Punctual Structures, Automatic Structures and Index Sets (Part II)

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The main theorem from part I

Theorem. There are uniformly computable sequences of computable structures $\{\mathcal{S}_n\}_{n\in\omega}$ and punctual structures $\{\mathcal{J}_n\}_{n\in\omega}$ such that:

- if $n \notin \mathcal{O}$, then $\mathcal{S}_n \cong \mathcal{J}_n$;
- if $n \in \mathcal{O}$, then the structure $\mathcal{S}_n$ has no punctual copies.

A rough idea: relativization

Theorem. There are uniformly computable sequences of $\emptyset'$-computable structures $\{\mathcal{S}_n\}_{n\in\omega}$ and $\emptyset'$-punctual structures$\{\mathcal{J}_n\}_{n\in\omega}$ such that:

- if $n \notin \mathcal{O}$, then $\mathcal{S}_n \cong \mathcal{J}_n$;
- if $n \in \mathcal{O}$, then the structure $\mathcal{S}_n$ has no $\emptyset'$-punctual copies.

A rough idea: $\emptyset'$-punctuality implies computability

Theorem. There are uniformly computable sequences of $\emptyset'$-computable structures $\{\mathcal{S}_n\}_{n\in\omega}$ and $\emptyset'$-punctual structures $\{\mathcal{J}_n\}_{n\in\omega}$ such that:

- if $n \notin \mathcal{O}$, then $\mathcal{S}_n \cong \mathcal{J}_n$;
- if $n \in \mathcal{O}$, then the structure $\mathcal{S}_n$ has no computable copies.

A rough idea: jump inversion

Now we need a uniform transformation $\Theta(\mathcal{S})$ of $\emptyset'$-computable structures $\mathcal{S}$ such that:

- $\Theta(\mathcal{S}_n)$ is computable;

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- $\Theta(\mathcal{S}_n)$ has a 1-decidable copy $\implies$ $\mathcal{S}_n$ has a computable copy;

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A rough idea: jump inversion

Now we need a uniform transformation $\Theta(\mathcal{S})$ of $\emptyset'$-computable structures $\mathcal{S}$ such that:

- $\Theta(\mathcal{S}_n)$ is computable;
- $\Theta(\mathcal{S}_n)$ has a 1-decidable copy $\implies$ $\mathcal{S}_n$ has a computable copy; and
- (this is most non-evident) $\Theta(\mathcal{J}_n)$ has an automatic copy.

The main theorem

Theorem. There are uniformly computable sequences of computable structures $\{\Theta(\mathcal{S}_n)\}_{n\in\omega}$ and automatic structures $\{\Theta(\mathcal{J}_n)\}_{n\in\omega}$ such that:

- if $n \notin \mathcal{O}$, then $\Theta(\mathcal{S}_n) \cong \Theta(\mathcal{J}_n)$;
- if $n \in \mathcal{O}$, then the structure $\Theta(\mathcal{S}_n)$ has no 1-decidable copies.

The main theorem

Theorem. There are uniformly computable sequences of computable structures $\{\Theta(\mathcal{S}_n)\}_{n\in\omega}$ and automatic structures $\{\Theta(\mathcal{J}_n)\}_{n\in\omega}$ such that:

- if $n \notin \mathcal{O}$, then $\Theta(\mathcal{S}_n) \cong \Theta(\mathcal{J}_n)$;
- if $n \in \mathcal{O}$, then the structure $\Theta(\mathcal{S}_n)$ has no 1-decidable copies.

Thus, we are also improving

Theorem (Harrison-Trainor, 2018). There are uniformly computable sequences of computable structures $\{\mathcal{C}_n\}_{n\in\omega}$ and decidable structures $\{\mathcal{D}_n\}_{n\in\omega}$ such that:

- if $n \notin \mathcal{O}$, then $\mathcal{C}_n \cong \mathcal{D}_n$;
- if $n \in \mathcal{O}$, then the structure $\mathcal{C}_n$ has no 2-decidable copies.

Original proof (relativized to $\emptyset'$)

The junk structure $\mathcal{J} := \mathcal{J}_n$ is a disjoint union of double-boxes

$$\boxed{\mathcal{A}_i^e} - \mathcal{Z}$$

(each repeated infinitely), for all $e = 0, 1, 2, \ldots$, $i = 1, 2, 3 \ldots$ and injection structures $\mathcal{Z} \in K = \{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_\omega, \mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_\omega\}$.

The structure $\mathcal{S} := \mathcal{S}_n$ is the junk $\mathcal{J}$ with added double-boxes

$$\boxed{\mathcal{A}_0^e} - \mathcal{D}^e$$

for all $e = 0, 1, 2, \ldots$, where the purpose of $\mathcal{D}^e \cong \mathcal{Z} \in K$ is to diagonalize against e-th computable structure if $e \in \mathcal{O}$.

If $e \notin \mathcal{O}$ then for every e there is an $i = 1, 2, \ldots$ such that $\mathcal{A}_0^e \cong \mathcal{A}_i^e$ so that $\mathcal{S} \cong \mathcal{J}$.

New proof

We need two special jump inversion transformations Φ and Ψ.

The junk structure $\Theta(\mathcal{J}) := \Theta(\mathcal{J}_n)$ is a disjoint union of double-boxes

$$\boxed{\Psi(\mathcal{A}_i^e)} - \Phi(\mathcal{Z})$$

(each repeated infinitely), for all $e = 0, 1, 2, \ldots,\ i = 1, 2, 3 \ldots$ and injection structures $\mathcal{Z} \in K = \{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_\omega, \mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_\omega\}$.

The structure $\Theta(\mathcal{S}) := \Theta(\mathcal{S}_n)$ is the junk $\Theta(\mathcal{J})$ with added double-boxes

$$\boxed{\Psi(\mathcal{A}_0^e)} - \Phi(\mathcal{D}^e)$$

for all $e = 0, 1, 2, \ldots,$ where the purpose of $\mathcal{D}^e \cong \mathcal{Z} \in K$ is to diagonalize against e-th 1-decidable structure if $e \in \mathcal{O}$.

If $e \notin \mathcal{O}$ then for every e there is an $i = 1, 2, \ldots$ such that $\mathcal{A}_0^e \cong \mathcal{A}_i^e$ so that $\Theta(\mathcal{S}) \cong \Theta(\mathcal{J})$.

The first key lemma (relativized to $\emptyset'$)

Key Lemma 1

Let $n \in \omega$ and $\{\mathcal{U}_i\}_{i\in\omega}$ be a computable sequence of (partial) $\emptyset'$-computable graphs. One can effectively construct a $\emptyset'$-computable function $f(x)$ and a sequence of $\emptyset'$-computable graphs $\{\mathcal{A}_i\}_{i\in\omega}$ with the following properties:

(a) If $n \notin \mathcal{O}$, then there exists $i = 1, 2, 3\dots$ such that $\mathcal{A}_i \cong \mathcal{A}_0$.

(b) Suppose that $n \in \mathcal{O}$ and the sequences $\{\mathcal{A}_i\}_{i\in\omega}$ and $\{\mathcal{U}_i\}_{i\in\omega}$ contain precisely the same isomorphism types. Then every $i = 1, 2, 3, \dots$ satisfies $\mathcal{A}_i \not\cong \mathcal{A}_0$. In addition, there exists a limit $k = \lim_s f(s)$, and $\mathcal{A}_0 \cong \mathcal{U}_k$.

The first key lemma after jump inversion

Key Lemma 1

Let $n \in \omega$ and $\{\Psi(\mathcal{U}_i)\}_{i\in\omega}$ be a computable sequence of (partial) 1-decidable graphs. One can effectively construct a $\emptyset'$-computable function $f(x)$ and a sequence of computable graphs $\{\Psi(\mathcal{A}_i)\}_{i\in\omega}$ with the following properties:

(a) If $n \notin \mathcal{O}$, then there exists $i = 1, 2, 3 \dots$ such that $\Psi(\mathcal{A}_i) \cong \Psi(\mathcal{A}_0)$.

(b) Suppose that $n \in \mathcal{O}$ and the sequences $\{\Psi(\mathcal{A}_i)\}_{i\in\omega}$ and $\{\Psi(\mathcal{U}_i)\}_{i\in\omega}$ contain precisely the same isomorphism types. Then every $i = 1, 2, 3, \dots$ satisfies $\Psi(\mathcal{A}_i) \not\cong \Psi(\mathcal{A}_0)$. In addition, there exists a limit $k = \lim_s f(s)$, and $\Psi(\mathcal{A}_0) \cong \Psi(\mathcal{U}_k)$.

The second key lemma (relativized to $\emptyset'$)

Consider the following computable injection structures:

- For a non-zero $\ell \leq \omega$, $\mathcal{X}_\ell$ contains infinitely many cycles of size m, for each $m < 1 + \ell$.
- The structure $\mathcal{Y}_\ell$ is a disjoint union of $\mathcal{X}_\ell$ and an ω-chain.

Put $K = \{\mathcal{X}_\ell, \mathcal{Y}_\ell : 1 \leq \ell \leq \omega\}$.

Key Lemma 2

Given a computable sequence of (total) $\emptyset'$-computable unary algebras $\{\mathcal{V}_i\}_{i\in\omega}$ and an $\emptyset'$-computable function $f(x)$, one can effectively construct a $\emptyset'$-computable unary algebra $\mathcal{D}$ with the following properties:

(1) $\mathcal{D} \cong \mathcal{Z}$ for some $\mathcal{Z} \in K$, and

(2) if the limit $k = \lim_s f(s)$ exists, then $\mathcal{D} \not\cong \mathcal{V}_k$.

The second key lemma after jump inversion

Consider the following computable injection structures:

- For a non-zero $\ell \leq \omega$, $\mathcal{X}_\ell$ contains infinitely many cycles of size m, for each $m < 1 + \ell$.
- The structure $\mathcal{Y}_\ell$ is a disjoint union of $\mathcal{X}_\ell$ and an ω-chain.

Put $K = \{\mathcal{X}_\ell, \mathcal{Y}_\ell : 1 \leq \ell \leq \omega\}$.

Key Lemma 2

Given a computable sequence of 1-decidable structures $\{\Phi(\mathcal{V}_i)\}_{i \in \omega}$ and an $\emptyset'$-computable function $f(x)$, one can effectively construct a computable structure $\Phi(\mathcal{D})$ with the following properties:

(1) $\mathcal{D} \cong \mathcal{Z}$ for some $\mathcal{Z} \in K$, and

(2) if the limit $k = \lim_s f(s)$ exists, then $\Phi(\mathcal{D}) \ncong \Phi(\mathcal{V}_k)$.

Thank you!