Birational geometry and Fano varieties

Conference dedicated to Yu. G. Prokhorov 60^{th} birthday

MIAN, March 12-15, 2024

Towards birational classification of 3-folds determined by empty lattice 4-simplices

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Lattice d-polytopes

Consider lattice $\Lambda \cong \mathbb{Z}^d$. Let $\Lambda_{\mathbb{R}} := \Lambda \otimes \mathbb{R} \cong \mathbb{R}^d$.
$\Delta \subset \Lambda_{\mathbb{R}}$ is called lattice d-polytope if Δ is convex hull of finitely many Λ-lattice points whose affine span equals $\Lambda_{\mathbb{R}}$.

Two lattice d-polytopes $\Delta_1, \Delta_2 \subset \Lambda_{\mathbb{R}}$ are called unimodular equivalent if there exists an affine linear transformation $\tau : \Lambda_{\mathbb{R}} \to \Lambda_{\mathbb{R}}$ such that $\tau(\Lambda) = \Lambda$ and $\tau(\Delta_1) = \Delta_2$.

$\Delta \subset \Lambda_{\mathbb{R}}$ is lattice d-simplex if there exist affine independent $v_0, \ldots, v_d \in \Lambda$ such that

$$\Delta = \mathrm{Conv}\{v_0, \ldots, v_d\}.$$

h^*-polynomial of Δ

The coefficients of the power series $\Psi_\Delta(t) := \sum_{k\geq 0} |k\Delta \cap \Lambda| t^k$ are invariants of Δ under the unimodular equivalence and

$$\Psi_\Delta(t) = \frac{h^*_\Delta(t)}{(1-t)^{d+1}} = \frac{\sum_{i=0}^d h^*_i(\Delta) t^i}{(1-t)^{d+1}},$$

where $h^*_\Delta(t)$ is called h^*-polynomial of Δ. All coefficients h^*_i are nonnegative intergers (unimodular invariants of Δ) and

$$h^*_0(\Delta) = 1,\ h^*_1(\Delta) = |\Lambda \cap \Delta| - d - 1,\ h^*_d(\Delta) = |\mathrm{Int}(\Delta) \cap \Lambda|.$$

The positive integer $h^*_\Delta(1) = \sum_{i=0}^d h^*_i(\Delta)$ is called lattice normalized volume of Δ and denoted by $\mathrm{Vol}(\Delta)$. This number equals $d! \times$(Euclidean volume of Δ).

Empty lattice d-simplices

Definition. Δ is called empty lattice d-simplex if $h_1^*(\Delta) = 0$, i.e., if all Λ-lattice points in Δ are only its vertices.

If $\Delta = \mathrm{Conv}\{v_0, v_1, \dots, v_d\}$ is empty lattice d-simplex, then

- $h_d^*(\Delta) = 0$;
- every k-dimensional face of Δ is an empty lattice k-simplex;
- $\mathrm{Vol}(\Delta) = \det \begin{pmatrix} v_0 & v_1 & \cdots & v_d \\ 1 & 1 & \cdots & 1 \end{pmatrix}$; (this holds true for any d-simplex)
- if $d \in \{1, 2\}$ then $h_\Delta^*(t) = 1 = \mathrm{Vol}(\Delta)$, i.e., Δ is Λ-basic;
- if $d = 3$, then $h_\Delta^*(t) = 1 + (\mathrm{Vol}(\Delta) - 1)t^2$;
- if $d = 4$, then $h_\Delta^*(t) = 1 + h_2^*(\Delta)t^2 + h_3^*(\Delta)t^3$, where $1 + h_2^*(\Delta) + h_3^*(\Delta) = \mathrm{Vol}(\Delta)$.

Unimodular classification of empty lattice 3-simplices

Theorem White (1964). Every empty lattice tetrahedron Δ with $\mathrm{Vol}(\Delta) = q$ is unimodular equivalent to

$$T(p,q) := \mathrm{Conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\}$$

for some $p \in (\mathbb{Z}/q\mathbb{Z})^*$. Moreover, one has

$$T(p,q) \cong_\Lambda T(p',q) \Leftrightarrow p' = \pm p^{\pm 1} (\bmod q).$$

Lattice width of Δ

Let $f \in \Lambda^*$ be a nonzero linear function $f : \Lambda \to \mathbb{Z}$ and $\Delta \subset \Lambda_{\mathbb{R}}$ a lattice polytope. The f-width of Δ is

$$width_f(\Delta) := \max_{\Delta}(f) - \min_{\Delta}(f).$$

The lattice width of Δ is

$$width_\Lambda(\Delta) := \min_{f \in \Lambda^* \setminus 0} width_f(\Delta).$$

Lattice width of empty lattice d-simplices Δ

if $d \in \{1,2,3\}$, then $width_\Lambda(\Delta) = 1$. The case $d = 3$ follows from theorem of White.

Conjectures of Haase-Ziegler (2000), Theorems of Iglesias-Santos (2018).

- if $d = 4$, then $width_\Lambda(\Delta) \leq 4$.
- Up to unimodular equivalence, there is a unique empty lattice 4-simplex Δ of width 4. For a Λ-basis e_1, e_2, e_3, e_4, one has

$$\Delta \cong_\Lambda \mathrm{Conv}\{e_1, e_2, e_3, e_4, 6e_1 + 14e_2 + 17e_3 + 65e_4\}.$$

The normalized volume $\mathrm{Vol}(\Delta)$ equals 101.

Empty lattice 4-simplices of width 3

Theorem of Haase-Ziegler (2000) If Δ is an empty lattice 4-simplex of width 3, then $\mathrm{Vol}(\Delta) \geq 41$. Up to unimodular equivalence, there is a unique empty lattice 4-simplex Δ of width 3 with $\mathrm{Vol}(\Delta) = 41$. For a Λ-basis e_1, e_2, e_3, e_4, one has

$$\Delta \cong_{\Lambda} \mathrm{Conv}\{e_1, e_2, e_3, e_4, 4e_1 + 23e_2 + 25e_3 - 10e_4\}.$$

Conjectures of Haase-Ziegler (2000), Theorems of Iglesias-Santos (2018).
Up to unimodular equivalence, there are exactly 178 empty lattice 4-simplices Δ of width 3. Their normalized volumes $\mathrm{Vol}(\Delta)$ are ranged between 41 and 179.

Birational geometry

Example. Let

$$X_k := \mathbb{P}_{\mathbb{P}^{d-1}}(\mathcal{O} \oplus \mathcal{O}(k)), \;\; 1 \leq k \leq d-1.$$

Then X_k is a smooth Fano d-fold. X_k is a $\mathbb{P}^1$-fibering over $\mathbb{P}^{d-1}$ and $\mathrm{Pic}(X_k) \cong \mathbb{Z}^2$. There exists a section $E \cong \mathbb{P}^{d-1} \hookrightarrow X_k$ with the normal bundle $\mathcal{O}_{\mathbb{P}^{d-1}}(-k)$ and birational morphism (extremal Mori contraction)

$$X_k \to Y_k := \mathbb{P}(\underbrace{1, \ldots, 1}_{d}, k)$$

which contracts the divisor $E \subset X_k$ to a **terminal point** $y \in Y_k$.

Toric geometry

X_k is a smooth toric Fano d-fold whose defining fan Σ_k is generated by

$$\Sigma_k[1] := \{e_1, \ldots, e_d, -e_d, ke_d - \sum_{i=1}^{d-1} e_i\}.$$

Y_k is a terminal toric Fano d-fold whose defining fan Σ'_k is generated by

$$\Sigma'_k[1] := \{e_1, \ldots, e_{d-1}, -e_d, ke_d - \sum_{i=1}^{d-1} e_i\}.$$

Terminal cyclic quotient singularities and empty simplices

In the considered example, the lattice d-simplex

$$\Delta = \mathrm{Conv}\{0, e_1, \ldots, e_d, ke_d - \sum_{i=1}^{d-1} e_i\}$$

is empty and $\mathrm{Vol}(\Delta) = k$. In particular, if we take $d = 3$ and $k = 2$, then we obtain an isolated terminal quotient singularity

$$\mathbb{C}^3/\mu_2 = \mathbb{C}^3/\pm id \subset \mathbb{P}(1, 1, 1, 2).$$

It follows follows from theorem of White that every terminal 3-dimensional cyclic quotient singularity is isomorphic to $\mathbb{C}^3/\mu_q$, $\mu_q \cong \langle(\zeta, \zeta^p, \zeta^{-p})\rangle \subset GL(3, \mathbb{C})$, ζ is a primitive q-th root of unity.

4-dimensional Gorenstein terminal quotient singularities

Theorem of Morrison-Stevens (1984) Every 4-dimensional terminal cyclic quotient singularity $\mathbb{C}^4/\Gamma$ ($\Gamma \cong \mu_q$) comes from some empty lattice 3-simplex Δ with $\mathrm{Vol}(\Delta) = q$. In particular,

$$\mu_q \cong \langle (\zeta, \zeta^{-1}, \zeta^p, \zeta^{-p}) \rangle \subset SL(4, \mathbb{C}).$$

The 4-dimensional Gorenstein toric variety $\mathbb{C}^4/\Gamma$ is determined by the 4-dimensional cone $C(\Delta)$ over the empty lattice 3-simplex Δ.

Results of Mori-Morrison-Morrison (1988)

Search for quintuples $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4) \in \mathbb{Z}^5$ such that $\sigma = (\zeta^{\alpha_1}, \zeta^{\alpha_2}, \zeta^{\alpha_3}, \zeta^{\alpha_4}\zeta^{\alpha_5}) \in SL(5, \mathbb{C})$, $\zeta^p = 1$ for a prime p.

An extensive computer study of terminal quotient singularities of prime index led us to the discovery of some large classes of stable quintuples.

THEOREM 1.3. *Let Q be a quintuple of integers summing to zero, and let p be a prime number. Suppose that either*

(a) $Q = (\alpha, -\alpha, \beta, \gamma, -\beta-\gamma)$ *with* $0 < |\alpha|, |\beta|, |\gamma| < p/2$, *and* $\beta + \gamma \neq 0$, *or*

(b) $Q = (\alpha, -2\alpha, \beta, -2\beta, \alpha+\beta)$ *with* $0 < |\alpha|, |\beta| < p/2$, *and* $\alpha + \beta \neq 0$, *or*

(c) Q *is one of the 29 quintuples listed in Table 1.9 and* $p > M_Q$.

Then Q is p-terminal.

Results of Mori-Morrison-Morrison (1988)

29 one-parameter families of quintuples with respect to $V = \mathrm{Vol}(\Delta)$:

$\frac{1}{V}(9, 1, -2, -3, -5)$
$\frac{1}{V}(9, 2, -1, -4, -6)$
$\frac{1}{V}(12, 3, -4, -5, -6)$
$\frac{1}{V}(12, 2, -3, -4, -7)$
$\frac{1}{V}(9, 4, -2, -3, -8)$
$\frac{1}{V}(12, 1, -2, -3, -8)$
$\frac{1}{V}(12, 3, -1, -6, -8)$
$\frac{1}{V}(15, 4, -5, -6, -8)$
$\frac{1}{V}(12, 2, -1, -4, -9)$
$\frac{1}{V}(10, 6, -2, -5, -9)$

$\frac{1}{V}(15, 1, -2, -5, -9)$
$\frac{1}{V}(12, 5, -3, -4, -10)$
$\frac{1}{V}(15, 2, -3, -4, -10)$
$\frac{1}{V}(6, 4, 3, -1, -12)$
$\frac{1}{V}(7, 5, 3, -1, -14)$
$\frac{1}{V}(9, 7, 1, -3, -14)$
$\frac{1}{V}(15, 7, -3, -5, -14)$
$\frac{1}{V}(8, 5, 3, -1, -15)$
$\frac{1}{V}(10, 6, 1, -2, -15)$
$\frac{1}{V}(12, 5, 2, -4, -15)$

$\frac{1}{V}(9, 6, 4, -1, -18)$
$\frac{1}{V}(9, 6, 5, -2, -18)$
$\frac{1}{V}(12, 9, 1, -4, -18)$
$\frac{1}{V}(10, 7, 4, -1, -20)$
$\frac{1}{V}(10, 8, 3, -1, -20)$
$\frac{1}{V}(10, 9, 4, -3, -20)$
$\frac{1}{V}(12, 10, 1, -3, -20)$
$\frac{1}{V}(12, 8, 5, -1, -24)$
$\frac{1}{V}(15, 10, 6, -1, -30)$

Conjecture of Mori-Morrison-Morrison

Conjecture: Up to unimodular equivalence, there exist only finitely many empty lattice 4-simplices Δ (they are called sporadic) of prime normalized volume $\mathrm{Vol}(\Delta) = p$ which are not contained in Theorem 1.3.

This conjecture was proved by Sankaran (1990), and Bober (2009). A more general case was considered by Iglesias-Valiño and Santos (2018+). The classification of Iglesias-Santos covers all empty lattice 4-simplices Δ, non necessarily of prime volume $\mathrm{Vol}(\Delta)$. The key tool of this classification is lattice projection for hollow lattice polytopes.

Lattice projections of hollow lattice polytopes

A d-dimensional lattice polytope $\Delta \subset \Lambda_{\mathbb{R}}$ is called hollow if Δ has no interior lattice points.

A k-dimensional lattice polytope $\Delta' \subset \Lambda'_{\mathbb{R}} \cong \mathbb{R}^k$ $(1 \leq k \leq d-1)$ is called lattice projection of Δ is there exist a surjective linear map $\varphi : \Lambda \to \Lambda'$ such that $\varphi(\Delta) = \Delta'$.

It is easy to show that if a hollow lattice polytope Δ' is a lattice projection of Δ, then Δ is also hollow. This provides a method for constructing infinitely many hollow lattice polytopes from lower dimensional hollow ones.

Lattice projection of hollow lattice polytopes

Theorem of Nill-Ziegler (2011) For all fixed d, all hollow d-polytopes project to a hollow polytope of dimension less than d except for finitely many cases.

Remark. A lattice polytope Δ has a lattice projection onto $[0,1] \subset \mathbb{R}$ if and only if $width_\Lambda(\Delta) = 1$.

The quotient group Λ/Λ_Δ

For an empty lattice d-simplex $\Delta = \mathrm{Conv}\{v_0, \ldots, v_d\}$ we denote Λ_Δ the sublattice in Λ spanned by differences $v_i - v_0$ $(1 \leq i \leq d)$.
The quotient Λ/Λ_Δ is a finite abelian group of order $\mathrm{Vol}(\Delta)$.
If $d \in \{1, 2\}$, then Λ/Λ_Δ is trivial.
If $d = 3$, then Λ/Λ_Δ is cyclic of order $\mathrm{Vol}(\Delta)$.

Theorem of Barile, Bernardi, Borisov Kantor (2011) If $d = 4$, then Λ/Λ_Δ is cyclic of order $\mathrm{Vol}(\Delta)$.

h^*-polynomial of an empty lattice 4-simplex Δ

If $n = p > 2$ is prime, then $h^*_\Delta(t) = 1 + \frac{p-1}{2}t^2 + \frac{p-1}{2}t^3$.
In general,

$$h^*_\Delta(t) = 1 + \sum_{1 \neq d, d|n} h_d(t), \quad n = \mathrm{Vol}(\Delta),$$

where

- $h_d(t) := \frac{\varphi(d)}{2}(t^2 + t^3)$ if elements γ of order d in the cyclic group $\Lambda/\Lambda_\Delta \cong \Gamma \subset SL(5, \mathbb{C})$ have only isolated fixed point $0 \in \mathbb{C}^5$.
- $h_d(t) := \varphi(d)t^2$ if the fixed point set of γ of order d has positive dimension.

Examples

There exactly two empty lattice 4-simplices Δ of width 3 and $\mathrm{Vol}(\Delta) = 55$. They have different h^*-polynomials

$$1 + 29t^2 + 25t^3 = 1 + 4t^2 + 25(t^2 + t^3)$$

and

$$1 + 27t^2 + 27t^3.$$

The classification of empty lattice 4-simplices

Theorem of Iglesias-Santos (2018) Let Δ be an arbitrary empty lattice 4-simplex with $\mathrm{Vol}(\Delta) = n$. Then Δ is determined by a 5-dimensional cyclic Gorenstein terminal singulary $\mathbb{C}^5/\mu_n$ for some diagonal embedding $\mu_n \hookrightarrow SL(5, \mathbb{C})$ given by a quintuple $A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4) \in \mathbb{Z}^5$. Denote by k the minimal dimension of a hollow polytope Δ' that Δ project to. Then

1. if $k = 1$, then there exists one 3-pameter family of A;
2. if $k = 2$, then there exist two 2-parameter families of A;
3. if $k = 3$, then there exist $46 = 29 + 17$ 1-parameter families.
4. if $k = 4$, then there exist 2461 sporadic examples.

Affine 3-folds determined by Δ

Let $\Delta = \mathrm{Conv}\{v_0, v_1, v_2, v_3, v_4\}$ be an empty lattice 4-simplex. Consider the affine variety

$$Z_\Delta := \{\mathbf{t} = (t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4 \mid \sum_{i=0}^{4} \mathbf{t}^{v_i} = 0\}.$$

Problems concerning birational properties of Z_Δ.

1. Find the Kodaira dimension of Z_Δ.
2. Find birational Fano-fibrations of Z_Δ.
3. Find equations of minimal models of Z_Δ.
4. Find 4-simplices Δ such that Z_Δ is a rational variety.

Rational hypersurfaces Z_Δ

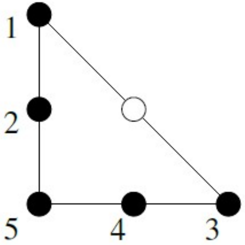
If $k = 1$, i.e., Δ has a lattice projection onto $[0, 1]$. In this case Z_Δ is rational. Two vertices of Δ are projected to $\{0\}$, the remaining three vertices are projected to $\{1\}$
These Δ corresponds to a Gorenstein terminal quotient singularity $\mathbb{C}^5/\mu_n$ for a cyclic subgroup with the splitting

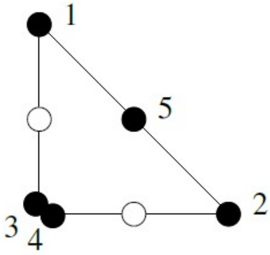
$$\mu_n \subset SL(2, \mathbb{C}) \times SL(3, \mathbb{C}) \subset SL(5, \mathbb{C}).$$

The intermediate jacobian of $\overline{Z_\Delta}$ is jacobian of curve $C \subset (\mathbb{C}^*)^2$ whose Newton polygon P is a lattice triangle with $\mathrm{Vol}(P) = n$ and the cone $\mathrm{Cone}(P)$ determines the Gorenstein quotient singularity $\mathbb{C}^3/\mu_n$ induced by the projection $SL(2, \mathbb{C}) \times SL(3, \mathbb{C}) \to SL(3, \mathbb{C})$.

Conic bundles stuctures on Z_Δ

If $k = 2$, then Δ has a lattice projection onto the hollow lattice triangle $\mathrm{Conv}\{(0,0),(0,2),(2,0)\} \subset \mathbb{R}^2$. There are two possibilities for this projection:

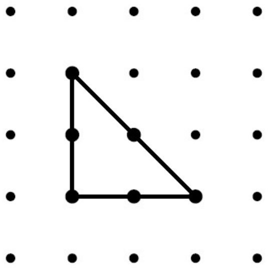




Example $\mathrm{Vol}(\Delta) = 11$

$$Z_\Delta = \{t_1^2 + t_1 t_2^2 + t_2 t_3^2 + t_3 t_4^2 + t_4 = 0\}, \quad \mathrm{Vol}(\Delta) = 11.$$

With respect to t_1 and t_3 we obtain a lattice projection of the first type on the lattice triangle



Example $\mathrm{Vol}(\Delta) = 11$

The projective closure of Z_Δ in $\mathbb{P}^4$ is the smooth Klein cubic:

$$\widehat{Z_\Delta} = \{z_0 z_1^2 + z_1 z_2^2 + z_2 z_3^2 + z_3 z_4^2 + z_4 z_0^2 = 0\} \subset \mathbb{P}^4$$

The lattice projection Δ with respect to t_1 and t_3 defines the conic bundle fibration $Z_\Delta \to (\mathbb{C}^*)^2$ whose ramification locus is the curve of genus 6 given by the equation

$$t_2 t_4 - \frac{t_2^5}{4} - \frac{t_4^4}{4} = 0.$$

Del Pezzo fibrations of Z_Δ

If $k = 3$, then Δ has a lattice projection onto one the hollow lattice polytope $\Delta' \subset \mathbb{R}^3$ is a bipyramid with five vertices which is contained in one of the following maximal hollow lattice 3-simplices Δ'_i:

$$\Delta'_1 := \mathrm{Conv}\{(0,0,0),(2,0,0),(0,3,0),(0,0,6)\};$$

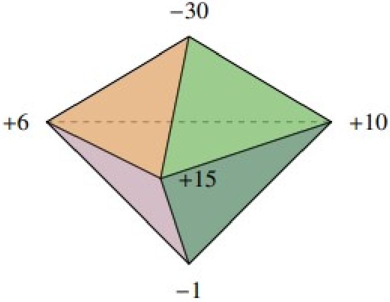
$$\Delta'_2 := \mathrm{Conv}\{(0,0,0),(2,0,0),(0,4,0),(0,0,4)\},$$

$$\Delta'_3 := \mathrm{Conv}\{(0,0,0),(3,0,0),(0,3,0),(0,0,3)\},$$

corresponding to Del Pezzo surfaces of anticanonical degrees $D \in \{1,2,3\}$.

Example: the stable quintuple $(15, 10, 6, -1, -30)$

This quintuple comes from lattice projection $\Delta \to \Delta'$, which is the bipyriamid:



The bipyramid Δ' is contained in Δ_1. The corresponding Del Pezzo fibration has monodromy contained in the Weyl group E_8.

Sporadic examples

Among 2461 *sporadic* empty lattice 4-simiplices Δ there exist exactly 552 ones such that the corresponding 3-folds $Z_\Delta \subset (\mathbb{C}^*)^4$ which have non-negative Kodaira dimension κ:

- $\kappa = 0$ (137 simplices),
- $\kappa = 1$ (124 simplices),
- $\kappa = 2$ (111 simplices),
- $\kappa = 3$ (180 simplices).

The remaining 1909 *sporadic* empty lattice 4-simplices determine terminal Fano 3-folds with Picard number 1 (may be non $\mathbb{Q}$-factorial).

Cyclotomic empty lattice 4-simplices

An empty lattice 4-simplex Δ is called cyclotomic if a cyclic unimodular symmetry of order 5 permuting its vertices. There are exactly four such 4-simplices Δ, where $\mathrm{Vol}(\Delta) \in \{11, 41, 61, 101\}$. It is more elegant to view these simplices as Newton polytopes of projective hypersurfaces in $\mathbb{P}^4$ or in fake projective spaces $\mathbb{P}^4/\mu_l$:

$$\mathrm{Vol}(\Delta) = 11,\ \mathit{width} = 2: \quad z_0^2 z_1 + z_1^2 z_2 + z_2^2 z_3 + z_3^2 z_4 + z_4^2 z_0 = 0.$$

$$\mathrm{Vol}(\Delta) = 61,\ \mathit{width} = 3: \quad z_0^3 z_1 + z_1^3 z_2 + z_2^3 z_3 + z_3^3 z_4 + z_4^3 z_0 = 0.$$

Cyclotomic empty lattice 4-simplices

or as Newton polytopes of projective hypersurfaces in fake projective spaces $\mathbb{P}^4/\mu_l$, $l \in \{5, 11\}$:

$\mathrm{Vol}(\Delta) = 41$, *width* $= 3$, $\kappa(Z_\Delta) = 0$:

$$\{z_0^4 z_1 + z_1^4 z_2 + z_2^4 z_3 + z_3^4 z_4 + z_4^4 z_0 = 0\}/\mu_5,$$

$$(1, \zeta, \zeta^2, \zeta^3, \zeta^4), \quad \zeta^5 = 1.$$

$\mathrm{Vol}(\Delta) = 101$, *width* $= 4$ (Haase-Ziegler), $\kappa(Z_\Delta) = 3$:

$$\{z_0^6 z_1 + z_1^6 z_2 + z_2^6 z_3 + z_3^6 z_4 + z_4^6 z_0 = 0\}/\mu_{11},$$

$$(\zeta, \zeta^5, \zeta^3, \zeta^4, \zeta^9), \quad \zeta^{11} = 1.$$

Happy Birthday, Yura!