CATEGORICITY-LIKE PROPERTIES IN THE FIRST ORDER REALM

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Plan of the talk

- PART 1 provides background definitions and results about interpretability theory.
- PART 2 focuses on categoricity-like features of canonical theories of arithmetic and set theory that come to light when they are viewed within an interpretability-theoretic framework.

Preamble: Interpretations

The notion of (relative) interpretation was first rigorously defined and developed in the 1953 monograph *Undecidable Theories* of Tarski, Mostowski, and Robinson as a tool for establishing undecidability results.

The most obvious and direct use of interpretations is for relative consistency. We now know that the notion of interpretation is fundamental to mathematical logic and the foundations of mathematics, with a much wider significance than consistency and undecidability results.

- Interpretations allow us to 'compare incomparables'.
- Interpretations explicate intuitions of 'reducible to' and 'sameness' of theories and structures.
- Interpretations transfer information and conceptual resources from one theory or structure to another.

H. Friedman, *Interpretations, according to Tarski*, online manuscript (2007).

A. Visser, *Categories of theories and interpretations*, **Logic in Tehran**, Lecture Notes in Logic. Cambridge University Press; 2006:284-341.

Interpretability between structures: Definition

- $A \trianglelefteq B$ is read as: A is interpretable in B, where A and B are both structures, or they are both theories.
- Given first order structures $\mathcal{M} = (M, R, ...)$ and $\mathcal{N} = (N, S, ...)$ (possibly with two different languages),

$$\mathcal{M} \trianglelefteq_{\mathrm{par}} \mathcal{N}$$

  means: $\mathcal{M} \cong (D/E, P, ...)$, where:
  D, E, P, ... are all parametrically first order definable in $\mathcal{N}$,
  $D \subseteq N^k$ for some finite k, and
  E is an equivalence relation on D.

Interpretability between structures: Examples

- $(\{0,1\}, +_{\text{mod } 2}, \cdot_{\text{mod}2}) \trianglelefteq (\mathbb{Z}, +, \cdot)$, since we can define:

$$x{=}y \quad as \quad \exists z[2z = x - y],$$

$$x{+}_{\text{mod2}}y \quad as \quad x + y, \qquad x{\cdot}_{\text{mod2}}y \quad as \quad xy.$$

  But not the other way around by cardinality considerations.

- $(\mathbb{R}, <) \trianglelefteq ([0,1], <)$ since $((0,1), <) \cong (\mathbb{R}, <)$. But not the other way around (by elimination of quantifiers), unless we allow parameters.

- $(\mathbb{C}, +, \cdot) \trianglelefteq (\mathbb{R}, +, \cdot)$ since $(\mathbb{C}, +, \cdot) \cong (\mathbb{R}^2, \oplus, \odot)$, where:

$$(x, y) \oplus (x', y') = (x + x', y + y'),$$

$$(x, y) \odot (x', y') = (xx' - yy', xy' + y'x).$$

  But not the other way around, by considerations involving stability theory, or by using automorphisms.

Interpretability between theories (1)

- Suppose U and V are first order theories formulated in relational languages $\mathcal{L}_U$ and $\mathcal{L}_V$ (respectively). An interpretation $\mathcal{I}$ of U in V, written $U \trianglelefteq^{\mathcal{I}} V$, is given by a translation τ of each $\mathcal{L}_U$-formula φ into an $\mathcal{L}_V$-formula φ^τ, with the requirement that $V \vdash \varphi^\tau$ for each $\varphi \in U$, where τ is determined by an $\mathcal{L}_V$-formula $\delta(x_1, ..., x_k)$ (referred to as a domain formula), and a mapping $P \mapsto_\tau F_P$ that translates each n-ary $\mathcal{L}_U$-predicate P into some kn-ary $\mathcal{L}_V$-formula F_P.
  The translation τ is then lifted to the full first order language in the obvious way by making it commute with propositional connectives, and subject to:

$$(\forall x \varphi)^\tau = \forall x_1, ..., x_n \ (\delta(x_1, ..., x_n) \rightarrow \varphi^\tau).$$

- If $U \trianglelefteq^{\mathcal{I}} V$, then $\mathcal{I}$ gives rise to an internal model construction that uniformly builds a model $\mathcal{M}^{\mathcal{I}} \models U$ for any $\mathcal{M} \models V$.

Interpretability between theories (2)

- U is interpretable in V, written $U \trianglelefteq V$, iff $U \trianglelefteq^{\mathcal{I}} V$ for some interpretation $\mathcal{I}$.
- **Theorem**. *Suppose $U \trianglelefteq V$. Then we have:*
  1. *If V is consistent, then so is U.*
  2. *If U is essentially undecidable (i.e., every extension of U is undecidable), then so is V.*
- We write $U \triangleleft V$ to indicate that $U \trianglelefteq V$, but not vice versa.
- **Fact.** $\mathsf{ACF}_0 \triangleleft \mathsf{RCF} \triangleleft \mathsf{Q} \triangleleft \mathsf{PA} \triangleleft \mathsf{ZF}$.
- **Fact.** $\mathsf{PA} \triangleleft \mathsf{ACA}_0$ and $\mathsf{ZF} \triangleleft \mathsf{GB}$.

Examples of Mutual interpretability

- We say that U and V mutually interpretable iff $U \trianglelefteq V$ and $V \trianglelefteq U$.
- **Theorem.** (1960s) (Cohen, Scott, Solovay)
  1. $\mathsf{ZFC} + \mathsf{CH}$ *is mutually interpretable with* $\mathsf{ZFC} + \neg\mathsf{CH}$.
  2. ZFC *is mutually interpretable with* $\mathsf{ZF} + \neg\mathsf{AC}$.
- **Proof idea**: Relativization to Gödel's constructible universe in one direction, and Boolean valued approach to forcing in the other.
- But the above pairs of theories are NOT bi-interpretable (as we shall see).

More Examples of Mutual interpretability

- **Theorem.** (1940s to now) *The following theories are pairwise mutually interpretable.* (Tarski-Mostowski-Robinson, Szmielew, Quine, Grzegorczyk, Solovay, Nelson, Wilkie, Švejdar, Visser, Ferreira, Damjanovič, ...)

  1. Q (*Robinson arithmetic*).
  2. AST (*Adjunctive set theory, whose axioms consist of* "there is an empty set" and $\forall x \forall y \; x \cup \{y\}$ exists).
  3. TC (Grzegorczyk's elementary theory of concatenation)
  4. S^1_2 (Buss arithmetic).

- **Proof Idea**: Ingenious coding + the method of "shortening cuts".

Arithmetical Completeness Theorem

- **Theorem.** (Hilbert-Bernays 1939, Kleene 1952).
  Suppose T is a consistent r.e. set of first order sentences with no finite models, then $(\omega, R, ...) \models T$, where ω is the set of natural numbers. *where each of* R, ... *is* Δ_2 (*in the arithmetical hierarchy*).

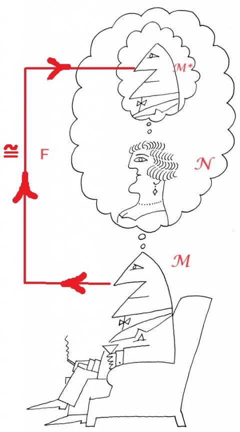
- **Theorem.** (Feferman 1961)
  1. PA is mutually interpretable with PA $+ \neg$Con(PA), but
  2. PA $\lhd$ PA $+$ Con(PA).

- As we shall see, PA and PA $+ \neg$Con(PA) are NOT bi-interpretable.

Strengthenings of mutual interpretability (1)

- U is a *retract* of V iff there are interpretations $\mathcal{I}$ and $\mathcal{J}$ with $U \trianglerighteq^{\mathcal{I}} V$ and $V \trianglerighteq^{\mathcal{J}} U$ (so up to here U and V are mutually interpretable), and a binary $\mathcal{L}_U$-formula F such that the following holds for every $\mathcal{M} \models U$:

$$F^{\mathcal{M}} : \mathcal{M} \xrightarrow{\cong} \mathcal{M}^* := \left(\mathcal{M}^{\mathcal{I}}\right)^{\mathcal{J}} .$$



Strengthenings of mutual interpretability (2)

- U and V are *bi-interpretable* iff there are interpretations $\mathcal{I}$ and $\mathcal{J}$ that witness that U is a retract of V, and additionally, there is a $\mathcal{L}_V$-formula G, such that for all $\mathcal{M} \models U$ and $\mathcal{N} \models V$, we have:
$$G^{\mathcal{N}} : \mathcal{N} \xrightarrow{\cong} \mathcal{N}^* := \left(\mathcal{N}^{\mathcal{J}}\right)^{\mathcal{I}}.$$
- U and V are *definitionally equivalent* iff U and V have a common definitional extension.

Def. Eq. $\Rightarrow$ Bi-int $\Rightarrow$ Mutual Retract $\Rightarrow$ Mutually Interpretable

Examples of Def. Equivalence and Bi-interpretability (1)

- **Theorem.** (J. Robinson 1948)
  $\mathrm{Th}(\mathbb{Q}, +, \cdot)$ *and* $\mathrm{Th}(\mathbb{N}, +, \cdot)$ *are definitionally equivalent*.

- **Theorem.** *Let* $\mathsf{ZF}_{\mathsf{fin}}$ *be the result of replacing the axiom of infinity in* ZF *by its negation, and let* TC *denote the statement that every set has a transitive closure*.
  1. (Ackermann 1940, Mycielski 1964, Kaye-Wong 2007)
     PA *and* $\mathsf{ZF}_{\mathsf{fin}} + \mathsf{TC}$ *are definitionally equivalent.*
  2. (Visser + Schmerl + E. 2011) $\mathsf{ZF}_{\mathsf{fin}}$ *is not even a sentential retract of* PA (but PA is a retract of $\mathsf{ZF}_{\mathsf{fin}}$).

More Examples of Def. Equivalence and Bi-interpretability

- **Theorem.** (Mostowski 1950s)
  1. $Z_2 + \Pi^1_\infty$-AC *is bi-interpretable with* $\mathsf{ZF} \setminus \{\text{Powerset}\} + \forall x\ |x| \leq \aleph_0$.
  2. $\mathsf{KM} + \Pi^1_\infty$-AC *is bi-interpretable with* $\mathsf{ZF} \setminus \{\text{Powerset}\} + \exists\kappa(\mathrm{Inacc}(\kappa) \wedge\ \forall x\ \ |x| \leq \kappa)$.
- As recently established in the following paper, in the above theorem of Mostowski, "bi-interpretable" cannot be improved to "definitionally equivalent". The proof uses various tools, including Solovay's construction of a model of ZFC in which every projective set is measurable.

A. Enayat and M. Łełyk, *Categoricity-like properties in the first-order realm*, available via researchgate platform.

END OF PART 1

Preamble: Categoricity gained and lost

- **Dedekind Categoricity Theorem.** (1888) *There is a sentence σ in second order logic of the form $\forall X\varphi(X)$, where $\varphi(X)$ only has first order quantifiers, such that σ holds in a structure $\mathcal{M}$ iff $\mathcal{M} \cong (\omega, S, 0)$, where S is the successor function.*
- **Zermelo Quasi-categoricity Theorem.** (1930) *There is a sentence θ in second order logic of the form $\forall X\psi(X)$, where $\psi(X)$ only has first order quantifiers, such that θ holds in a structure $\mathcal{M}$ iff $\mathcal{M} \cong (V_\kappa, \in)$, where κ is a strongly inaccessible cardinal.*
- **Skolem Anti-categoricity Theorems.**
  1. (1934) *There is a structure $\mathcal{M}$ whose first order theory coincides with the first order theory of $(\mathbb{N}, +, \cdot)$, but $\mathcal{M}$ contains an 'infinite' element. Thus $\mathcal{M} \not\cong (\omega, +, \cdot)$, but they share the same first order theory.*
  2. (1922) *For any inaccessible cardinal κ there is a countable structure $\mathcal{M}$ whose first order theory coincides with the first order theory of $(V_\kappa, \in)$. Thus $\mathcal{M} \not\cong (V_\kappa, \in)$, but they share the same first order theory.*

Solidity

- T is *solid* iff the following holds for all models $\mathcal{M}$, $\mathcal{M}^*$, and $\mathcal{N}$ of T:

  If $\mathcal{M} \trianglerighteq_{\mathrm{par}} \mathcal{N} \trianglerighteq_{\mathrm{par}} \mathcal{M}^*$ and there is a parametrically $\mathcal{M}$-definable somorphism $i_0 : \mathcal{M} \to \mathcal{M}^*$, then there is an $\mathcal{M}$-definable isomorphism $i : \mathcal{M} \to \mathcal{N}$.

- Intuitively: T is solid if for all models $\mathcal{M}$ and $\mathcal{N}$ of T we have:

$$\mathcal{M} \trianglerighteq_{\mathrm{par}} \mathcal{N} \trianglerighteq_{\mathrm{par}} \mathcal{M} \implies \mathcal{M} \models \text{“I am isomorphic to } \mathcal{N}\text{”}.$$

Consequences of solidity

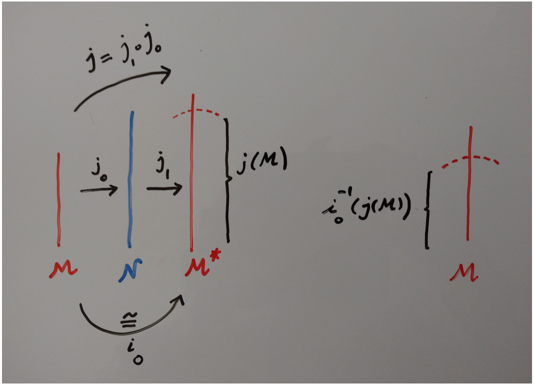
- T is *neat* iff for any two deductively closed extensions U and V of T (both of which are formulated in the language of T), if U is a retract of V, then $U \supseteq V$ (equivalently: $\mathrm{Mod}(U) \subseteq \mathrm{Mod}(V)$).
- T is *tight* iff for any two deductively closed extensions U and V of T (both of which are formulated in the language of T), if U and V are bi-interpretable, then $U = V$.
- In other words, T is tight if distinct deductively closed extensions of T are never bi-interpretable.
- solidity $\Rightarrow$ neatness $\Rightarrow$ tightness.
- Solidity, neatness, and tightness are all preserved under bi-interpretations.

Solidity of PA

- **Theorem.** (Visser 2006) PA *is solid.*
- **Proof Outline.** Suppose $\mathcal{M}$, $\mathcal{M}^*$, and $\mathcal{N}$ are models of PA such that:

$$\mathcal{M} \trianglerighteq_{\text{par}} \mathcal{N} \trianglerighteq_{\text{par}} \mathcal{M}^*, \text{ and}$$

there is an $\mathcal{M}$-definable isomorphism $i_0 : \mathcal{M} \to \mathcal{M}^*$.



Solidity of other theories

- **Theorem.** (E 2016) *The following theories are solid*:
  1. Z_2 (*second order arithmetic*).
  2. ZF (*Zermelo-Fraenkel set theory).*
  3. KM (*Kelley-Morse theory of classes*).
  4. Higher order analogues of Z_2 and KM.
- *Other examples of solid theories include*:
  1. CT[PA] (PA with an inducive truth predicate).
  2. $\mathsf{ZF} \setminus \{\mathsf{Powerset}\} + \forall x \ |x| \leq \aleph_0$.
  3. $\mathsf{ZF} \setminus \{\mathsf{Powerset}\} + \exists\kappa(\mathrm{Inacc}(\kappa) \wedge \ \forall x \ |x| \leq \kappa)$.
  4. $\mathsf{ZF} \setminus \{\mathsf{Infinity}\} + \mathsf{TC}$.

Optimality Questions (E. 2016)

Question A (open ended). *Are the known proofs of solidity of* PA, ZF, *etc. optimal in some sense*?

Question B. *Is there a finitely axiomatizable tight/solid sequential theory*?

Question C. *Is there an example of a subtheory of* PA *that is solid but whose deductive closure does not include all of* PA?

Published Optimality Results

- **Theorem A.** (E. 2016) *Neither* ZF $\setminus$ {Extensionality} *nor* ZF $\setminus$ {Foundation} *is tight.*
- **Theorem B.** (Freire + Hamkins 2020) *Neither of the theories* Z (*Zermelo set theory*) *nor* ZF $\setminus$ {Powerset} *is tight.*
- **Theorem C.** (Freire + Williams 2023) *For each fixed* $n \in \omega$ the Π^1_n-CA *subsystem of* Z_2 and KM *are not tight.*

A new optimality result: Theorem D

- Theorems D, E, and F appear in the paper co-authored with Mateusz Łełyk that was mentioned on slide 14.
- **Theorem D.** *Let T be any of the theories* PA, ZF, Z_2, KM, *etc. whose solidity has been asserted in this talk.*
  1. *No finitely axiomatizable subtheory of T is tight.*
  2. *Indeed, for each $n \in \omega$, T_{Π_n} fails to be tight, where T_{Π_n} is the set of Π_n-consequences of T.*

Proof idea of Theorem D for $T = \mathrm{PA}$.

- For $\mathcal{M} \models \mathrm{PA}$, $K_n(\mathcal{M})$ is the submodel of $\mathcal{M}$ whose universe consists of elements of $\mathcal{M}$ that are definable in $\mathcal{M}$ by a Σ_n-formula.
- **Theorem.** (Kirby-Paris, Lessan 1978) *Suppose $n \in \omega$, $n > 0$, and $\mathcal{M}$ is a nonstandard model of* PA, then:
  1. $K_n(\mathcal{M}) \prec_{\Pi_n} \mathcal{M}$, hence $K_n(\mathcal{M}) \models \mathrm{Th}_{\Pi_{n+1}}(\mathcal{M})$.
  2. $K_n(\mathcal{M}) \models \mathrm{PA}_{\Pi_{n+1}} + \mathrm{I}\Sigma_{n-1} + \neg \mathrm{B}\Sigma_n$.
- **Theorem.** (Lessan 1978) *The standard cut ω is first order definable in $K_n(\mathcal{M})$.*

Proof idea of Theorem D for $T = \mathsf{PA}$, continued

- For each $n \in \omega$, a model of the form $K_n(\mathcal{M})$ is interpretable in the standard model of arithmetic by using the arithmetical completeness theorem.
- **Theorem.** *For each $n \in \omega$ the standard model $(\omega, +, \cdot)$ of* PA *is bi-interpretable with a nonstandard model of the form $K_n(\mathcal{M})$, where $\mathcal{M}$ is a nonstandard model of* PA.
- Since $K_n(\mathcal{M})$ is a model of $\mathsf{PA}_{\Pi_{n+1}}$, this shows that $\mathsf{PA}_{\Pi_{n+1}}$ is not solid.
- With a little more work, the above argument can be refined to yield the failure of tightness of PA_{Π_n} for each $n \in \omega$.
- A similar idea works for models of ZF + V = L, i.e., which allows one to show that ZF_{Π_n} is not tight for each $n \in \omega$.

Theorem E

- **Theorem E.** *For each* $n \in \omega$, $\mathsf{PA}_{\Pi_n} + \mathsf{Ref}$ *is not tight*, where Ref (the scheme of LOCAL reflection) consists of implications of the form (φ ranges over arithmetical sentences).

$$\varphi \to \mathsf{Con}(\varphi).$$

- **Proof Idea**: This is really a corollary of the PROOF of Theorem D since, the the proof of Theorem D shows that the Π_n-fragment of TRUE ARITHMETIC is not tight. This fact together with observation (essentially due to Feferman) that Ref is a subtheory of $\mathsf{I}\Delta_0 + \mathsf{Exp}$ plus the Π_1-fragment of TRUE ARITHMETIC implies Theorem E.
- The failure of tightness of $\mathsf{PA}_{\Pi_n} + \mathsf{Ref}$ should be contrasted with the fact that $\mathsf{I}\Delta_0 + \mathsf{Exp} + \mathsf{REF}$ is solid (since it axiomzatises PA by Kreisel-Levy), where REF is the GLOBAL reflection scheme, which consists of implications of the form

$$\forall x \, [\varphi(x) \to \mathsf{Con}(\varphi(\dot{x}))].$$

Theorem F (1)

- **Theorem F.** PA^- + Collection *is not solid*. (*Indeed even* IOpen + Collection *fails to be solid.*)

- **Proof Idea.** Let $\mathbb{Z}$ be the ring of integers and $(I, <_I)$ be a linear order. $\mathbb{Z}[X_i : i \in I]$ is the ring of polynomials with coefficients in $\mathbb{Z}$, whose indeterminates come from the I-indexed collection $\{X_i : i \in I\}$ of indeterminates. Each nonconstant $p \in \mathbb{Z}[X_i : i \in I]$ can be written as:

$$p(X_{i_1}, X_{i_2}, \cdots, X_{i_n}) \text{ where } i_1 <_I i_2 <_I \cdots <_I i_n,$$

  for some finite subset $\{i_1, i_2, \cdots, i_n\}$ of I, with the understanding that $\{i_1, i_2, \cdots, i_n\}$ is the least (in the sense of inclusion) subset of $A \subseteq I$ such that $p \in \mathbb{Z}[X_i : i \in A]$. With this notation in mind, we refer to $\{i_1, i_2, \cdots, i_n\}$ as the *support* of p.

- The ordering on $\mathbb{Z}[X_i : i \in I]$ is defined by first declaring $p(X_{i_1}, X_{i_2}, \cdots, X_{i_n}) > 0$, provided the coefficient of X_{i_n} is positive; and then given p and q in $\mathbb{Z}[X_i : i \in I]$, we define $p > q$ iff $p - q > 0$.

Theorem F (2)

- For every linear order $(I, <_I)$ the substructure $\mathbb{Z}[X_i : i \in I]^{\geq 0}$ of non-negative elements of $\mathbb{Z}[X_i : i \in I]$ is a model of PA^-.
- Key Fact: Collection holds in the model $\mathbb{Z}[X_i : i \in \mathbb{Q}]^{\geq 0}$.
- The model $\mathbb{Z}[X_i : i \in \mathbb{Q}]^{\geq 0}$ can be described in the standard model of arithmetic. Also, the standard cut is definable in the model (as the longest initial segment of objects with parity).
- Putting all the above makes it clear that $\mathbb{Z}[X_i : i \in \mathbb{Q}]^{\geq 0}$ is bi-interpretable with the standard model of arithmetic.
- The failure of solidity of IOpen + Collection should be contrasted with the fact that $\mathrm{I}\Delta_0$ + Collection is solid, since it axiomatizes PA.

A surprise from Warsaw

- The following theorems were obtained recently as the result of joint work of Piotr Gruza, Leszek Kołodziejczyk, and Mateusz Łełyk.
- **Theorem A.** *There exists a proper r.e. subtheory of* PA *that is solid. Moreover, such a subtheory can be arranged so that it does not even interpret* PA. This provides a strong negative answer to Question C.
- **Theorem B.** *There exists an r.e. subtheory of* PA *that is tight but not neat* (*and therefore not solid*).

As reported talk "Tightness and solidity in fragments of Peano Arithmetic" in the MOPA seminar (CUNY) by Piotr Gruza on March 5, 2024; for the abstract and the video see:

https://nylogic.github.io/mopa/2024/03/05/tightness-and-solidity-in-fragments-of-peano-arithmetic.html

Thanx!

Drawing by Saul Steinberg (1962)