Higher-Order Three-Term Recurrences and Asymptotics of Multiple Orthogonal Polynomials

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A. I. Aptekarev, V. A. Kalyagin, and E. B. Saff :

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Definition, motivation

- We consider families of polynomials defined by recurrences

$$Q_{n+1} = zQ_n - a_{n-p+1}Q_{n-p}, \quad n \geq p, \quad z \in \mathbb{C}, \quad p \in \mathbb{N},$$

with initial conditions

$$Q_j(z) = z^j, \quad j = 0, 1, \ldots, p\,.$$

- Main motivations:
  - *difference operators* of order $p+1$
  - *Hermite-Padé* rational approximants of a vector of analytic functions

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Generalization of Tchebychev, Faber polynomials

- the constant coefficients case,

$$a_n \; := \; a > 0\,, \qquad n = 1, 2, \ldots,$$

Eiermann-Varga (1993) and He-Saff (1994)

- Properties of zeros –> Star-like set

$$S_0 := [0, \alpha] \cup [0, \alpha_1] \cup [0, \alpha_2] \ldots \cup [0, \alpha_p]$$

with end points

$$\mathcal{A} := \{\alpha,\; \alpha_1 := \varepsilon^{(1)}_{(p+1)}\alpha,\; \alpha_2 := \varepsilon^{(2)}_{(p+1)}\alpha, \quad \ldots, \alpha_p := \varepsilon^{(p)}_{(p+1)}\alpha\},$$

where $\alpha := \alpha(a, p)$ and $\varepsilon^{(k)}_{(p+1)}$ are roots of unity

$$\varepsilon^{(k)}_{(p+1)} := \exp(k2\pi i/(p+1)),\;\; k = 1, \ldots, p$$

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Generalization of Stieltjes continued fractions

- Multiple-Orthogonal polynomials –> Favard-type theorem
- Theorem (Aptekarev, Kalyagin and Van Iseghem - 2000)

$$a_n > 0, \quad n \geq 1, \qquad \Rightarrow \qquad \{\mu_j\}_{j=1}^{p} \ :$$

- multiple-orthogonal (non-Hermitian) relations:

$$\int_S Q_n(t) t^r d\mu_j(t) = 0, \quad r = 0, 1, \ldots, k, \ \ j = 1, 2, \ldots, d \ ,$$

$$\int_S Q_n(t) t^r d\mu_j(t) = 0, \quad r = 0, 1, \ldots, k-1, \ \ j = d+1, \ldots, p \ ,$$

where $n = kp + d$, $0 \leq d \leq (p-1)$,

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Bogoyavlenskii lattice

- $$\dot{a}_n = a_n\Big(\sum_{k=1}^{p} a_{n+k} - \sum_{k=1}^{p} a_{n-k}\Big), \qquad n \geq 1, \quad a_{-n} = 0, \qquad n \geq 0,$$

- Inverse problem method, existence of global solution, ...

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Our goals

- Direct spectral problems

$$\{a_n\} \qquad \to \qquad \{Supp(\mu_j)\}_{j=1}^{p}$$

and Asymptotics of Q_n as $n- > \infty$
and Structure of $\{\mu_j\}_{j=1}^{p}$.

- the *Blumenthal-Nevai* class

$$\lim_{n\to\infty} a_n = a(> 0)\,.$$

- *Weyl-type* theorem characterizing $\{Supp(\mu_j)\}_{j=1}^{p}$.
- l^1 *perturbation* class

$$\sum_{n=1}^{\infty} |a_n - a| < \infty,$$

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Algebraic curve, the constant rec.coeff. case

- $$w(z) := \{w_j(z)\}_{j=0}^{p} \quad : \quad w^{p+1} - zw^p + a = 0.$$
- Structure of the Riemann surface
$$w_0 \in \mathrm{H}(\Omega_0), \quad \Omega_0 := \mathbb{C} \setminus S_0, \quad w_0(z) = z + \cdots \ ,$$
$$w_j(z) = \mathcal{O}(1/z^{1/p}) \ , \ j = 1, \ldots, p.$$
- The constant rec. coef. case $a_n = a$
$$U_n(z) := Q_n(z)$$
- Vector of the rational Hermite-Padé approximants
$$\left(\frac{U_{n-1}}{U_n}, \frac{U_{n-2}}{U_n}, \ldots, \frac{U_{n-p}}{U_n} \right)$$
for the vector of functions
$$\left(\frac{1}{w_0}, \frac{1}{w_0^2}, \ldots, \frac{1}{w_0^p} \right),$$

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Asymptotics in l_1 class perturbation

- Gelfand-Levitan (comparison) equation

$$Q_n = U_n + \sum_{k=0}^{n-1} (1-a_{k+1}) U_{n-p-1-k} Q_k \,, \quad U_{-1} = \ldots = U_{-p} = 0.$$

- Analog of the Szegő function

$$F_0(z) := \frac{w_0(z)^p \, \Phi_0(z)}{\prod_{k=1}^{p}(w_0 - w_k)} \,, \quad \Phi_0(z) := 1 + \sum_{k=0}^{\infty} \frac{1 - a_{k+1}}{w_0^{p+1}} \cdot \frac{Q_k(z)}{w_0^k(z)}$$

- Asymptotics

$$\lim_{n\to\infty} \frac{Q_n(z)}{w_0^n(z)} = F_0(z) \,, \quad z \in K \subset \Omega_0 := \mathbb{C} \setminus S_0 \,,$$

$$\frac{Q_n(t)}{|w_0(t)|^n} = \left(\frac{w_{0,+}(t)}{|w_0(t)|}\right)^n F_{0,+}(t) + \left(\frac{w_{0,-}(t)}{|w_0(t)|}\right)^n F_{0,-}(t) + o(1) \,,$$

on $t \in \widetilde{K} \subset S_0$.

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Direct "spectral" problem

- Basis of the solutions

$$Y_{n+1} = zY_n - a_{n-p+1} Y_{n-p}, \quad n \geq p \ ,$$

with initial conditions

$$P_n^{(j)} = 0, \ 0 \leq n < j; \quad P_n^{(j)} = z^{n-j}, \ j \leq n \leq p.$$

- Analytic functions in Ω_0

$$\Phi_j(z) := 1 + \sum_{k=0}^{\infty} \frac{1 - a_{k+1}}{w_0^{p+1}} \cdot \frac{P_k^{(j)}(z)}{w_0^{k-j}(z)}, \quad j = 1, 2, \ldots, p \ .$$

- On S_0, measures μ_j are absolutely continuous

$$d\mu_j(t) = \rho_j(t)|dt|, \quad t \in S_0,$$

where

$$\rho_j(t) = \left[\frac{1}{w_0(t)^j} \frac{\Phi_j(t)}{\Phi_0(t)}\right]^+ - \left[\frac{1}{w_0(t)^j} \frac{\Phi_j(t)}{\Phi_0(t)}\right]^-, \quad j = 1, 2, \ldots, p \ .$$

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Nikishin system on the stars

- "Stars" <– the Riemann surface $\mathfrak{R}$ of w

$$S_{j-1} := \pi(\partial\mathfrak{R}_{j-1,j}) \ , \quad j = 1, \dots, p.$$

- Generator of a Nikishin system

$$\{\sigma_j\}_{j=1}^{p} \quad : \quad \operatorname{supp} \sigma_j = S_{j-1} \ , \quad j = 1, \dots, p.$$

- Definition of a *Nikishin system*

$$\{f_j\} \ , \quad f_j \in H(\overline{\mathbb{C}} \setminus S_0) \ , \quad j = 1, \dots, p \ ,$$

if $\{f_j\}$ satisfy BVP on S_0

$$f_{j+} - f_{j-}\Big|_{S_0} = \sigma_1 \cdot f_j^{(2)} \ , \quad j = 1, \dots, p \ ,$$

where

$$f_1^{(2)} \equiv 1 \ , \quad f_j^{(2)} \in H(\overline{\mathbb{C}} \setminus S_1) \ , \quad j = 2, \dots, p \ ,$$

and the functions $\{f_j^{(2)}\}_{j=2}^{p}$ in their turn form the Nikishin system with respect to $\{\sigma_k\}_{k=2}^{p}$ on $\{S_{k-1}\}_{k=2}^{p}$.

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Nikishin's Hierarchy

Nikishin system starting from $f_j^{(1)} := f_j\ ,\ j = 1, \ldots, p$, defines a hierarchy of analytic functions

$$\{f_j^{(k)}\}\ , \quad k = 1, \ldots, p\ , \quad j = k, \ldots, p\ ,$$

by means of analytic continuation of their jumps

$$f_{j+}^{(k)} - f_{j-}^{(k)}\Big|_{S_{k-1}} = \sigma_k \cdot f_j^{(k+1)}\ ,$$

from S_{k-1} to the whole domain of their holomorphicity :

$$f_k^{(k+1)} \equiv 1\ , \quad f_j^{(k+1)} \in H(\overline{\mathbb{C}} \setminus S_k)\ , \quad k = 1, \ldots, p\ , \quad j = k, \ldots, p\ .$$

Weights of Multiple orthogonality and Nikishin system

- Multiple orthogonality measure:

$$d\mu_j(t) = \rho_j(t)|dt|, \quad t \in S_0,$$

where

$$\rho_j(t) = \left[\frac{1}{w_0(t)^j}\frac{\Phi_j(t)}{\Phi_0(t)}\right]^+ - \left[\frac{1}{w_0(t)^j}\frac{\Phi_j(t)}{\Phi_0(t)}\right]^-, \ \ j = 1, 2, \ldots, p \ .$$

- **Theorem.** Consider

$$f_j = \frac{1}{w_0^j}\,\frac{\Phi_j}{\Phi_0}\,, \quad j = 1, \ldots, p\,,$$

then $\{f_j\}_{j=1}^p$ forms a Nikishin system on the stars $\{S_{j-1}\}_{j=1}^p$, with respect to some $\{\sigma_j\}_{j=1}^p$

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$$f_j = \frac{1}{w_0^j}\ \frac{\Phi_j}{\Phi_0}\ , \quad j = 1, \ldots, p\ ,$$

then $\{f_j\}_{j=1}^p$ forms a Nikishin system on the stars $\{S_{j-1}\}_{j=1}^p$. with respect to some $\{\sigma_j\}_{j=1}^p$

BVP on $\mathfrak{R}$ for the Szegő functions

- Asymptotics and the Szegő function (recall)

$$\lim_{n\to\infty} \frac{Q_n}{w_0^n} = F_0 \;, \quad \text{on} \;\; \mathbb{C}\setminus S_0$$

- BVP on $\mathfrak{R}$:

$$1) \;\; \mathcal{F} \in H\left(\mathfrak{R}\setminus \bigcup_{l=1}^{p} \partial\mathfrak{R}_{l-1,l}\right) \;, \; \exists\; \mathcal{F}_\pm \in L^1_{loc}\left(\bigcup_{l=1}^{p} \partial\mathfrak{R}_{l-1,l}\right) \;,$$

$$2) \;\; \mathcal{F}_+ = \mathcal{F}_- \frac{1}{\overset{\circ}{\sigma}_l} \quad \text{on} \quad \partial\mathfrak{R}_{l-1,l} \;, \;\; l = 1,\ldots,p \;,$$

$$3) \;\; \mathcal{F}(\infty^{(0)})\,\mathcal{F}(\infty^{(1)})\cdots\mathcal{F}(\infty^{(p)}) = 1 \;.$$

- The Szegő function

$$F_0 = \mathcal{F}\Big|_{\mathfrak{R}_0} \;.$$

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Functions of the second kind

- BVP for hierarchy of the functions of the second kind

$$\Psi_n^{(0)} = Q_n, \quad \begin{cases} \Psi_n^{(k)} \in H(\overline{\mathbb{C}} \setminus S_{k-1}) \\ \left. \Psi_{n+}^{(k)} - \Psi_{n-}^{(k)} \right|_{S_{k-1}} = \sigma_k \Psi_n^{(k-1)} \\ \Psi_n^{(k)}(z) = O\left(z^{-\frac{n+p-k+1}{p}}\right), \quad z \to \infty \end{cases}, \quad k = 1 \dots p$$

- System of the Szegő function

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Asymptotics of the functions of the second kind

as $n \to \infty$:

$$1)\Psi_n^{(l)} = \frac{w_l^n\, F_l}{\prod\limits_{m=0}^{l-1}(w_m - w_l)}\,(1 + o(1))\,, \quad \text{on} \quad \overline{\mathbb{C}} \setminus \{S_{l-1} \cup S_l\}\,;$$

$$2)\frac{\Psi_n^{(l)}}{|w_l|^n} = \frac{1}{|w_l|^n}\left(\frac{w_l^n\, F_l}{\prod\limits_{m=0}^{l-1}(w_m - w_l)}\right)_+ + \frac{1}{|w_l|^n}\left(\frac{w_l^n\, F_l}{\prod\limits_{m=0}^{l-1}(w_m - w_l)}\right)_- + o($$

on S_l,

$$3)\Psi_{n\pm}^{(l)} = \left(\frac{w_l^n\, F_l}{\prod\limits_{m=0}^{l-1}(w_m - w_l)}\right)_\pm (1 + o(1))\,, \quad \text{on} \quad S_{l-1}\,,$$