega+1 > \omega > 7 > 2$$

(pre-)

An ordinal denotable system consists of:

terms $t(x_0, \dots, x_{n-1})$

comparison rules, determining for given terms

$t(x_0, \dots, x_{n-1})$, $s(y_0, \dots, y_{n-1})$

and parameters $\alpha_0, \dots, \alpha_{n-1}$; $\beta_0, \dots, \beta_{n-1}$,

which of $t(\alpha_0, \dots, \alpha_{n-1})$, $s(\beta_0, \dots, \beta_{n-1})$

is bigger, and this depends only on t, s ,
and the relative ordering of the parameters.

This means that if

$(\alpha_0, \dots, \alpha_{n-1}; \beta_0, \dots, \beta_{n-1}) \cong (\alpha'_0, \dots, \alpha'_{n-1}; \beta'_0, \dots, \beta'_{n-1})$

then $t(\vec{\alpha}) < s(\vec{\beta}) \Leftrightarrow t(\vec{\alpha}') < s(\vec{\beta}')$

This is a functor: given $\alpha \in \text{WO}$,

$D(\alpha) = \{ t(\vec{\alpha}) : t \text{ is a term, } \alpha_i < \alpha \}$

$\neq \emptyset \quad f: \alpha \rightarrow \beta$

$D(f): D(\alpha) \rightarrow D(\beta)$

$t(\vec{\alpha}) \mapsto t(f(\alpha_0), \dots, f(\alpha_{n-1}))$

In fact, a pre-dilator.

Other examples:

$$d \mapsto \omega^d$$

CNF

exp

$$d \mapsto \varepsilon_d$$

sexp

$$d \mapsto \varphi_d 0$$

VEB

$\mathcal{L}$ (AFRW)

$$d \mapsto \psi(d + \varepsilon_{d+1})$$

BH

$B_{\text{IACA} \circ \text{I} \Pi_2}$ (APW)

Joint
with
Fedor
Pakhomov

$$|\text{IACA} \circ \text{I} \Pi_2| = \sum_{n \in \mathbb{N}} \exp^n \circ \text{Add}(1)$$

$$|\text{IACA}^+ \circ \text{I} \Pi_2| = \sum_{n \in \mathbb{N}} \text{sexp}^n \circ \text{Add}(1)$$

$$|\text{ATRO} \circ \text{I} \Pi_2| = \sum_{n \in \mathbb{N}} \mathcal{L}^n \circ \text{Add}(1)$$

$|\text{I} \Pi_2| = \text{least } D \text{ such that } F \leq D \text{ whenever } T \vdash F \text{eDIL. } \text{DIL} = \text{set of (codes of) recursive dilators}$

$\forall \lambda$
 $F(\lambda) \leq \lambda$

An embedding $F \hookrightarrow D$ consists of an assignment of D -terms to F -terms of the same arity

$$t^F(x_0, \dots, x_n) \mapsto s^D(x_0, \dots, x_n)$$

in such a way that the comparison rules between t^F, s^F are the same as those between $e(\vec{t}), d(\vec{s})$

$$\left(\begin{array}{c} \bigcirc \\ \bigcirc \end{array} \right) \xrightarrow{e} \left(\begin{array}{c} \bigcirc \quad \bigcirc \\ \bigcirc \quad \bigcirc \end{array} \right)$$

$$\begin{aligned} (d \mapsto 2^d) &\hookrightarrow (d \mapsto \omega^d) \\ (d \mapsto d+1) &\hookrightarrow (d \mapsto d+2) \end{aligned}$$

$(T) \upharpoonright_{n_2} = D$ iff $\forall F (T \vdash F \in D \Rightarrow F \in D)$
 and $\forall D^*$ iff $\forall F (T \vdash F \in D \Rightarrow F \in D^*)$
 then $D \in D^*$

Theorem (Kechris - Woodin; Girard - Normann)

Suppose $S \in D_K$ is Σ_1^1 . Then there is D_0 ^{recursive}
 such that $F \in D_0$ for all $F \in S$

Theorem (with Pakhomov) Suppose T is D_2^1 -sured.
 Then $(T) \upharpoonright_{n_2}$ exists and is recursive.

Proof. Let $D_0, D_1, \dots$, list all dilators $\tilde{D}_i$ such that T
 proves " $\tilde{D}_i \in D_K$ ". Let $D = \sum_{i \in \mathbb{N}} D_i$

$$\sum_{i \in \mathbb{N}} D_i : A \mapsto \sum_{i \in \mathbb{N}} D_i(A)$$

$$f : A \rightarrow B$$

$$\left(\sum_{i \in \mathbb{N}} D_i \right) (f) : \left(\sum_{i \in \mathbb{N}} D_i \right) (A) \rightarrow \left(\sum_{i \in \mathbb{N}} D_i \right) (B)$$

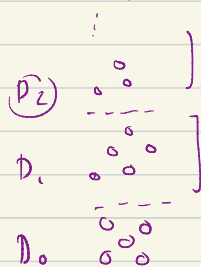
$$\sum_{i < j^*} D_i(A) + x \mapsto \sum_{i < j^*} D_i(B) + D_{j^*}(f)(x) \quad (x \in D_{j^*}(A))$$

$$| \text{---} | \text{---} | \text{---} \dots | \text{---} |$$

$$D_0(A) \quad D_1(A) \quad \dots \quad D_{i+1}(A)$$

$$D \in \Sigma_1^0$$

$$\text{trans}(D) = \bigcup_i \text{trans}(D_i)$$



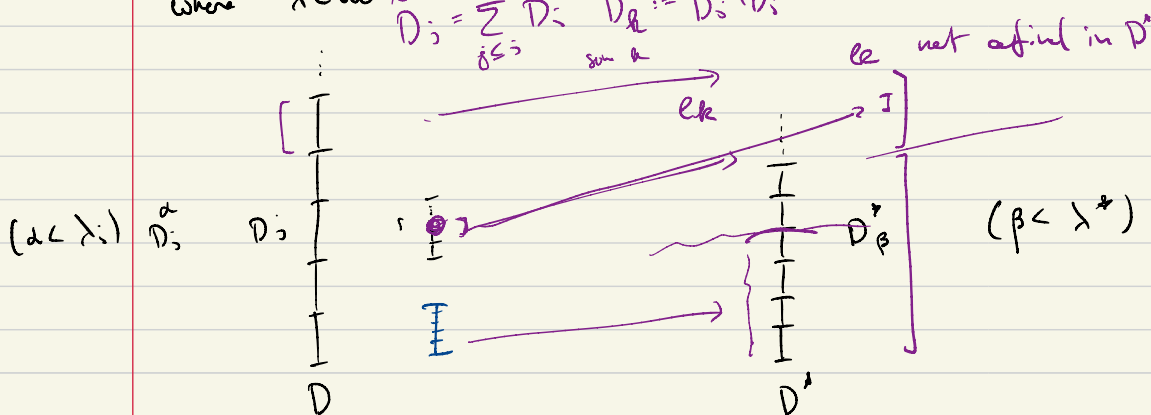
① Clearly if $T \vdash F \in \mathcal{K}$, then $F \hookrightarrow D$

(2) Suppose D^* is such that $F \subset D^*$ whenever $T \vdash F \in K$. Show $D \subset D^*$.

A dilator F is indecomposable if $F = F' + F''$ implies $F \cong F'$ or $F \cong F''$

Decomposition theorem (Givens) For every dilator F there is a uniquely determined family of nonzero indecomposable dilators such that $F = \sum_{i \in I} F_i$, where $\lambda \in \mathbb{Q}$.

$$\tilde{D}_i = \sum_{j \leq i} D_j \quad D_k := \hat{D}_i + \hat{D}_i$$



B₇ hypothesis: $e_i: D_i \hookrightarrow D'$, so $e_i = \sum_{\alpha \in \lambda_i} e_i^\alpha$

Build embedding $f: D \rightarrow D'$ by induction:

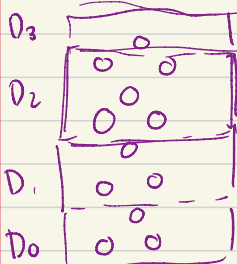
for each $i \in \mathbb{N}$, $\alpha < \beta$, choose $f_{i\beta}^\alpha: D_i^\alpha \rightarrow D_i^\beta$ ←
 where β is least above its range of $\bigcup_{i \leq \beta} f_i^\alpha$]

Thus $f = \sum f_i$ is as desired.

This D is r.e. With some extra work

D can be made recursive. $-$

decomposition theorem



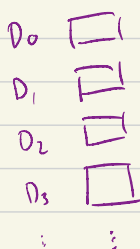
$t \leq s$ if $t(\vec{x}) < s(\vec{y})$ for some arguments

$$D_2 \cong D' + D'' \rightarrow D_2 \cong D' \text{ or } D_2 \cong D''$$

indek set is well ordered

$$D/\leq$$

$$D = \sum_{i \in \omega} D_i$$



$$D(\omega) \neq \omega \cdot \omega$$

$$t_0(\vec{x})$$

$$t_1(\vec{x})$$

$$t_2(\vec{x})$$

$|T|_{n_2}$ is defined up to bi-embeddability:

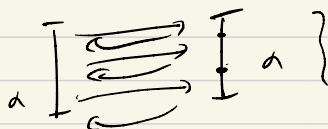
if $D = |T|_{n_2}$ & $F = |T|_{n_2}$, then

$D \cong F$. However we may have $D \not\cong F$.

$$\begin{array}{ccc} \beta & D(\beta) & \xrightarrow{e_\beta} F(\beta) \\ \uparrow \beta & \uparrow D(\beta) & \sim \uparrow F(\beta) \\ \alpha & D(\alpha) & \xrightarrow{e_\alpha} F(\alpha) \end{array}$$

implies

$$\forall \alpha \quad D(\alpha) \cong F(\alpha)$$



Corollary: $|T|_{\Pi_2^1} = \sum_{j \in \mathbb{N}} D_j$, where D_j ranges over the T -provable dilemmas.

This definition of $|T|_{\Pi_2^1}$ makes sense even if T is not Π_2^1 -sound.

$|T|_{\Pi_1^1}$ also makes sense, but is not interesting for Π_1^1 -unsound T because $|T|_{\Pi_1^1} = \omega_1^{ck}$.

However there are things to say about $|T|_{\Pi_2^1}$.

$$|T|_{\Pi_2^1} = \sum \underbrace{D_j}_{\substack{\vdots \\ \bigcirc \\ \bigcirc \\ \bigcirc}} (k) \longrightarrow \underbrace{\quad}_{\quad}$$

Definition. $\alpha_2^1(T) =$ least α such that $|T|_{\Pi_2^1}(\alpha)$ is illfounded.

• T is Π_2^1 -sound $\Rightarrow \alpha_2^1(T) = \infty$

Question: What are the possible values of $\alpha_2^1(T)$?

The Π_2^1 -Spectrum Conjecture: The following are equivalent:

- i) $\alpha = \alpha_2^1(T)$ for some r.e. extension T of ACA_0
- ii) $\alpha = \omega^d$ & α is Σ_1^1 -definable.

When α is Σ_1^1 -definable means $\exists \varphi \in \Sigma_1^1$ such that $\alpha \models \varphi$ & $\forall \bar{\alpha} < \alpha$ $\bar{\alpha} \not\models \varphi$.

Theorem (with Pakhomov) Assume ATR_0 . Suppose that the Π_2^1 -Spectrum Conjecture fails. Then, Second-Order Arithmetic is consistent.

In particular, the Π_2^1 -SC is consistent with Z_2 .

Definition: the Pakhomov-Walsh norm of T is

$|T|_{pw} : \omega_1 \rightarrow \omega_1$ given by $|T|_{pw}(\alpha) = |T + \omega_1(\alpha)|_{n_1^1}$.

Theorem (w. Pakhomov). $|T|_{pw} = |T|_{n_2^1}$, i.e.

$$|T|_{n_2^1}(\alpha) = |T + \omega_1(\alpha)|_{n_1^1}.$$

Proof. Fix $\alpha < \omega_1^{ck}$. Suppose $\gamma < |T|_{n_2^1}(\alpha)$.

$$\begin{aligned} \text{We saw } |T|_{n_2^1}(\alpha) &= \left(\sum \{ \beta : T \vdash \text{wcl}(\beta) \} \right) \alpha \\ &= \sum \{ \beta(\alpha) : T \vdash \text{wcl}(\beta) \}. \end{aligned}$$

Thus, $\gamma < \sum_{j < j^*} D_j(\alpha)$ for some $j^* \in \mathbb{N}$.

$$T \vdash \bigwedge_{j < j^*} D_j \in \text{DIL}, \text{ so}$$

$$T \vdash \sum_{j < j^*} D_j \in \text{DIL}, \text{ so}$$

$$T + w(\alpha) \vdash \sum_{j < j^*} D_j(\alpha) \in w(\alpha), \text{ and } \vdash$$

$$\gamma < \sum_{j < j^*} w_j(\alpha) \leq |T + w(\alpha)|_{n_1},$$

$$\text{hence } |T|_{n_2}(\alpha) \leq |T + w(\alpha)|_{n_1}.$$

Suppose now that $\gamma < |T + w(\alpha)|_{n_1}$, so
 $T \vdash w(\alpha) \rightarrow w(\gamma)$.

Lemma. There is a ^{recursive} dilator $D_{\alpha \rightarrow \gamma}$ so that
 provably in ACA_0 , we have

- i) $D_{\alpha \rightarrow \gamma} \in \text{DIL} \Leftrightarrow w(\alpha) \rightarrow w(\gamma)$ holds.
- ii) $\gamma \leq D_{\alpha \rightarrow \gamma}(\alpha)$.

By the lemma, $T \vdash D_{\alpha \rightarrow \gamma} \in \text{DIL}$ and $D_{\alpha \rightarrow \gamma}(\alpha) \geq \gamma$.
 Therefore, $|T|_{n_2}(\alpha) \geq \underline{D_{\alpha \rightarrow \gamma}(\alpha)} \geq \gamma$, so $|T|_{n_2}(\alpha) \geq |T + w(\alpha)|_{n_1}$.

Lemma. There is a ^{recursive} dilator $D_{\alpha \rightarrow \gamma}$ so that
 provably in ACA_0 , we have

- (i) $D_{\alpha \rightarrow \gamma} \in \text{ON} \Leftrightarrow \text{wo}(\alpha) \rightarrow \text{wo}(\gamma)$ holds. $\leftarrow$
- (ii) $\gamma \leq D_{\alpha \rightarrow \gamma}(\alpha)$ (if $\gamma \in \text{wo}$)

Proof of the lemma. Given $x \in \text{WO}$, we need to
 define $D_{\alpha \rightarrow \gamma}(x)$. First, construct a tree T_x .

T_x is the tree of simultaneous attempts to
 construct i) an i.d.s. through γ
 ii) an embedding $e: \alpha \rightarrow x$

If b is a branch through T_x , then b computes
 an i.d.s. through γ and an embedding $e: \alpha \rightarrow x$
 witnessing $\text{wo}(\alpha) \wedge \neg \text{wo}(\gamma)$. Thus

$(\text{wo}(\alpha) \wedge \neg \text{wo}(\gamma)) \rightarrow \text{wf}(T_x)$ for all $x \in \text{WO}$.

Conversely, if $\text{wo}(\alpha) \wedge \neg \text{wo}(\gamma)$. Then, for $x \geq \alpha$
 there is an embedding $e: \alpha \rightarrow x$ and an i.d.s.
 through γ , so T_x has a branch (for this x).

$(f|s, f|s, g|s, (a, \gamma))$
 $a = s^{\text{th}}$ element
 of α in
 its IN-enumeration

If we take

$D_{\alpha \rightarrow \gamma}(x) := \text{KB}(T_x)$, then

$D_{\alpha \rightarrow \gamma}$ is a dilator, and it
 satisfies (i). For (ii), consider T_α .

T_α is wellfounded. Let $e = \text{id}: \alpha \rightarrow \alpha$,
 let $T^* = T_\alpha \cap \{(f|s, \text{id}(s)) : f|s \text{ satisfies (i)}\}$

$\text{KB}(T_\alpha) \geq \text{KB}(T^*) \geq \text{mb}(T^*) \geq \gamma$

(*) $f: s \rightarrow \gamma$
 descending
 $g: \text{ONS} \rightarrow x$
 order-preserving

In particular, $|T|_{n_1^2}(0) = |T|_{n_1}$.

Next topic: ideas behind the proof of Theorem (v. Palbomay). $|ACA_0|_{n_1^2} = \sum_{i \in \mathbb{N}} \exp^i \circ \text{Add}(i)$,

where $\text{Add}(i): A \mapsto A+i$

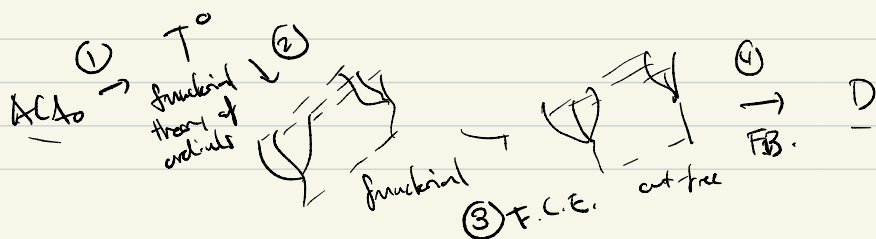
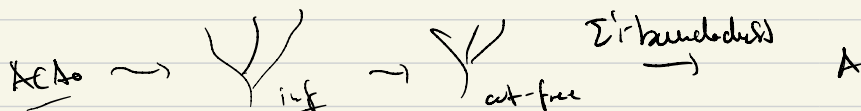
$\exp: A \rightarrow \omega^A$

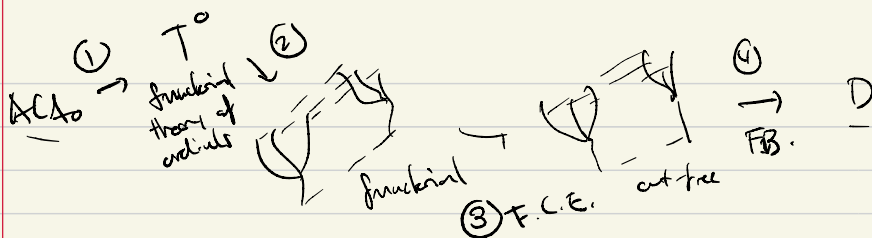
for $f: A \rightarrow B$, $\exp(f): \omega^A \rightarrow \omega^B$
 $\exp(f)(\omega^{a_0} \cdot m_0 + \dots + \omega^{a_m} \cdot m_m) = \omega^{f(a_0)} \cdot m_0 + \dots + \omega^{f(a_m)} \cdot m_m$

Example. $|ACA_0|_{n_1^2}(0) = \sum_{i \in \mathbb{N}} \exp^i(1) = 1 + \omega + \omega^\omega + \omega^{\omega^\omega} + \dots = \epsilon_0$

$|ACA_0|_{n_1^2}(\epsilon_0) = \sum_{i \in \mathbb{N}} \exp^i(\epsilon_0 + 1) = \epsilon_0 + 1 + \omega^{\epsilon_0 + 1} + \omega^{\omega^{\epsilon_0 + 1}} + \dots = \epsilon_1$

Recall: for $|ACA_0|_{n_1}$





① T^0 fractal theory of wellorders.

Intended models of T^0 :

$(V, U, <, \in)$

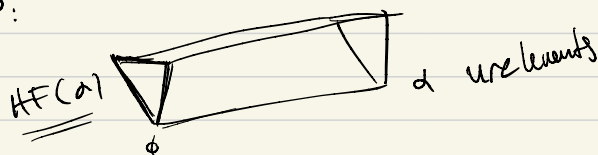
axioms: $U(x) \rightarrow y \notin x$

adjunction: if x, y are things, $x \cup \{y\}$ exists.

induction: schema of $\in$ -induction

induction for $<$ (" $<$ is a total order" on U)

language: $(U, \in, \cup, X)$



Theorem. Suppose $ACA_0 \vdash D(U)$. Then $T^0 \vdash WO(D(U))$,
 i.e., $T^0 \vdash X \subset D(U) \wedge \forall x \in D(U) (\forall y \subset D(U) x \not\supset y \rightarrow x \in X) \rightarrow X = \emptyset \vee X = D(U)$

② ranked β -proofs: function $P: WO \rightarrow \text{inf-proof}$.

$P(A)$ is an HFCA-proof

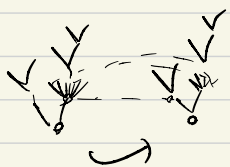
$Kx \vdash \varphi, \pi$ φ true sentence about elements of $HF(A)$ } HFCA-proof;
 usual rules $\vee, \wedge, \rightarrow, \exists$ } HFCA-proof if
 infinitary HFCA-rules: } wellfounded
 $\vdash \pi, \varphi(a)$ for all $a \in HF(A)$
 $\vdash \pi, \forall x \varphi(x)$

P is a β -pre-proof if $P(A)$ is an $HFF(A)$ -pre-proof for all A and P is functorial:

each $f: A \rightarrow B$ induces a mapping

$$\boxed{P(f)}: P(A) \rightarrow P(B)$$

$$\begin{array}{ccc} HFF(A) & \xrightarrow{HFF(f)} & HFF(B) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$



preserve structure
reverse constants
according to the
lift $HFF(f): HFF(A) \rightarrow HFF(B)$

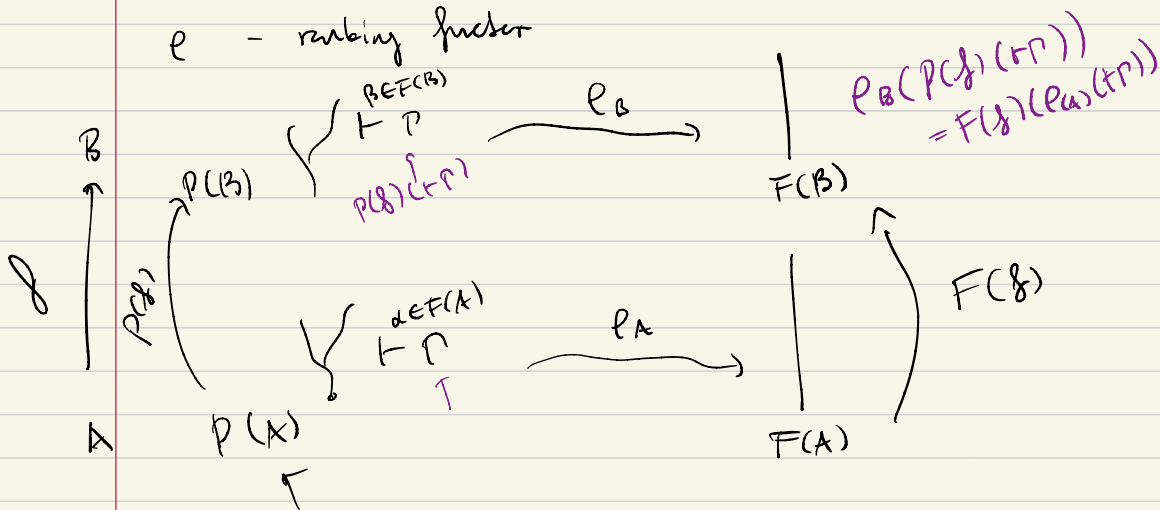
P is a β -proof if $P(A) \in WFF$ for all A .

Ranked β -proof (P, F, e)

P - β -proof

F - rule dilator

e - ranking functor



Theorem. If $T^0 \vdash \varphi$, then $\vdash_n^{\exp^2 + c} \varphi(x)$

(3) Theorem. Suppose $\vdash_{n+1}^F \Gamma$. Then $\vdash_n^{\exp^0 F} \Gamma$

(4) Theorem. Suppose $\vdash_0^F \text{WO}_X(D(A))$.
 Then there is a natural transformation
 $T: D \rightarrow \exp^2 \circ F$.

(5) Conclusion. Suppose $\text{ACA}_0 \vdash \text{DCL}(D)$.

B₁ (1) $T^0 \vdash D(u) \in \text{WO}$

B₁ (2) $\vdash_{n+4}^{\exp^2 + c} D(A) \in \text{WO}$ } formalized appropriately

B₁ (3) $\vdash_0^{\exp^{n+4} + 1} D(A) \in \text{WO}$

B₁ (4) $D \hookrightarrow \exp^{n+7} + 1$

Therefore: $\|\text{ACA}_0\|_{\Pi_2^1} = \sum_{n \in \mathbb{N}} \exp^n \circ \text{Add}(1)$ }

Corollary: $\|\text{ACA}_0\|_{\text{pw}} = \omega \mapsto \varepsilon^+(\omega)$.

