Punctual Structures, Automatic Structures and Index Sets (Part I)

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Introduction: Computable structures

In the 1960s, Mal'tsev and Rabin initiated the systematic development of *computable structure theory* (or the theory of constructive models).

A structure $\mathcal{S}$ in the signature

$$\sigma = \{P_0^{n_0}, P_1^{n_1}, \dots, P_k^{n_k};\ f_0^{m_0}, f_1^{m_1}, \dots, f_\ell^{m_\ell};\ c_0, c_1, \dots, c_t\}$$

is a computable structure if:

- the domain of $\mathcal{S}$ is a (Turing) computable subset of ω,
- the predicates $P_i^{\mathcal{S}}$ and the operations $f_j^{\mathcal{S}}$ are computable.

Automatic structures

The studies of structures presented by finite automata go back to the works of Hodgson (1976), Khoussainov and Nerode (1995), Blumensath and Grädel (2000).

The roots of this approach go back to the works of Büchi, Elgot, and Rabin from the 1960s.

Let Σ be a finite alphabet.

A structure $\mathcal{S} = (S; P_0, \ldots, P_k; f_0, \ldots, f_\ell; c_0, \ldots, c_t)$ is automatic if:

- the domain of $\mathcal{S}$ is a regular subset of Σ^*;
- every relation P_i is recognizable by a finite automaton;
- for every function f_j, its graph is recognizable by a finite automaton.

Some examples of automatic structures

Theorem (Delhommé 2004)

An ordinal α has an automatic copy if and only if $\alpha < \omega^\omega$.

Theorem (Khoussainov, Nies, Rubin, and Stephan 2004)

A Boolean algebra $\mathcal{B}$ has an automatic copy if and only if either $\mathcal{B}$ is finite, or $\mathcal{B}$ is a finite direct power of the algebra $\mathbf{B}_f(\mathbb{N})$ containing all finite and all cofinite subsets of $\mathbb{N}$.

Theorem (Tsankov 2011)

The abelian group $(\mathbb{Q}, +)$ does not have automatic isomorphic copies.

Some examples of automatic structures

In general, automatic structures could be quite complex:

Theorem (Khoussainov and Minnes 2009)

For every computable ordinal α, there exists an automatic structure of Scott rank at least α.

There exist automatic structures of Scott rank ω_1^{CK} and $\omega_1^{CK} + 1$.

The index sets approach

Let σ be a finite signature. By using a universal Turing machine, one can uniformly effectively list all (partial) computable σ-structures: $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \dots$

Let P be a certain property of an algebraic structure.
In computable structure theory, the complexity of a property P is often measured via the complexity of its index set

$$I_P = \{e \in \omega : \text{the } e\text{-th computable structure } \mathcal{M}_e \text{ has property } P\}.$$

Examples. (1) For $P =$ "being a linear order", the set I_P is m-complete Π^0_2 (folklore).

(2) For $P =$ "being a well-order", the set I_P is m-complete Π^1_1 (Kleene, Spector).

The question of Khoussainov and Nerode

Question 4.9 in [KN-2008].

For the property $\mathrm{Aut} =$"having an isomorphic automatic copy", is the corresponding index set I_{Aut} Σ^1_1-complete?

[KN-2008] B. Khoussainov and A. Nerode, Open questions in the theory of automatic structures, *Bulletin of the EATCS*, no. 94 (Feb. 2008), pp. 181–204.

The main result

Question 4.9 in [KN-2008].

For the property $\mathrm{Aut} =$"having an isomorphic automatic copy", is the corresponding index set I_{Aut} Σ^1_1-complete?

Our main result discussed in these two talks (April 1 and April 8) is the following

Theorem [BHKMN-2019]

The index set I_{Aut} is Σ^1_1-complete.

[KN-2008] B. Khoussainov and A. Nerode, Open questions in the theory of automatic structures, *Bulletin of the EATCS*, no. 94 (Feb. 2008), pp. 181–204.

[BHKMN-2019] N. Bazhenov, M. Harrison-Trainor, I. Kalimullin, A. Melnikov, and K. M. Ng, Automatic and polynomial-time algebraic structures, *Journal of Symbolic Logic*, vol. 84 (2019), no. 4, pp. 1630–1669.

Part I. The index set of punctually presentable structures.

Primitive recursive structures

The following definition goes back to Mal'tsev (1961).

A structure $\mathcal{S}$ in the signature

$$\sigma = \{P_0, P_1, \ldots, P_k;\, f_0, f_1, \ldots, f_\ell;\, c_0, c_1, \ldots, c_t\}$$

is a primitive recursive structure if:

- the domain of $\mathcal{S}$ is a primitive recursive subset of ω, and
- the predicates $P_i^{\mathcal{S}}$ and the operations $f_j^{\mathcal{S}}$ are primitive recursive.

Punctual structures

Kalimullin, Melnikov, and Ng (2017) started systematic development of the theory of punctual structures:

Definition (Kalimullin, Melnikov, Ng)

A countably infinite structure $\mathcal{S}$ is punctual if the structure $\mathcal{S}$ is primitive recursive and $\mathrm{dom}(\mathcal{S}) = \omega$.

Since 2017, the theory of punctual structures has become a vast research area with a lot of interesting results and applications.

Punctual structures: Some examples

In each of the following classes, every computable structure has a punctual isomorphic copy:

1. linear orders [Grigorieff 1990],
2. torsion-free abelian groups [Kalimullin, Melnikov, Ng 2017],
3. Boolean algebras [Kalimullin, Melnikov, Ng 2017],
4. abelian p-groups [Kalimullin, Melnikov, Ng 2017].

In most of these cases, one gets a polynomial-time isomorphic copy almost for free.

In each of the following classes, there exists a computable structure which *does not have* punctual isomorphic copies:

a. torsion abelian groups [Cenzer and Remmel 2000],
b. Archimedean ordered abelian groups [Kalimullin, Melnikov, Ng 2017],
c. undirected graphs [Kalimullin, Melnikov, Ng 2017].

Recall that we fix a uniformly effective list of all computable σ-structures: $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \ldots$

In order to illustrate the ideas of the proof for automatic structures, here we give a proof for the following, simpler result:

Theorem 1 [BHKMN-2019]

For the property $\mathrm{Pun} =$"having a punctual isomorphic copy", the index set

$$I_{\mathrm{Pun}} = \{e \in \omega : \text{the } e\text{-th computable structure } \mathcal{M}_e \text{ is isomorphic to a punctual structure}\}$$

is m-complete Σ^1_1.

[BHKMN-2019] N. Bazhenov, M. Harrison-Trainor, I. Kalimullin, A. Melnikov, and K. M. Ng, Automatic and polynomial-time algebraic structures, *Journal of Symbolic Logic*, vol. 84 (2019), no. 4, pp. 1630–1669.

Proof of Theorem 1: Ideas

Preliminary observations

$$I_{\mathrm{Pun}} = \{e \in \omega : \text{the } e\text{-th computable structure } \mathcal{M}_e \text{ is isomorphic to a punctual structure}\}.$$

It is not hard to observe that the set I_{Pun} is Σ^1_1. Therefore, in order to establish Σ^1_1-completeness of I_{Pun}, it is sufficient to build a uniformly computable sequence of computable structures $(\mathcal{S}_n)_{n\in\omega}$ such that:

- if $n \notin \mathcal{O}$, then the structure $\mathcal{S}_n$ has a punctual isomorphic copy (Σ^1_1 outcome),
- if $n \in \mathcal{O}$, then the structure $\mathcal{S}_n$ has no punctual copies (Π^1_1 outcome).

In order to build the sequence $(\mathcal{S}_n)_{n\in\omega}$, firstly we fix a uniformly computable sequence of computable linear orders $(\mathcal{L}_n)_{n\in\omega}$ with the following properties [Harrison 1968]:

- if $n \notin \mathcal{O}$, then $\mathcal{L}_n$ is isomorphic to the Harrison linear order $\mathcal{H} = \omega_1^{CK} \cdot (1+\eta)$,
- if $n \in \mathcal{O}$, then $\mathcal{L}_n$ is isomorphic to an ordinal.

Outcomes of the construction

The signature of our constructed structures $\mathcal{S}_n$, $n \in \omega$, will be equal to $\sigma_0 = \{R^2; f^1, g^1\}$.

We fix a uniformly computable list $(\mathcal{N}_e)_{e \in \omega}$ of *all* punctual structures in the signature σ_0.

We have two possible outcomes for a given $n \in \omega$:

(Σ^1_1) If the given order $\mathcal{L}_n$ is isomorphic to the Harrison order $\mathcal{H}$, then the constructed $\mathcal{S}_n$ has a punctual copy.

(Π^1_1) If $\mathcal{L}_n$ is isomorphic to an ordinal, then $\mathcal{S}_n \not\cong \mathcal{N}_e$ for all $e \in \omega$.

In order to diagonalize against a fixed $\mathcal{N}_e$, we assume that our construction builds a separate component of $\mathcal{S}_n$, call this component $\mathcal{S}^e_n$.

In what follows, we will use the notation $\mathcal{S} = \mathcal{S}_n$.

First attempt: Dealing with Σ^1_1 outcome

Fix some naive diagonalization strategy $\mathbf{D}$ (working for Π^1_1 outcome) which ensures that $\mathcal{S} \not\cong \mathcal{N}_e$ for some $e \in \omega$ in isolation.

We assume that one can *uniformly primitively recursively* list all possible isomorphism types of substructures of $\mathcal{S}$ which can be produced by $\mathbf{D}$. Denote this list by $(\mathcal{D}_k)_{k\in\omega}$.

Double-boxes. Our key coding gadgets are "double-boxes" of the form:

$$\boxed{\mathcal{A}}\text{-}(\mathcal{B})$$

For now, assume that every square box $\boxed{\mathcal{A}}$ is distinguished by the unary predicate U_S, every round box $\mathcal{B}$ is distinguished by the unary predicate U_R, and each element of $\boxed{\mathcal{A}}$ is connected to each element of $\mathcal{B}$ by an edge.

We define the "junk structure" J^e as follows. For each $k \in \omega$, put inside J^e infinitely many double-boxes of isomorphism type $\boxed{\mathcal{H}}\text{-}(\mathcal{D}_k)$, where $\mathcal{H}$ is the Harrison linear order.

Notice that $\mathcal{H} = \omega_1^{CK} \cdot (1 + \eta)$ has a punctual copy. Therefore, the junk structure J^e also has a punctual copy.

In order to successfully deal with the Σ^1_1 outcome, we build a special double-box of the form

$$\boxed{\mathcal{L}_n}\text{-}(\mathrm{D}),$$

where D is the substructure produced by the diagonalization strategy $\mathbf{D}$ which works against the corresponding round box inside the opponent $\mathcal{N}_e$.

Note that the special double-box does not need to be primitive recursive.

We build the resulting component $\mathcal{S}^e$ (inside our structure $\mathcal{S}$) as a disjoint union of the junk structure J^e with the special double-box.

- In the Σ^1_1 outcome, we have $\mathcal{L}_n \cong \mathcal{H}$, and the diagonalization strategy $\mathbf{D}$ fails. Therefore, the special double-box is isomorphic to one of the junk double-boxes. This can be used to show that $\mathcal{S} = \mathcal{S}_n$ has a punctual copy.

- In the Π^1_1 outcome, $\mathcal{L}_n \cong \alpha$ for some computable ordinal α. Then α in the double-box $\boxed{\alpha}\text{-}(\mathrm{D})$ *uniquely determines* the spot where we could diagonalize against $\mathcal{N}_e$ (i.e., ensure that $\mathcal{S} \not\cong \mathcal{N}_e$).

  If we knew the image of $\boxed{\alpha}$ inside $\mathcal{N}_e$, then we would be able to successfully diagonalize. But our first construction attempt *does not allow* to computably find this image.

Second attempt: Preliminaries on labels

The following label technique goes back to the work of Goncharov (1977). A typical application of the technique is as follows.

A constructed structure $\mathcal{M}$ is built as the disjoint union of its components $\mathcal{M}^e$, $e \in \omega$. The component $\mathcal{M}^e$ is equal to, say, the restriction of $\mathcal{M}$ to the unary predicate P_e. Among others, one aims to satisfy the following requirements:

$$\mathcal{M}^e \cong \mathcal{N}_e \upharpoonright P_e \ \Rightarrow\ \mathcal{M}^e \cong_{\Delta^0_1} \mathcal{N}_e \upharpoonright P_e.$$

This requirement is ensured via the following *local rigidity* property:

For every stage s, there is a unique isomorphic embedding from $\mathcal{M}^e[s]$ into $\mathcal{M}^e[s+1]$.

In order to guarantee local rigidity, one uses labels: a typical label $\boxed{k}$ is an undirected cycle of size $(k+3)$ attached to some node of the component $\mathcal{M}^e$.

Example of the label technique

For simplicity, assume that the points of P_e do not have edges between them.

(1) Add a node x_0 into $\mathcal{M}^e = \mathcal{M} \upharpoonright P_e$. Attach a label $\boxed{0}$ to x_0. Wait for the opponent $\mathcal{N}_e$ to show a node $y_0 \in P_e$ labelled by $\boxed{0}$.

(2) Add a node x_1 into $\mathcal{M}^e$ and label it by $\boxed{1}$. Wait until $\mathcal{N}_e$ shows a node $y_1 \in P_e$ labelled by $\boxed{1}$ such that $y_1 \neq y_0$.

Whenever the opponent $\mathcal{N}_e$ responds by copying our actions, we say that the respective stage of the construction is e-expansionary.

Our *final goal* is to ensure the following: if $\mathcal{N}_e \upharpoonright P_e \cong \mathcal{M}^e$, then there exists an isomorphism from $\mathcal{M}^e$ onto $\mathcal{N}_e \upharpoonright P_e$ which maps x_i to y_i.

Our *local goal* is to ensure that at each finite stage, the points x_0 and x_1 do not look automorphic inside the structure $\mathcal{M}$.

(3) Label x_0 with $\boxed{2}$ and x_1 with $\boxed{3}$. Wait for the opponent $\mathcal{N}_e$ to respond by putting $\boxed{2}$ on y_0 and $\boxed{3}$ on y_1.

(4) Now label x_0 with $\boxed{1}$ and x_1 with $\boxed{0}$. Wait for $\mathcal{N}_e$ to respond by putting $\boxed{1}$ on y_0 and $\boxed{0}$ on y_1.

...

(2n+3) Label x_0 with $\boxed{2n+2}$ and x_1 with $\boxed{2n+3}$. Wait for $\mathcal{N}_e$ to respond by putting $\boxed{2n+2}$ on y_0 and $\boxed{2n+3}$ on y_1.

(2n+4) Label x_0 with $\boxed{2n+1}$ and x_1 with $\boxed{2n}$. Wait for $\mathcal{N}_e$ to respond by putting $\boxed{2n+1}$ on y_0 and $\boxed{2n}$ on y_1.

...

Observation. If the opponent $\mathcal{N}_e$ fails to respond at some stage, then $\mathcal{N}_e \upharpoonright P_e \ncong \mathcal{M}^e = \mathcal{M} \upharpoonright P_e$.

Otherwise, each of the nodes x_0, x_1 has all the labels $\boxed{k}$, $k \in \omega$. Hence, the nodes x_0 and x_1 are automorphic. If $\mathcal{N}_e \cong \mathcal{M}$, then one can safely (isomorphically) map x_i to y_i.

Second attempt: Pressing an opponent via labels

Recall that in the Π^1_1 outcome (when $\mathcal{L}_n \cong \alpha$), we need to successfully find the special double-box $\boxed{\alpha}$-(D) inside a given opponent $\mathcal{N}_e$.

This is achieved via modifying the construction of square boxes $\boxed{\mathcal{A}}$ in double-boxes $\boxed{\mathcal{A}}$-($\mathcal{B}$). Instead of adding a linear order $\mathcal{A}$, we put a labelled linear order $\tilde{\mathcal{A}}$ inside a square box. The labels are carefully placed similarly to the example above.

- If there are infinitely many e-expansionary stages, then inside the component $\mathcal{S}^e$, all elements from the square boxes (in the limit) obtain all labels from some fixed computable set. Thus, these labels *do not disturb* the behavior intended for the Σ^1_1 outcome.

- If there are infinitely many e-expansionary stages, then we can search for the least m such that the m-th double-box in the opponent $\mathcal{N}_e$ copies our special double-box.

If $\mathcal{N}_e \cong \mathcal{S}$, then the contents of the 'special' square box of $\mathcal{N}_e$ has to be isomorphic to the contents of our square box from $\boxed{\alpha}$-(D). So, in the Π^1_1 outcome, (albeit the order α is 'complicated') we can recover the corresponding 'diagonalization spot' and diagonalize successfully.

There are further technical obstacles involved:

- If there are only finitely many e-expansionary stages, then this means that $\mathcal{N}_e \ncong \mathcal{S}$.

  In the Σ^1_1 outcome, this is OK for us, but here the constructed special double-box $\boxed{\mathcal{L}}$-D will be a finite structure. This gives a situation different from the first attempt: we need to additionally ensure that the constructed $\mathcal{S}$ will have a punctual isomorphic copy.

- ...

Proof of Theorem 1: A formal outline

The arrangement of double-boxes

Given a computable linear order $\mathcal{L}_n$, we will construct a computable structure $\mathcal{S}_n$ in the signature $\sigma_0 = \{R^2; f^1, g^1\}$.

The structure $\mathcal{S} := \mathcal{S}_n$ is a disjoint union of components $\mathcal{S}^e$, $e \in \omega$.

Each $\mathcal{S}^e$ is a disjoint union of double-boxes $\boxed{\mathcal{U}}$-$\mathcal{V}$, where:

- The square box $\boxed{\mathcal{U}}$ is a directed graph in the signature $\{R\}$. One can think of $\boxed{\mathcal{U}}$ as a *labelled linear order*, where each label has form $\boxed{\langle e, j \rangle}$ for some $j \in \omega$.
- The round box $\mathcal{V}$ is a unary algebra in the signature $\{f\}$.
- The connection between the square box and the round box is realized as follows:
  - We introduce two additional elements $c = c[\mathcal{U}, \mathcal{V}]$ and $d = d[\mathcal{U}, \mathcal{V}]$. For $x \in \{c, d\}$, put $f(x) = g(x) = d$.
  - For $x \in \boxed{\mathcal{U}}$, $f(x) = x$ and $g(x) = c$.
  - For $x \in \mathcal{V}$, $g(x) = d$.

Preprocessing of a given $\mathcal{N}_e$

We fix a computable list $(\mathcal{N}_e)_{e\in\omega}$ of all punctual σ_0-structures.

Recall our outcomes:

(Σ^1_1) If $\mathcal{L}_n$ is isomorphic to the Harrison order $\mathcal{H}$, then the constructed $\mathcal{S}$ has a punctual copy.

(Π^1_1) If $\mathcal{L}_n \not\cong \mathcal{H}$, then $\mathcal{S} \not\cong \mathcal{N}_e$ for all $e \in \omega$.

We treat a given punctual structure $\mathcal{N}_e$ as a collection of disjoint double-boxes $\boxed{\mathcal{U}}$-$\mathcal{V}$. Hence, given $\mathcal{N}_e$, we produce two uniformly computable lists:

- $(\mathcal{U}_i)_{i\in\omega}$ is a list of computable directed graphs, and
- $(\mathcal{V}_i)_{i\in\omega}$ is a list of computable unary algebras.

If the structure $\mathcal{N}_e$ is 'nice enough', then $\boxed{\mathcal{U}_i}$-$\mathcal{V}_i$, $i \in \omega$, are precisely all the double-boxes inside $\mathcal{N}_e$.

The first key lemma

Key Lemma 1

Let $\mathcal{L}$ and $\mathcal{H}$ be computable linear orders. Let $(\mathcal{U}_i)_{i\in\omega}$ be a computable sequence of (partial) computable graphs. One can effectively construct a computable function $f(x)$ and a computable sequence of graphs $(\mathcal{A}_i)_{i\in\omega}$ with the following properties:

(a) If $\mathcal{L} \cong \mathcal{H}$, then there exists $i > 0$ such that $\mathcal{A}_i \cong \mathcal{A}_0$.

(b) Suppose that $\mathcal{L} \not\cong \mathcal{H}$, and the sequences $(\mathcal{A}_i)_{i\in\omega}$ and $(\mathcal{U}_i)_{i\in\omega}$ contain precisely the same isomorphism types. Then every $i > 0$ satisfies $\mathcal{A}_i \not\cong \mathcal{A}_0$. In addition, there exists a limit $k = \lim_s f(s)$, and $\mathcal{A}_0 \cong \mathcal{U}_k$.

Constructing $\mathcal{S}^e$, part 1. Given a punctual structure $\mathcal{N}_e$, we recover its sequence of square boxes $(\mathcal{U}_i)_{i\in\omega}$. We apply Lemma 1 to the orders $\mathcal{L}_n, \mathcal{H}$ and to the sequence $(\mathcal{U}_i)_{i\in\omega}$.

We get a computable function $f(x)$ and a computable sequence of graphs $(\mathcal{A}_i)_{i\in\omega}$.

The second key lemma

Consider the following computable injection structures:

- For a non-zero $\ell \leq \omega$, $\mathcal{X}_\ell$ contains infinitely many cycles of size m, for each $m < 1 + \ell$.
- The structure $\mathcal{Y}_\ell$ is a disjoint union of $\mathcal{X}_\ell$ and an ω-chain.

Put $K = \{\mathcal{X}_\ell, \mathcal{Y}_\ell : 1 \leq \ell \leq \omega\}$.

Key Lemma 2

Given a computable sequence of unary algebras $(\mathcal{V}_i)_{i \in \omega}$ and a computable function $f(x)$, one can effectively construct a computable unary algebra $\mathcal{C}$ with the following properties:

(1) $\mathcal{C}$ is isomorphic to some structure from K, and

(2) if the limit $k = \lim_s f(s)$ exists, then $\mathcal{C} \not\cong \mathcal{V}_k$.

Constructing $\mathcal{S}^e$, part 2. Given a punctual structure $\mathcal{N}_e$, we recover its sequence of round boxes $(\mathcal{V}_i)_{i \in \omega}$. We apply Lemma 2 to the sequence $(\mathcal{V}_i)_{i \in \omega}$ and to the computable function $f(x)$ from the previous slide.

We get a computable unary algebra $\mathcal{C}$.

Constructing $\mathcal{S}^e$, part 3. The structure $\mathcal{S}^e$ is a disjoint union of the following double-boxes:

- the special double-box $\boxed{\mathcal{A}_0}$-$\mathcal{C}$,
- infinitely many copies of each of the 'non-special' double-box $\boxed{\mathcal{A}_{i+1}}$-$\mathcal{X}_j$, $\boxed{\mathcal{A}_{i+1}}$-$\mathcal{Y}_j$, for $i \in \omega$ and $1 \leq j \leq \omega$.

The resulting structure $\mathcal{S}$ is a disjoint union of all $\mathcal{S}^e$.

Verification. **Lemma 1 (correctness of the Π^1_1 outcome)**

If $\mathcal{L}_n \not\cong \mathcal{H}$, then the constructed $\mathcal{S}$ does not have punctual copies.

Proof. Towards a contradiction, assume that $\mathcal{S} \cong \mathcal{N}_e$. By Key Lemma 1, since the sequences of square boxes $(\mathcal{U}_i)_{i\in\omega}$ and $(\mathcal{A}_i)_{i\in\omega}$ contain precisely the same isomorphism types, we have:

- $\mathcal{A}_0 \not\cong \mathcal{A}_{i+1}$ for every $i \in \omega$. Thus, the special double-box $\boxed{\mathcal{A}_0}$-$\mathcal{C}$ *must be isomorphic* to a unique double-box inside $\mathcal{N}_e$.
- $\mathcal{A}_0 \cong \mathcal{U}_{k^*}$, for $k^* = \lim_s f(s)$. Hence, $\boxed{\mathcal{A}_0}$-$\mathcal{C}$ is isomorphic to the box $\boxed{\mathcal{U}_{k^*}}$-$\mathcal{V}_{k^*}$ inside $\mathcal{N}_e$.

On the other hand, by Key Lemma 2, we obtain that $\mathcal{C} \not\cong \mathcal{V}_{k^*}$.

We obtain a contradiction. □

Verification continued

Lemma 2 (correctness of the Σ^1_1 outcome)

If $\mathcal{L}_n \cong \mathcal{H}$, then the constructed $\mathcal{S}$ has a punctual copy.

Proof Sketch. If $\mathcal{L}_n \cong \mathcal{H}$, then by Key Lemma 1, the structure $\mathcal{A}_0$ is isomorphic to $\mathcal{A}_{i+1}$ for some $i \in \omega$. Therefore, (for each $\mathcal{S}^e$) the special double-box $\boxed{\mathcal{A}_0}$-$\mathcal{C}$ is isomorphic to some non-special double-box.

Hence, $\mathcal{S}$ is isomorphic to the disjoint union of all non-special double-boxes, denote this union by $\mathcal{S}_{ns}$.

In the proof of Key Lemma 1, one can choose the structure $\mathcal{A}_1$ as a(n appropriately) labelled punctual copy of the Harrison order $\mathcal{H}$. Using this observation, we can build a punctual copy of $\mathcal{S}_{ns}$. (One 'promptly' produces a copy of Harrison order, while waiting for the convergence of various computations for $\mathcal{A}_j$, $j \in \omega$.) $\square$

This finishes the proof of Theorem 1.

A careful analysis of the proof of Theorem 1 allows to obtain the following

Theorem

The index set

$$I_{\mathrm{Poly}} = \{e \in \omega : \text{the } e\text{-th computable structure } \mathcal{M}_e \text{ is isomorphic to a polynomial-time computable structure}\}$$

is m-complete Σ^1_1.

Thank you!