Graphical generalisation of operads and Generating series

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Graphic Configuration spaces

- For a manifold M, define graphic configuration space

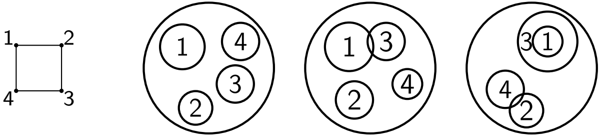
$$\mathrm{Conf}_\Gamma(M) = \{(x_v) \in X^{V_\Gamma} | (v,w) \in E_\Gamma \Rightarrow x_v \neq x_w\}.$$

- What we can say about homology groups $H_\bullet(\mathrm{Conf}_\Gamma(M))$?
- More generally, what we can say about rational homotopy type of $\mathrm{Conf}_\Gamma(M)$?

Little disks

The Little n-discs contractad $\mathcal{D}_n$

- For a graph Γ, $\mathcal{D}_n(\Gamma)$ consists of configurations of n-discs labeld by Γ: if vertices are adjacent, then related discs don't intersect.



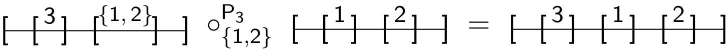
- Centering map $\pi\colon \mathcal{D}_n(\Gamma) \xrightarrow{\simeq} \mathrm{Conf}_\Gamma(\mathbb{R}^n)$ is a homotopy equivalence.

Little disks

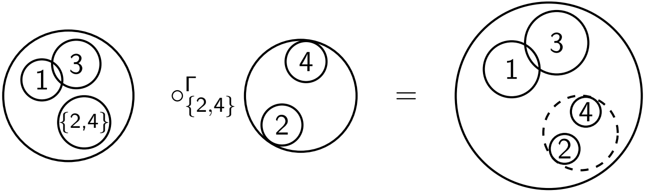
Disk insertions

$$\circ^{\Gamma}_{G} \colon\; \mathcal{D}_n(\Gamma\,/\,G) \times \mathcal{D}_n(\Gamma\,|_G) \to \mathcal{D}_n(\Gamma)$$

For n=1



For n=2



Contractads

A contractad consists of

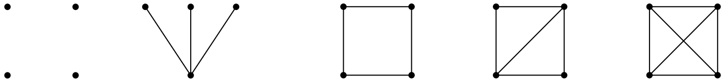
- A graphical collection $\mathcal{P}\colon \mathsf{CGr}^{\mathrm{op}} \to \mathcal{C}$
- Infinitesimal compositions

$$\circ^{\Gamma}_{G}\colon\ \mathcal{P}(\Gamma/G) \otimes \mathcal{P}(\Gamma|_{G}) \to \mathcal{P}(\Gamma)$$

- These maps satisfy certain associative and equivarience axioms.

Generating functions

For a partition $\lambda \vdash n$, let K_λ be complete multipartite graph



$\mathrm{K}_{(4)}$ $\mathrm{K}_{(3,1)} \cong \mathrm{St}_3$ $\mathrm{K}_{(2^2)} \cong \mathrm{C}_4$ $\mathrm{K}_{(2,1^2)}$ $\mathrm{K}_{(1^4)} \cong \mathrm{K}_4$

Young generating function $F_{\mathrm{Y}}(\mathcal{P}) \in \Lambda_{\mathbb{Q}}[[z]]$

$$F_{\mathrm{Y}}(\mathcal{P})(z) = \sum_{l(\lambda)\geq 2} \dim \mathcal{P}(\mathrm{K}_\lambda)\frac{m_\lambda}{\lambda!} + \sum_{n\geq 1}\sum_{|\lambda|\geq 0} \dim \mathcal{P}(\mathrm{K}_{(1^n)\cup\lambda})\frac{z^n}{n!}\frac{m_\lambda}{\lambda!}$$

Theorem

For a Koszul contractad $\mathcal{P}$, we have

$$-F_{\mathrm{Y}}(\mathcal{P}^!)(-F_{\mathrm{Y}}(\mathcal{P})(z)) = z$$

Commutative contractad

The commutative contractad gcCom

- Graphical collection: $\mathrm{gcCom}(\Gamma) = \{*\}$.
- Contractad structure:

$$\circ^{\Gamma}_{G}\colon\ \mathrm{gcCom}(\Gamma/G) \times \mathrm{gcCom}(\Gamma|_G) \to \mathrm{gcCom}(\Gamma), \quad (*,*) \mapsto *$$

- gcCom is binary and quadratic. Moreover it is Koszul.
- The generating function

$$F_{\mathsf{Y}}(\mathrm{gcCom}) = e^{z+p_1} - 1 - \sum_{n\geq 1} \frac{p_n}{n!}$$

Hamiltonian Paths

The Hamiltonian contractad Ham

- Graphical collection:
  $\mathrm{Ham}(\Gamma) = \langle \mathrm{P} \subset \Gamma \,|\, \mathrm{P}\ -\text{directed Hamiltonian path} \rangle.$
- Contractad structure=substitution of paths

$$\circ^{\Gamma}_{G} \colon \ \mathrm{Ham}(\Gamma/G) \otimes \mathrm{Ham}(\Gamma|_{G}) \to \mathrm{Ham}(\Gamma)$$

- Ham is binary, quadratic and Koszul
- The generating function of Hamiltonian paths

$$F_{\mathsf{Y}}(\mathrm{HP}) = \frac{1}{1-(z+\sum_{n\geq 1}(-1)^{n-1}p_n)} - p_1 - 1.$$

Hamiltonian Cycles*

Module of Hamiltonian cycles CycHam

- Graphical collection:
  $\mathrm{CycHam}(\Gamma) = \langle \mathsf{C} \subset \Gamma \mid \mathsf{C} -\text{directed Hamiltonian cycle}\rangle.$
- Module structure=substitution of paths

$$\circ_G^\Gamma : \ \mathrm{CycHam}(\Gamma / G) \otimes \mathrm{Ham}(\Gamma|_G) \to \mathrm{CycHam}(\Gamma)$$

- CycHam is quadratic Ham-module and Koszul
- The generating function of Hamiltonian cycles

$$F_{\mathsf{Y}}(\mathrm{HC}) = -\log(1-(z+\sum_{n\geq 1}(-1)^{n-1}p_n)) - \sum_{n\geq 1}(-1)^{n-1}\frac{p_n}{n}.$$

Homology of Little disks

The Gerstenhaber contractad $\mathsf{gcGerst} := H_\bullet(\mathcal{D}_2)$

- gcGerst is binary, quadratic and Koszul
- We have decomposition

$$\mathsf{gcGerst} \cong \mathsf{gcCom} \circ \mathsf{gcCom}^!$$

- Relation to chromatic polynomials

$$\sum_i (-q)^i \dim H^i(\mathrm{Conf}_\Gamma(\mathbb{C})) = q^{|V_\Gamma|}\chi_\Gamma(\frac{1}{q})$$

- The generating function of chromatic polynomials

$$\sum_\lambda \chi_{\mathsf{K}_\lambda}(q)\frac{m_\lambda}{\lambda!} = (1+\sum_{n\geq 1}\frac{p_n}{n!})^q$$

Modular compactifications

There is a contractad in the category of projective smooth varieties $\overline{\mathcal{M}}$, such that

- For the complete graph K_n, $\overline{\mathcal{M}}(\mathsf{K}_n) \cong \overline{\mathfrak{M}}_{0,n+1}$ is the Deligne-Mumford moduli space of $(n+1)$-pointed genus 0 stable curves.
- For the stellar graph $\mathsf{St}_n = \mathsf{K}_{(n,1)}$, $\overline{\mathcal{M}}(\mathsf{St}_n) \cong \overline{\mathcal{L}}_{0,n}$ is the Losev-Manin moduli space of stable chains.
- For K_λ, $\overline{\mathcal{M}}(\mathsf{K}_\lambda) \cong \overline{\mathfrak{M}}_{0,\mathsf{K}_\lambda}$, where $\overline{\mathfrak{M}}_{0,\mathsf{K}_\lambda}$ is a variation of moduli space of stable curves obtained from $\overline{\mathfrak{M}}_{0,n+1}$ by contractions of certain curves.
- For a tree T, $\overline{\mathcal{M}}(T)$ is a toric variety with dual polytope is a graph associahedron.
- For arbitrary Γ, $\overline{\mathcal{M}}_{\Bbbk}(\Gamma)$ is a compatification of $\mathrm{Conf}_\Gamma(\mathbb{A}^1_{\Bbbk})/\,\mathrm{Aff}_1(\Bbbk)$

Homology of the Wondeful contractad

- The Hypercommutative contractad $\mathrm{gcHyper} := H_\bullet(\overline{\mathcal{M}}_{\mathbb{C}})$ is quadratic and Koszul. Generators are fundamental classes $\nu_\Gamma := [\overline{\mathcal{M}}_{\mathbb{C}}(\Gamma)]$.
- The Odd Poisson contractad $\mathrm{gcPois}_{\mathrm{odd}} := H_\bullet(\overline{\mathcal{M}}_{\mathbb{R}}; \mathbb{Q})$ is quadratic and Koszul. It has one binary m_{P_2} and two ternary $p_{\mathsf{P}_3}, p_{\mathsf{K}_3}$ generators.

Complex points

The generating series for Betti numbers of Modular compactifications $\overline{\mathcal{M}}_{0,\mathsf{K}_\lambda}(\mathbb{C})$

$$F_{\mathsf{Y}}(\overline{\mathcal{M}}(\mathbb{C})) = \sum_{l(\lambda)\geq 2}\left[\sum_{i=0}^{|\lambda|-2} \dim H^{2i}(\overline{\mathcal{M}}_{0,\mathsf{K}_\lambda}(\mathbb{C}))q^i\right]\frac{m_\lambda}{\lambda!} + $$

$$+ \sum_{n\geq 1, |\lambda|\geq 0}\left[\sum_{i=0}^{|\lambda|+n-2} \dim H^{2i}(\overline{\mathcal{M}}_{0,\mathsf{K}_{(1^n)\cup\lambda}}(\mathbb{C}))q^i\right]\frac{m_\lambda}{\lambda!}\frac{z^n}{n!},$$

is functional inverse (with respect to the variable z) of the following function

$$G(z) = \frac{q}{q-1}z - \frac{1}{q(q-1)}\left[\left(1+z+\sum_{n\geq 1}\frac{p_n}{n!}\right)^q - 1 - \sum_{n\geq 1}\frac{p_n q^n}{n!}\right].$$

Real locus

The generating series for (rational) Betti numbers of Modular compactifications $\overline{\mathcal{M}}_{0,\mathsf{K}_\lambda}(\mathbb{R})$

$$F_{\mathsf{Y}}(\overline{\mathcal{M}}(\mathbb{R})) = \sum_{l(\lambda)\geq 2} \left[\sum_i (-q)^i \dim H^i(\overline{\mathcal{M}}_{0,\mathsf{K}_\lambda}(\mathbb{R}); \mathbb{Q}) \right] \frac{m_\lambda}{\lambda!} + \\ + \sum_{n\geq 1, |\lambda|\geq 0} \left[\sum_i (-q)^i \dim H^i(\overline{\mathcal{M}}_{0,\mathsf{K}_{(1^n)\cup\lambda}}(\mathbb{R}); \mathbb{Q}) \right] \frac{m_\lambda}{\lambda!} \frac{z^n}{n!}$$

has a coincise presentation

$$F_{\mathsf{Y}}(\overline{\mathcal{M}}(\mathbb{R})) = \left[\sqrt{q}(z + \mathrm{SINH}_q) + \sqrt{q(z + \mathrm{SINH}_q)^2 + 1} \right]^{\frac{1}{\sqrt{q}}} - 1 - \sum_{n\geq 1} \frac{p_n}{n!},$$

where $\mathrm{SINH}_q = \sum_{n\geq 0} \frac{p_{2n+1} q^n}{(2n+1)!}$.