

Punctual Structures, Automatic Structures and Index Sets (Part II)

Nikolay Bazhenov¹ and Iskander Kalimullin²

¹ — Sobolev Institute of Mathematics, Novosibirsk, Russia

² — Kazan Federal University, Kazan, Russia

Logic Online Seminar,
Steklov Mathematical Institute,
Moscow, Russia,
April 8, 2024

The main theorem from part I

Theorem. There are uniformly computable sequences of computable structures $\{\mathcal{S}_n\}_{n \in \omega}$ and punctual structures $\{\mathcal{J}_n\}_{n \in \omega}$ such that:

- ▶ if $n \notin \mathcal{O}$, then $\mathcal{S}_n \cong \mathcal{J}_n$;
- ▶ if $n \in \mathcal{O}$, then the structure \mathcal{S}_n has no punctual copies.

A rough idea: relativization

Theorem. There are uniformly computable sequences of \emptyset' -computable structures $\{\mathcal{S}_n\}_{n \in \omega}$ and \emptyset' -punctual structures $\{\mathcal{T}_n\}_{n \in \omega}$ such that:

- ▶ if $n \notin \mathcal{O}$, then $\mathcal{S}_n \cong \mathcal{T}_n$;
- ▶ if $n \in \mathcal{O}$, then the structure \mathcal{S}_n has no \emptyset' -punctual copies.

A rough idea: \emptyset' -punctuality implies computability

Theorem. There are uniformly computable sequences of \emptyset' -computable structures $\{\mathcal{S}_n\}_{n \in \omega}$ and \emptyset' -punctual structures $\{\mathcal{J}_n\}_{n \in \omega}$ such that:

- ▶ if $n \notin \mathcal{O}$, then $\mathcal{S}_n \cong \mathcal{J}_n$;
- ▶ if $n \in \mathcal{O}$, then the structure \mathcal{S}_n has no computable copies.

A rough idea: jump inversion

Now we need a uniform transformation $\Theta(\mathcal{S})$ of \emptyset' -computable structures \mathcal{S} such that:

- ▶ $\Theta(\mathcal{S}_n)$ is computable;

A rough idea: jump inversion

Now we need a uniform transformation $\Theta(\mathcal{S})$ of \emptyset' -computable structures \mathcal{S} such that:

- ▶ $\Theta(\mathcal{S}_n)$ is computable;
- ▶ $\Theta(\mathcal{S}_n)$ has a 1-decidable copy $\implies \mathcal{S}_n$ has a computable copy;

A rough idea: jump inversion

Now we need a uniform transformation $\Theta(\mathcal{S})$ of \emptyset' -computable structures \mathcal{S} such that:

- ▶ $\Theta(\mathcal{S}_n)$ is computable;
- ▶ $\Theta(\mathcal{S}_n)$ has a 1-decidable copy $\implies \mathcal{S}_n$ has a computable copy; and

A rough idea: jump inversion

Now we need a uniform transformation $\Theta(\mathcal{S})$ of \emptyset' -computable structures \mathcal{S} such that:

- ▶ $\Theta(\mathcal{S}_n)$ is computable;
- ▶ $\Theta(\mathcal{S}_n)$ has a 1-decidable copy $\implies \mathcal{S}_n$ has a computable copy; and
- ▶ (this is most non-evident) $\Theta(\mathcal{J}_n)$ has an automatic copy.

The main theorem

Theorem. There are uniformly computable sequences of computable structures $\{\Theta(\mathcal{S}_n)\}_{n \in \omega}$ and **automatic** structures $\{\Theta(\mathcal{J}_n)\}_{n \in \omega}$ such that:

- ▶ if $n \notin \mathcal{O}$, then $\Theta(\mathcal{S}_n) \cong \Theta(\mathcal{J}_n)$;
- ▶ if $n \in \mathcal{O}$, then the structure $\Theta(\mathcal{S}_n)$ has no 1-decidable copies.

The main theorem

Theorem. There are uniformly computable sequences of computable structures $\{\Theta(\mathcal{S}_n)\}_{n \in \omega}$ and **automatic** structures $\{\Theta(\mathcal{J}_n)\}_{n \in \omega}$ such that:

- ▶ if $n \notin \mathcal{O}$, then $\Theta(\mathcal{S}_n) \cong \Theta(\mathcal{J}_n)$;
- ▶ if $n \in \mathcal{O}$, then the structure $\Theta(\mathcal{S}_n)$ has no 1-decidable copies.

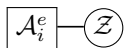
Thus, we are also improving

Theorem (Harrison-Trainor, 2018). There are uniformly computable sequences of computable structures $\{\mathcal{C}_n\}_{n \in \omega}$ and **decidable** structures $\{\mathcal{D}_n\}_{n \in \omega}$ such that:

- ▶ if $n \notin \mathcal{O}$, then $\mathcal{C}_n \cong \mathcal{D}_n$;
- ▶ if $n \in \mathcal{O}$, then the structure \mathcal{C}_n has no 2-decidable copies.

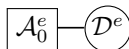
Original proof (relativized to \emptyset')

The junk structure $\mathcal{J} := \mathcal{J}_n$ is a disjoint union of double-boxes



(each repeated infinitely), for all $e = 0, 1, 2, \dots$, $i = 1, 2, 3, \dots$ and injection structures $Z \in K = \{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_\omega, \mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_\omega\}$.

The structure $\mathcal{S} := \mathcal{S}_n$ is the junk \mathcal{J} with added double-boxes



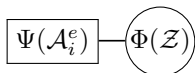
for all $e = 0, 1, 2, \dots$, where the purpose of $\mathcal{D}^e \cong Z \in K$ is to diagonalize against e -th computable structure if $e \in \mathcal{O}$.

If $e \notin \mathcal{O}$ then for every e there is an $i = 1, 2, \dots$ such that $\mathcal{A}_0^e \cong \mathcal{A}_i^e$ so that $\mathcal{S} \cong \mathcal{J}$.

New proof

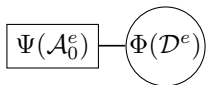
We need two special jump inversion transformations Φ and Ψ .

The junk structure $\Theta(\mathcal{J}) := \Theta(\mathcal{J}_n)$ is a disjoint union of double-boxes



(each repeated infinitely), for all $e = 0, 1, 2, \dots$, $i = 1, 2, 3, \dots$ and injection structures $\mathcal{Z} \in K = \{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_\omega, \mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_\omega\}$.

The structure $\Theta(\mathcal{S}) := \Theta(\mathcal{S}_n)$ is the junk $\Theta(\mathcal{J})$ with added double-boxes



for all $e = 0, 1, 2, \dots$, where the purpose of $\mathcal{D}^e \cong \mathcal{Z} \in K$ is to diagonalize against e -th 1-decidable structure if $e \in \mathcal{O}$.

If $e \notin \mathcal{O}$ then for every e there is an $i = 1, 2, \dots$ such that $\mathcal{A}_0^e \cong \mathcal{A}_i^e$ so that $\Theta(\mathcal{S}) \cong \Theta(\mathcal{J})$.

The first key lemma (relativized to \emptyset')

Key Lemma 1

Let $n \in \omega$ and $\{\mathcal{U}_i\}_{i \in \omega}$ be a computable sequence of (partial) \emptyset' -computable graphs. One can effectively construct a \emptyset' -computable function $f(x)$ and a sequence of \emptyset' -computable graphs $\{\mathcal{A}_i\}_{i \in \omega}$ with the following properties:

- (a) If $n \notin \mathcal{O}$, then there exists $i = 1, 2, 3, \dots$ such that $\mathcal{A}_i \cong \mathcal{A}_0$.
- (b) Suppose that $n \in \mathcal{O}$ and the sequences $\{\mathcal{A}_i\}_{i \in \omega}$ and $\{\mathcal{U}_i\}_{i \in \omega}$ contain precisely the same isomorphism types. Then every $i = 1, 2, 3, \dots$ satisfies $\mathcal{A}_i \not\cong \mathcal{A}_0$. In addition, there exists a limit $k = \lim_s f(s)$, and $\mathcal{A}_0 \cong \mathcal{U}_k$.

The first key lemma after jump inversion

Key Lemma 1

Let $n \in \omega$ and $\{\Psi(\mathcal{U}_i)\}_{i \in \omega}$ be a computable sequence of (partial) 1-decidable graphs. One can effectively construct a \emptyset' -computable function $f(x)$ and a sequence of computable graphs $\{\Psi(\mathcal{A}_i)\}_{i \in \omega}$ with the following properties:

- (a) If $n \notin \mathcal{O}$, then there exists $i = 1, 2, 3, \dots$ such that $\Psi(\mathcal{A}_i) \cong \Psi(\mathcal{A}_0)$.
- (b) Suppose that $n \in \mathcal{O}$ and the sequences $\{\Psi(\mathcal{A}_i)\}_{i \in \omega}$ and $\{\Psi(\mathcal{U}_i)\}_{i \in \omega}$ contain precisely the same isomorphism types. Then every $i = 1, 2, 3, \dots$ satisfies $\Psi(\mathcal{A}_i) \not\cong \Psi(\mathcal{A}_0)$. In addition, there exists a limit $k = \lim_s f(s)$, and $\Psi(\mathcal{A}_0) \cong \Psi(\mathcal{U}_k)$.

The second key lemma (relativized to \emptyset')

Consider the following computable injection structures:

- ▶ For a non-zero $\ell \leq \omega$, \mathcal{X}_ℓ contains infinitely many cycles of size m , for each $m < 1 + \ell$.
- ▶ The structure \mathcal{Y}_ℓ is a disjoint union of \mathcal{X}_ℓ and an ω -chain.

Put $K = \{\mathcal{X}_\ell, \mathcal{Y}_\ell : 1 \leq \ell \leq \omega\}$.

Key Lemma 2

Given a computable sequence of (total) \emptyset' -computable unary algebras $\{\mathcal{V}_i\}_{i \in \omega}$ and an \emptyset' -computable function $f(x)$, one can effectively construct a \emptyset' -computable unary algebra \mathcal{D} with the following properties:

- (1) $\mathcal{D} \cong \mathcal{Z}$ for some $\mathcal{Z} \in K$, and
- (2) if the limit $k = \lim_s f(s)$ exists, then $\mathcal{D} \not\cong \mathcal{V}_k$.

The second key lemma after jump inversion

Consider the following computable injection structures:

- ▶ For a non-zero $\ell \leq \omega$, \mathcal{X}_ℓ contains infinitely many cycles of size m , for each $m < 1 + \ell$.
- ▶ The structure \mathcal{Y}_ℓ is a disjoint union of \mathcal{X}_ℓ and an ω -chain.

Put $K = \{\mathcal{X}_\ell, \mathcal{Y}_\ell : 1 \leq \ell \leq \omega\}$.

Key Lemma 2

Given a computable sequence of 1-decidable structures $\{\Phi(\mathcal{V}_i)\}_{i \in \omega}$ and an \emptyset' -computable function $f(x)$, one can effectively construct a computable structure $\Phi(\mathcal{D})$ with the following properties:

- (1) $\mathcal{D} \cong \mathcal{Z}$ for some $\mathcal{Z} \in K$, and
- (2) if the limit $k = \lim_s f(s)$ exists, then $\Phi(\mathcal{D}) \not\cong \Phi(\mathcal{V}_k)$.

Thank you!