ORTHOGONAL POLYNOMIALS

FINITE BAND JACOBI MATRICES AND RANDOM MATRICES

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OUTLINE

- Polynomials orthogonal on a system σ of intervals
- Polynomials orthogonal with respect to the varying weights e^{-nV}
- Links between the asymptotics and the limiting Jacobi matrices J_σ and J_V
- Fluctuation laws for linear eigenvalue statistics of random matrices

1 Generalities

Given $\sigma \in \mathbb{R}$ consider $w : \sigma \to \mathbb{R}_+$ and $L^2(\sigma, wd\lambda)$ to obtain orthogonal polynomials

$$\{P_l(\lambda)\}_{l\in\mathbb{Z}_+}, \ \int_\sigma P_l(\lambda)P_m(\lambda)w(\lambda)d\lambda = \delta_{l,m}, \ \psi_l = w^{1/2}P_l$$

and the corresponding Jacobi matrices

$$\lambda\psi_l(\lambda) = r_l\psi_{l+1}(\lambda) + s_l\psi_l(\lambda) + r_{l-1}\psi_{l-1}(\lambda), \ l \geq 0, \ r_{-1} = 0$$

or

$$\lambda(\Psi(\lambda))_l = (J\Psi(\lambda))_l, \quad \Psi(\lambda) = \{\psi_l(\lambda)\}_{l\in\mathbb{Z}_+}$$

J is a symmetric operator in $l^2(\mathbb{Z}_+)$. It is selfadjoint under certain conditions and then σ is its spectrum, $\{\Psi(\lambda)\}_{\lambda\in\sigma}$ is its complete family of generalized eigenfunctions and

$$\mathcal{E}_J(d\lambda) = \{(\mathcal{E}_{J_\sigma}(d\lambda))_{lm}\}_{l,m=\mathbb{Z}_+}$$

is its resolution of identity, where

$$(\mathcal{E}_J(d\lambda))_{lm} = e_{lm}(\lambda)d\lambda,\ e_{lm}(\lambda) = \chi_\sigma(\lambda)\psi_l(\lambda)\psi_m(\lambda).$$

see e.g. *N.I.Akhiezer, Classical Moment Problem, Haffner, N. Y., 1965*

2 Polynomials orthogonal on several intervals

2.1 Asymptotics

Asymptotics is a kind of ”scattering theory” or/and ”semiclassical approximation”. Consider

$$-\infty < a_1 < b_1 < ... < a_q < b_q < \infty,$$

$$\sigma = \bigcup_{l=1}^{q} [a_l, b_l], \quad w, \quad \log w \in L^1(\sigma, d\lambda)$$

and corresponding orthogonal polynomials.

The case $q = 1$. Set $[a_1, b_1] = [-1, 1],\ \lambda = \cos \pi\nu(\lambda)$, then (*Szego, Bernstein, Akhiezer 30'-40'*)

$$\psi_n(\lambda) = \sqrt{\frac{2}{\pi \sin \nu(\lambda)}} \cos\big(\pi n \nu(\lambda) + \gamma(\lambda)\big) + o(1),\ n \to \infty,$$

and also

$$r_n = 1/2 + o(1),\ s_n = o(1),\ n \to \infty,$$

i.e., J is asymptotically the one-dimensional discrete Laplacian.

The case $q \geq 2$. Consider the set $\mathcal{M}_1(\sigma)$ of non-negative measures of unit mass on σ and their logarithmic energy functional

$$\mathcal{E}_\sigma[m] = -\int_\sigma \int_\sigma \log|\lambda - \mu| m(d\lambda) m(d\mu),\ m \in \mathcal{M}_1(\sigma).$$

The functional has a unique minimizer ν:

$$\min_{m \in \mathcal{M}_1(\sigma)} \mathcal{E}_\sigma[m] = \mathcal{E}_\sigma[\nu].$$

This is a standard variational problem of potential theory, that admits a simple electrostatic interpretation in which m is a distribution of positive charges on a conductor σ and ν is the equilibrium distribution of charges.

Set

$$\nu(\lambda) = \nu((\lambda, \infty)),\ \alpha = \{\alpha_l\}_{l=1}^{q-1} \in \mathbb{R}^{q-1},\ \alpha_l = \nu(a_{l+1}).$$

Then there exist (*Akhiezer-Tomchuk 60', Widom 69, Aptekarev 84, Peherstorfer-Yuditski 03*):

(i) continuous $\mathcal{D}_\sigma : \sigma \times \mathbb{T}^{q-1} \to \mathbb{R}_+,\ \mathcal{G}_\sigma : \sigma \times \mathbb{T}^{q-1} \to \mathbb{R}$ such that if $\lambda \in \mathrm{int}\ \sigma$, then ($a_n \simeq b_n \Leftrightarrow a_n = b_n + o(1),\ n \to \infty$)

$$\psi_n(\lambda) \simeq (2\mathcal{D}_\sigma(\lambda, n\alpha))^{1/2} \cos\Big(\pi n \nu(\lambda) + \mathcal{G}_\sigma(\lambda, n\alpha)\Big)$$

where the remainder vanishes in the $L^2(\sigma)$-norm as $n \to \infty$, and

$$n\alpha = (n\alpha_1, ...n\alpha_{q-1});$$

(ii) continuous $\mathcal{R}_\sigma : \mathbb{T}^{q-1} \to \mathbb{R}_+,\ \mathcal{S}_\sigma : \mathbb{T}^{q-1} \to \mathbb{R}$, such that

$$r_n \simeq \mathcal{R}_\sigma(n\alpha),\ s_n \simeq \mathcal{S}_\sigma(n\alpha),\ n \to \infty.$$

Functions $\mathcal{D}_\sigma, \mathcal{G}_\sigma, \mathcal{R}_\sigma$, and $\mathcal{S}_\sigma$ can be expressed via the $(q-1)$ -dimensional Riemann theta-function associated with the two-sheeted Riemann surface. The surface is obtained by gluing together two copies of the complex plane slit along the ”gaps”

$$(b_1, a_2), \ldots, (b_{q-1}, a_q), (b_q, a_1)$$

of the support of the measure ν, the last gap goes through the infinity.

The components of $\alpha = \{\alpha_l\}_{l=1}^{q-1}$ are rationally independent ”generically” in σ, thus the sequences

$$\{\mathcal{D}_\sigma(\lambda, n\alpha)\}_{n\in\mathbb{Z}},\ \{\mathcal{G}_\sigma(\lambda, n\alpha)\}_{n\in\mathbb{Z}}$$

for any fixed λ and the sequences

$$\{\mathcal{R}_\sigma(n\alpha)\}_{n\in\mathbb{Z}},\ \{\mathcal{S}_\sigma(n\alpha)\}_{n\in\mathbb{Z}}$$

are quasiperiodic in n.

An early precursor (*Akhiezer 33*): if σ consists of two intervals, then a certain characteristic of corresponding extremal polynomials of degree n can be expressed via the Jacobi elliptic functions as $n \longrightarrow \infty$, hence the characteristic does not converge as $n \longrightarrow \infty$ but has a set of limit points that fill a specific interval generically in σ.

2.2 Limiting quasiperiodic Jacobi matrices

Introduce the double-infinite Jacobi matrix $J_{\sigma,n}$, setting

$$r_{k,n} = \begin{cases} r_{k+n}, & k \geq -n, \\ 0, & k < -n, \end{cases} \qquad s_{l,n} = \begin{cases} s_{k+n}, & k \geq -n, \\ 0, & k < -n. \end{cases}$$

We denote by the same symbol $J_{\sigma,n}$ the selfadjoint operator in $l^2(\mathbb{Z})$, defined by the matrix.

Assume that the components of $\alpha \in \mathbb{R}^{q-1}$ are rationally independent. Then for any $x = (x_1, ..., x_{q-1}) \in \mathbb{T}^{q-1}$ there exists $\{n_i(x)\}_{i\geq 1}$, such that

$$\lim_{i\to\infty}\{n_i(x)\alpha_l\} = x_l,\ l = 1, ..., q-1,$$

where $\{t\}$ is the fractional part of $t \in \mathbb{R}$. We have for any $k \in \mathbb{Z}$:

$$\lim_{i\to\infty} r_{n_i(x)+k} = \mathcal{R}_\sigma(k\alpha + x),\ \lim_{i\to\infty} s_{n_i(x)+k} = \mathcal{S}_\sigma(k\alpha + x),$$

i.e., the selfadjoint operators $\{J_{\sigma,n_i(x)}\}_{i\geq 1}$ converge strongly to the operator in $l^2(\mathbb{Z})$ defined by the double - infinite Jacobi matrix $J_\sigma(x)$

with coefficients

$$\mathcal{R}_\sigma(k\alpha + x), \quad \mathcal{S}_\sigma(k\alpha + x),\ k \in \mathbb{Z}.$$

The matrices $J_\sigma(x)$ arise in spectral theory and integrable systems and known as finite band Jacobi matrices (*Novikov 70', Date-Tanaka 70', Aptekarev 80'*).

Set $\Psi_j(\lambda, x) = e^{i\pi\nu(\lambda)j} u_j(\lambda, x)$,

$$u_j(\lambda, x) = \mathcal{U}(\lambda, j\alpha + x),\ \mathcal{U}(\lambda, x) = \mathcal{D}_\sigma^{1/2}(\lambda, x) e^{i\mathcal{G}_\sigma(\lambda, x)},$$

then for $\lambda \in \operatorname{int} \sigma$ the sequences $\{\Psi_j(\lambda, x)\}_{j\in\mathbb{Z}},\ \{\overline{\Psi_j(\lambda, x)}\}_{j\in\mathbb{Z}}$ are the "quasi-Bloch" generalized eigenfunctions of the "limiting" selfadjoint operator $J_\sigma(x)$, acting in $l^2(\mathbb{Z})$.

2.3 Meanings of ν

(i) mathematical physics: quasi-"quasi"-momentum of J_σ;

(ii) spectral theory (ergodic operators): integrated density of states of J_σ;

(iii) analysis:

(a) harmonic measure for σ,

(b) limiting normalized (contracted) zero distribution of P_n as $n \longrightarrow \infty$.

3 Orthogonal polynomials with respect to varying weights

3.1 Definition and asymptotics

Consider $V : \mathbb{R} \to \mathbb{R}_+, \quad \lim_{|\lambda|\to\infty} V(\lambda)/\log(\lambda^2+1) = \infty$ and polynomials $\{P_l^{(n)}\}_{l\in\mathbb{Z}_+}$ orthogonal on $\mathbb{R}$ with respect to $w_n = e^{-nV}$. Set for $m \in \mathcal{M}_1(\mathbb{R})$ (*Gonchar-Rakhmanov 80', Mhaskar-Saaf-Totik 80'*)

$$\mathcal{E}_V[m] = -\int\int \log|\lambda-\mu| m(d\lambda)m(d\mu) + \int V(\lambda)m(d\lambda).$$

The functional has a unique minimizer N:

$$\min_{m \in \mathcal{M}_1(\mathbb{R})} \mathcal{E}_V[m] = \mathcal{E}_V[N].$$

The variational problem determines both the (compact) support σ_V of the measure and the form of the measure. Its meanings:

(i) integrated density of states of certain random matrices,

(ii) limiting zero distribution of $P_n^{(n)}$ (*Gonchar-Rakhmanov 84,...,* *P.-Shcherina 97*).

$\sigma_V = [a, b]$ if V is convex and $\sigma_V = \bigcup_{l=1}^{q} [a_l, b_l],\ 1 \le q < \infty$ if V is real analytic. Set

$$N(\lambda) = N((\lambda, \infty)),\ \beta = \{\beta_l\}_{l=1}^{q-1} \in \mathbb{R}^{q-1},\ \beta_l = N(a_{l+1}).$$

Then (*Deift et al 99*, see also *David et al 00*) there exist:

(i) continuous $\mathcal{D}_V : \sigma_V \times \mathbb{T}^{q-1} \to \mathbb{R}_+$, $\mathcal{G}_V : \sigma_V \times \mathbb{T}^{q-1} \to \mathbb{R}$ such that if $\lambda \in \mathrm{int}\ \sigma_V$, then

$$\begin{aligned} \psi_n^{(n)}(\lambda) &\simeq (2\mathcal{D}_V(\lambda, n\beta))^{1/2} \\ &\times \cos\Big(\pi n N(\lambda) + \mathcal{G}_V(\lambda, n\beta)\Big), \end{aligned}$$

where $n\beta = (n\beta_1, ..., n\beta_{q-1})$; analogous asymptotics for $\psi_{n-1}^{(n)}$; if $\lambda \in \mathrm{ext}\ \sigma_V$, then $\psi_n^{(n)}$ decays exponentially in n as $n \to \infty$; analogous asymptotics for $\psi_{n-1}^{(n)}$;

(ii) continuous $\mathcal{R}_V : \mathbb{T}^{q-1} \to \mathbb{R}_+$, $\mathcal{S}_V : \mathbb{T}^{q-1} \to \mathbb{R}$ such that

$$r_n^{(n)} \simeq \mathcal{R}_V(n\beta),\ s_n^{(n)} \simeq \mathcal{S}_V(n\beta),\ n \to \infty.$$

3.2 Limiting quasiperiodic Jacobi matrices

The above asymptotics are for $l = n - 1, n$. To construct an analog of J_σ we need them for $l = n + k,\ k = O(1)$. Replace V by $V/g,\ g > 0$ keeping V fixed and varying g. Thus orthonormal polynomials and related quantities will depend on g, e.g.

$$\sigma_{V/g} = \bigcup_{l=1}^{q} [a_l(g), b_l(g)].$$

Assume again that the frequency vector β has independent components, a generic case in g (*Kuijlaars-McLaughlin 00*).

Introduce $\{n_i(x)\}$:

$$\lim_{i\to\infty}\{n_i(x)\beta_l\} = x_l,\ l = 1, .., q-1,\ x = \{x_l\}_{l=1}^{q-1} \in \mathbb{T}^{q-1} :$$

and write

$$n\frac{V}{g} = (n+k)\frac{V}{g(1+k/n)},$$

and

$$r_{n+k}^{(n)}(g) = r_{n+k}^{(n+k)}(g(1+k/n)).$$

Then the above asymptotics imply

$$\lim_{i\to\infty} r^{(n_i(x))}_{n_i(x)+k}(g)$$

$$= \lim_{i\to\infty} \mathcal{R}_V \left(\frac{n_i(x)+k}{n_i(x)} g, (n_i(x)+k)\beta\left(\frac{n_i(x)+k}{n_i(x)} g\right)\right)$$

$$= \mathcal{R}_V \left(g, k\widetilde{\alpha}(g) + x\right).$$

where

$$\widetilde{\alpha}(g) := (g\beta(g))'$$

. Analogously:

$$\lim_{i\to\infty} s^{(n_i(x))}_{n_i(x)+k}(g) = \mathcal{S}_V \left(g, k\widetilde{\alpha}(g) + x\right).$$

$$\alpha,\ \beta \quad \leftrightarrow \quad \widetilde{\alpha}?$$

Consider the variational problem for the logarithmic energy for $\sigma = \sigma_{V/g}$ and denote ν_g its solution. We have (*P. 96, Buyarov-Rakhmanov 99*)

$$N_g = g^{-1} \int_0^g \nu_{g'} dg'.$$

”Adiabatic” regime: spectral theory (*Buslaev 80’*), applied mathematics (*Whitham 70’*), integrable systems (*Kriechever-Novikov 80’*).

The formula implies

$$\widetilde{\alpha}(g) = \alpha(g),$$

We obtain:

(i) the strong limit $J_{V/g}(x)$ of $J_{V/g,n}$ along $\{n_i(x)\}$ (analog of $J_{\sigma,n}$) and $J_{V/g}(x)$ has the same frequency vector as J_{σ_g},

(ii) the coefficients $\mathcal{D}_{\sigma_g},\ \mathcal{G}_{\sigma_g},\ \mathcal{R}_{\sigma_g}$, and $\mathcal{S}_{\sigma_g}$ differ from $\mathcal{D}_{V/g},\ \mathcal{G}_{V/g},\ \mathcal{R}_{V/g}$, and $\mathcal{S}_{V/g}$, by a shift in x,

(iii) the spectra of (generically) quasiperiodic matrices $J_{V/g}(x)$ and J_{σ_g} coincide.

$J_{V/g}(x)$ is "tangent" to $J_{V/g,n}(x)$ at $l = n$

3.3 Periodic Jacobi matrices

Let v be a monic polynomial of degree q with real coefficients, $g > 0$ be such that all zeros of $v^2 - 4g$ are real and simple. Set

$$V(\lambda) = \frac{v^2(\lambda)}{2q}.$$

Then $J_{\sigma_g}(x)$ and $J_{V/g}(x)$ are q-periodic, their spectrum is

$$\sigma_g = \{\lambda : v^2(\lambda) - 4g \leq 0\},$$

and their Integrated Density of States is (*Buslaev-P. 02*):

$$\nu_g(d\lambda) = d_g(\lambda)d\lambda,\ d_g(\lambda) = \frac{|v'(\lambda)|}{\pi q}|v^2(\lambda) - 4g|^{-1/2}\chi_{\sigma_g}(\lambda).$$

One can use a representation important in the inverse problem for periodic operators of second order *Marchenko-Ostrovsky 86*, cf polynomial image

Let u be a polynomial of degree q with real coefficients and such that all zeros of $u^2 - 1$ are real and simple. Then

$$u(z) = \cos\theta(z),$$

where θ is the conformal map of the open upper half-plane onto (comb)

$$\{\theta \;:\; q_1\pi < \Re\theta < q_2\pi,\ \Im\theta > 0\} \setminus \bigcup_{q_1<l<q_2,} \{\theta : \Re\theta = l\pi,\ q_1 < l < q_2,\ 0 < \Im\theta \le h_l\},$$

Here $q_1 < q_2$ are integers, $q_2 - q_1 = q$, $0 \leq h_l < \infty$ and $\theta(\infty) = \infty$. θ is the analytic continuation of the quasimomentum.

Set $\theta_+(\lambda) = \Re\theta(\lambda + i0)$, then

$$\nu_g(\lambda) = \frac{1}{\pi q}\theta_+(\lambda).$$

and

$$N_g(\lambda) = \frac{1}{\pi q}\left(\theta_+(\lambda) - \frac{\sin\theta_+(\lambda)}{2}\right).$$

4 Eigenvalue distribution of random matrices

4.1 Generalities

Consider $n \times n$ Hermitian random matrices

$$M_n = \{M_{jk} \in \mathbb{C},\ M_{kj} = \overline{M_{jk}}\}_{j,k=1}^{n},$$

whose probability law is

$$P(dM_n) = Z_n^{-1} \exp\{-n\mathrm{Tr}V(M_n)\}dM_n.$$

Here Z_n is the normalization constant, $V : \mathbb{R} \to \mathbb{R}_+$ is continuous, bounded below, and growing at least logarithmically at infinity and

$$dM_n = \prod_{j=1}^{n} dM_{jj} \prod_{1 \le j < k \le n}^{n} d\Re M_{jk} d\Im M_{jk}.$$

Let

$$\mathcal{N}_n[\varphi] = \sum_{l=1}^{n} \varphi(\lambda_l^{(n)}).$$

be a linear eigenvalue statistics, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a test function.

Problems:

(i) $\lim_{n\to\infty} n^{-1}\mathcal{N}_n$, selfaveraging, ”LLN”

(ii) $\mathbf{Var}\{\mathcal{N}_n\}$, fluctuations, ”CLT”

It is known that if N is a unique minimizer of the minimum energy problem in the external field V, then for $V' \in Lip_{loc}1$ we have in probability in weak sense

$$\lim_{n\to\infty} n^{-1}\mathcal{N}_n[\varphi] = \int \varphi(\lambda)N(d\lambda),$$

i.e., the ”LLN” *(Wigner 52, Brezin et al 79, A. Boutet de Monvel - P. - Shcherbina 95, Deift et al 98, Johansson 98)*

see e.g. *L. Pastur, M. Shcherbina ”Eigenvalue Distribution of Large Random Matrices”, AMS, 2011.*

4.2 Link with orthogonal polynomials

Consider polynomials $\{P_l^{(n)}\}_{l\geq 0}$ with varying weigh e^{-nV} and the reproducing kernel of $\{\psi_l^{(n)}\}_{l\geq 0},\ \psi_l^{(n)} = e^{-nV/2}P_l^{(n)}$:

$$K_n(\lambda,\mu) = \sum_{l=0}^{n-1} \psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu).$$

Then:

(i) $\mathbf{E}\{n^{-1}\mathcal{N}_n[\varphi]\} = \int_{\mathbb{R}} \varphi(\lambda)\rho_n(\lambda)d\lambda,\ \rho_n(\lambda) = n^{-1}K_n(\lambda,\lambda),$

Mehta-Godin 60'

(ii) $\mathbf{Var}\{\mathcal{N}_n[\varphi]\} = \int_{\mathbb{R}^2} (\Delta\varphi)^2 K_n^2(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2,$

where

$$\Delta\varphi := \varphi(\lambda_1) - \varphi(\lambda_2),$$

P.-Shcherbina 97

(iii) $Z_n[\varphi] = \mathbf{E}_V\left\{e^{-\mathcal{N}_n[\overline{\varphi}]}\right\} = e^{\Phi_n[\varphi]},\ \overline{\varphi} := \varphi - \mathbf{E}_n\{\varphi\}$

where

$$\Phi_n[\varphi] = \int_0^1 (1-s)\mathbf{Var}_{V+s\varphi/n}\{\mathcal{N}_n[\varphi]\} ds,$$

(free energy via susceptibility) *P. 06*

"$1/n$" perturbation of potential!

4.3 "CLT"

(i) By asymptotic formulas for $\psi^{(n)}_{n+k},\ k=0,-1$ we have for $\varphi \in C^1$ as $n \to \infty$:

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \simeq \int\int \Big(\frac{\Delta\varphi}{\Delta\lambda}\Big)^2 \mathcal{V}(\beta n, \lambda_1, \lambda_2) d\lambda_1 d\lambda_2 := \mathcal{V}_\varphi(n\beta),$$

$$\begin{aligned}\mathcal{V}(x,\lambda_1,\lambda_2) &= \mathcal{R}^2(x)\big(e_{0,0}(\lambda_1,x)e_{-1,-1}(\lambda_2,x) \\ &\quad - e_{0,-1}(\lambda_1,x)e_{-1,0}(\lambda_2,x)\big),\end{aligned}$$

where $\mathcal{E}_{j,k}(d\lambda,x) = e_{j,k}(\lambda,x)d\lambda$ is the resolution of identity of $J_V(x)$.

(ii)

$$\Phi_x[\varphi] := \lim_{n_j(x)\to\infty} \log Z_{n_j(x)}[\varphi] = \int_0^1 (1-s)\mathcal{V}_\varphi(x + s\dot{\beta}[\varphi])ds,$$

where

$$\dot{\beta} = \frac{\delta\beta}{\delta V}.$$

According to (i) $\mathcal{V}_\varphi$ is a quadratic functional of φ. Hence Φ_x is not quadratic in general in φ because of the term $s\dot{\beta}[\varphi]$ in the argument of the integrand. Thus we have here a ”limiting law”, that is not necessarily Gaussian.

4.4 Example

$\varphi = t\lambda,\ t \in \mathbb{R}$, i.e., $\mathcal{N}_n = t\big(\lambda_1^{(n)} + \ldots + \lambda_n^{(n)}\big)$,
$\mathrm{supp} N = [-b, -a] \cup [a, b]$, hence the 2-periodic variance

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \simeq t^2(b^2 + a^2 - 2(-1)^n ab)/4,$$

and

$$\begin{aligned}
\Phi &= \frac{d_0 t^2}{2} + A(x + \alpha_1 t) - A(x) - tA'(x),\ \{x = n/2\} \\
d_0 &= \frac{a^2 + b^2}{4} \neq \text{"}\lim\text{"}_{n\to\infty} \mathbf{Var}\{\mathcal{N}_n\}, \\
A(x) &= \sum_{m \in \mathbb{Z}\setminus\{0\}} d_m (2\pi i \alpha_1 m)^{-2} e^{2\pi i m x}
\end{aligned}$$

where

$$\alpha_1 = \alpha[\varphi]|_{\varphi=\lambda} = ta\,/K(a/b)\ ,$$

and K is the complete elliptic integral of first kind.

Thus α_1 is irrational for almost all (a, b) and Φ is not quadratic in t.

Conclusion: There are two limiting laws (for $n \longrightarrow \infty$ along even and odd n's), both non-Gaussian.