Calculation of multiloop Feynman diagrams and diagonalization of commuting operators

S. Derkachov

PDMI, St.Petersburg

(based on the works with A.Isaev, L.Shumilov, V.Kazakov, E.Olivicci, G.Ferrando)

Mathematical and Theoretical Physics, dedicated to Ludwig Faddeev
28-31 May 2024, PDMI

Plan

- D.Broadhurst and D.Kreimer conjecture for zig-zag diagrams
- Basso-Dixon diagrams

The whole story was initiated by the works

L.N. Lipatov, *Asymptotic behavior of multicolor QCD at high energies in connection with exactly solvable spin models,* JETP Lett. 59 (1994) 596

L.D. Faddeev and G.P. Korchemsky, *High-energy QCD as a completely integrable model,* Phys. Lett. B 342 (1995) 311

L.N. Lipatov, *Integrability of scattering amplitudes in $N = 4$ SUSY,* J. Phys. A 42 (2009) 304020

The rich source of iterative Feynman diagrams is fishnet conformal field theory

O.Gurdogan, V.Kazakov, *New Integrable 4D Quantum Field Theories from Strongly Deformed Planar $N = 4$ Supersymmetric Yang-Mills Theory* Phys.Rev.Lett. 117 (2016) 20, 201602,e-Print: 1512.06704 [hep-th]

V.Kazakov, E.Olivucci, *Biscalar Integrable Conformal Field Theories in Any Dimension,* Phys.Rev.Lett. 121 (2018) 13, 131601 , e-Print: 1801.09844 [hep-th]

N.Gromov, V.Kazakov, G.Korchemsky *Exact Correlation Functions in Conformal Fishnet Theory,* JHEP 08 (2019) 123 , e-Print: 1808.02688 [hep-th]

Fishnet

A.B. Zamolodchikov, *"Fishing-net" diagrams as a completely integrable system,* Phys. Lett.B 97 (1980) 63.

Plan

Heavily based on quantum inverse scattering method

E.K. Sklyanin, L.D. Faddeev *Quantum Mechanical Approach to Completely Integrable Field Theory Models,* Sov.Phys.Dokl. 23 (1978) 902-904, Dokl.Akad.Nauk Ser.Fiz. 243 (1978) 1430-1433

E.K. Sklyanin, L.A. Takhtadzhyan, L.D. Faddeev, *Quantum inverse problem method. I,* Theor.Math.Phys. 40 (1979) 2, 688-706, Teor.Mat.Fiz. 40 (1979) 194-220

L. D. Faddeev, *Quantum completely integrable models in field theory,* Sov. Sci. Rev., Sect. C, 1 (1980), 107-155

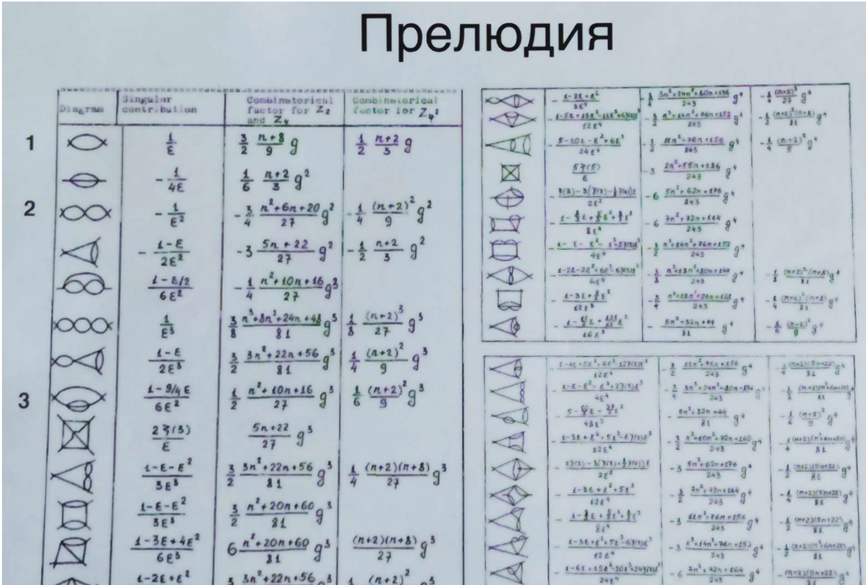
L. D. Faddeev, *How the algebraic Bethe ansatz works for integrable models*, Symetries quantiques, North-Holland, Amsterdam, 1998, 149-219 , arXiv: hep-th/9605187

and the representation of separated variables

E.K. Sklyanin, *The Quantum Toda Chain,* Lect.Notes Phys. 226 (1985) 196-233

E.K. Sklyanin, *Separation of variables - new trends,* Prog.Theor.Phys.Suppl. 118 (1995) 35-60 , e-Print: solv-int/9504001 [nlin.SI]

Distinguished class of diagramms



Прелюдия

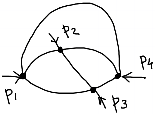
Diagram	Singular contribution	Combinatorical factor for Z_2 and Z_4	Combinatorical factor for Z_{φ^2}
1	$\frac{1}{\varepsilon}$	$\frac{3}{2}\,\frac{n+8}{9}\,g$	$\frac{1}{2}\,\frac{n+2}{3}\,g$
	$-\frac{1}{4\varepsilon}$	$\frac{1}{6}\,\frac{n+2}{3}\,g^2$	
2	$-\frac{1}{\varepsilon^2}$	$-\frac{3}{4}\,\frac{n^2+6n+20}{27}\,g^2$	$-\frac{1}{4}\,\frac{(n+2)^2}{9}\,g^2$
	$-\frac{1-\varepsilon}{2\varepsilon^2}$	$-3\,\frac{5n+22}{27}\,g^2$	$-\frac{1}{2}\,\frac{n+2}{3}\,g^2$
	$\frac{1-\varepsilon/2}{6\varepsilon^2}$	$-\frac{1}{4}\,\frac{n^2+10n+16}{27}\,g^3$	
	$\frac{1}{\varepsilon^3}$	$\frac{3}{8}\,\frac{n^3+8n^2+24n+48}{81}\,g^3$	$\frac{1}{8}\,\frac{(n+2)^3}{27}\,g^3$
	$\frac{1-\varepsilon}{2\varepsilon^3}$	$\frac{3}{2}\,\frac{3n^2+22n+56}{81}\,g^3$	$\frac{1}{4}\,\frac{(n+2)^2}{9}\,g^3$
3	$\frac{1-9/4\,\varepsilon}{6\varepsilon^2}$	$\frac{1}{2}\,\frac{n^2+10n+16}{27}\,g^3$	$\frac{1}{6}\,\frac{(n+2)^2}{9}\,g^3$
	$\frac{2\zeta(3)}{\varepsilon}$	$\frac{5n+22}{27}\,g^3$	
	$\frac{1-\varepsilon-\varepsilon^2}{3\varepsilon^3}$	$\frac{3}{2}\,\frac{3n^2+22n+56}{81}\,g^3$	$\frac{1}{4}\,\frac{(n+2)(n+8)}{27}\,g^3$
	$\frac{1-\varepsilon-\varepsilon^2}{3\varepsilon^3}$	$\frac{3}{2}\,\frac{n^2+20n+60}{81}\,g^3$	
	$\frac{1-3\varepsilon+4\varepsilon^2}{6\varepsilon^3}$	$6\,\frac{n^2+20n+60}{81}\,g^3$	$\frac{(n+2)(n+8)}{27}\,g^3$
	$1-2\varepsilon+\varepsilon^2$ …	$3n^2+22n+56$ … g^3	$(n+2)^2$ … g^3

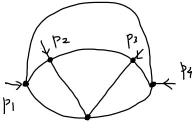
Diagram	Singular contribution	Combinatorical factor for Z_2 and Z_4	Combinatorical factor for Z_{φ^2}
	$-\frac{1-2\varepsilon+\varepsilon^2}{3\varepsilon^4}$	$-\frac{3}{4}\,\frac{3n^3+24n^2+60n+136}{243}\,g^4$	$-\frac{1}{4}\,\frac{(n+2)^3}{27}\,g^4$
	$-\frac{1-5\varepsilon+13\varepsilon^2-11\varepsilon^3+6\zeta(3)\varepsilon^3}{12\varepsilon^4}$	$-\frac{3}{2}\,\frac{n^3+14n^2+76n+152}{243}\,g^4$	$-\frac{1}{2}\,\frac{(n+2)^2(n+8)}{81}\,g^4$
	$-\frac{5-10\varepsilon-\varepsilon^2+6\varepsilon^3}{24\varepsilon^4}$	$-\frac{1}{2}\,\frac{11n^2+76n+156}{243}\,g^4$	$-\frac{1}{4}\,\frac{(n+2)^2}{9}\,g^4$
	$\frac{5\zeta(5)}{\varepsilon}$	$-3\,\frac{2n^2+55n+186}{243}\,g^4$	
	$-\frac{\zeta(3)-3(\zeta(3)-\frac{1}{2}\zeta(4))\varepsilon}{2\varepsilon^2}$	$-6\,\frac{5n^2+62n+176}{243}\,g^4$	
	$-\frac{1-\frac{4}{3}\varepsilon+\frac{5}{3}\varepsilon^2+[illegible]\varepsilon^3}{8\varepsilon^4}$	$-6\,\frac{7n^2+72n+164}{243}\,g^4$	
	$-\frac{1-\varepsilon-\varepsilon^2-\varepsilon^3+2\zeta(3)\varepsilon^3}{4\varepsilon^4}$	$-\frac{3}{2}\,\frac{n^3+14n^2+76n+152}{243}\,g^4$	
	$-\frac{1-2\varepsilon-2\varepsilon^2+6\varepsilon^3-6\zeta(3)\varepsilon^3}{6\varepsilon^4}$	$-\frac{3}{8}\,\frac{n^3+13n^2+[illegible]}{243}\,g^4$	$-\frac{1}{8}\,\frac{(n+2)^2(n+8)}{81}\,g^4$
	$-\frac{1-3\varepsilon+\frac{3}{2}\varepsilon^2}{12\varepsilon^3}$	$-\frac{3}{4}\,\frac{n^3+18n^2+[illegible]}{243}\,g^4$	$-\frac{1}{4}\,\frac{(n+2)^2(n+8)}{81}\,g^4$
	$-\frac{1-\frac{11}{2}\varepsilon+[illegible]\varepsilon^2}{16\varepsilon^3}$	$-\frac{5n^2+32n+44}{81}\,g^4$	$-\frac{1}{6}\,\frac{(n+2)^2}{9}\,g^4$

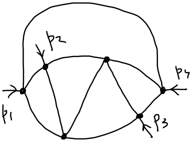
Diagram	Singular contribution	Combinatorical factor for Z_2 and Z_4	Combinatorical factor for Z_{φ^2}
	$-\frac{1-4\varepsilon+5\varepsilon^2+6\varepsilon^3-12\zeta(3)\varepsilon^3}{12\varepsilon^4}$	$-\frac{3}{2}\,\frac{11n^2+76n+156}{243}\,g^4$	$-\frac{1}{4}\,\frac{(n+2)(5n+22)}{81}\,g^4$
	$-\frac{1-\varepsilon-\varepsilon^2-\varepsilon^3+2\zeta(3)\varepsilon^3}{4\varepsilon^4}$	$-\frac{3}{4}\,\frac{3n^3+24n^2+80n+136}{243}\,g^4$	$-\frac{1}{8}\,\frac{(n+2)(n^2+6n+20)}{81}\,g^4$
	$-\frac{5-\frac{10}{3}\varepsilon-\frac{27}{4}\varepsilon^2}{48\varepsilon^3}$	$-\frac{5n^2+32n+44}{81}\,g^4$	$-\frac{1}{6}\,\frac{(n+2)^2}{9}\,g^4$
	$-\frac{1-3\varepsilon+\varepsilon^2+5\varepsilon^3-6\zeta(3)\varepsilon^3}{12\varepsilon^4}$	$-\frac{3}{2}\,\frac{n^3+10n^2+72n+160}{243}\,g^4$	$-\frac{1}{4}\,\frac{(n+2)(n^2+6n+20)}{81}\,g^4$
	$-\frac{2\zeta(3)-3(\zeta(3)+\frac{1}{2}\zeta(4))\varepsilon}{2\varepsilon^2}$	$-3\,\frac{5n^2+62n+176}{243}\,g^4$	$-\frac{1}{2}\,\frac{(n+2)(5n+22)}{81}\,g^4$
	$-\frac{1-3\varepsilon+\varepsilon^2+5\varepsilon^3}{12\varepsilon^4}$	$-\frac{3}{2}\,\frac{7n^2+72n+164}{243}\,g^4$	$-\frac{1}{4}\,\frac{(n+2)(5n+22)}{81}\,g^4$
	$-\frac{1-\frac{4}{3}\varepsilon+\frac{5}{3}\varepsilon^2+[illegible]\varepsilon^3}{8\varepsilon^4}$	$-3\,\frac{11n^2+76n+156}{243}\,g^4$	$-\frac{1}{2}\,\frac{(n+2)(5n+22)}{81}\,g^4$
	$-\frac{1-3\varepsilon+\varepsilon^2+5\varepsilon^3-6\zeta(3)\varepsilon^3}{12\varepsilon^4}$	$-3\,\frac{n^3+14n^2+76n+152}{243}\,g^4$	$-\frac{1}{2}\,\frac{(n+2)(n^2+6n+20)}{81}\,g^4$
	$-\frac{1-6\varepsilon+15\varepsilon^2-30\varepsilon^3+24\zeta(3)\varepsilon^3}{24\varepsilon^4}$	$-6\,\frac{3n^2+42n+164}{243}\,g^4$	$-\frac{(n+2)(5n+22)}{81}\,g^4$

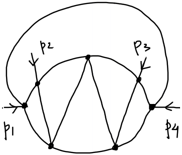
Reduction to two-point function

Massless scalar φ^4-theory, $d = 4-2\varepsilon$, Euclidean metric

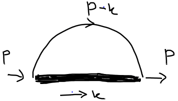






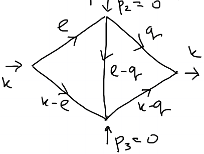


Singular in ε part does not depend on p_k $\Rightarrow$ $p_2 = p_3 = 0$ $\quad p_1 = p = -p_4$



$$\Rightarrow \int dk^{4-2\varepsilon} \frac{A_n(\varepsilon)}{(p-k)^2 k^{2(1+n\varepsilon)}}$$

two-loop example

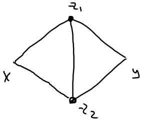


$$= \frac{\overset{\text{finite}}{A_2(\varepsilon)}}{k^{2(1+2\varepsilon)}}$$

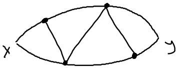
Iterative structure and graph-building operator

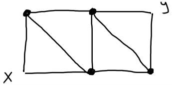
Momentum representation → Coordinate representation

conformal invariance

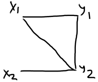


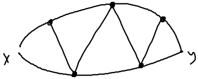
$$\int d^4 z_1\, d^4 z_2 \, \frac{1}{(x-z_1)^2 (x-z_2)^2 (z_1-z_2)^2 (z_1-y)^2 (z_2-y)^2}$$

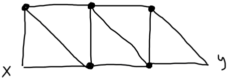




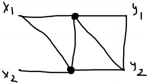
$H \leftrightarrow$



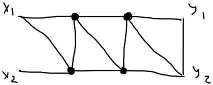




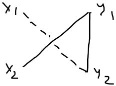
$H^2 \leftrightarrow$



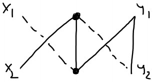
$H^3 \leftrightarrow$



$Q \leftrightarrow$



$Q^2 \leftrightarrow$



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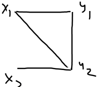

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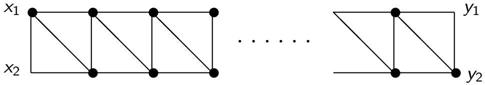
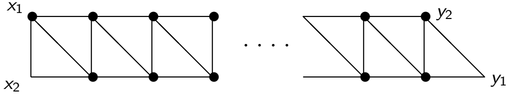
Even number of loops $M = 2N$



$$= \int d^4x_1 d^4y_2 \frac{\langle x_1, x_2 | Q^{2N} | y_1, y_2 \rangle}{(x_1 - x_2)^2}$$

Odd number of loops $M = 2N + 1$



$$= \int d^4x_1 d^4y_2 \frac{\langle x_1, x_2 | Q^{2N+1} | y_1, y_2 \rangle}{(x_1 - x_2)^2}$$

The two-point function, which is represented by the M-loop zig-zag diagram, has the general form

$$G_2(x_2, y_1) = \frac{\pi^{2M}}{(x_2 - y_1)^2} Z(M + 1)$$

The M-loop zig-zag diagram gives the (M+1)-loop contribution to the β-function.

D.Broadhurst and D.Kreimer conjecture

The first nontrivial terms $Z(3) = 6\zeta_3$ and $Z(4) = 20\zeta_5$ were analytically evaluated in

K. G. Chetyrkin, A. L. Kataev, F. V. Tkachov, *New Approach to Evaluation of Multiloop Feynman Integrals: The Gegenbauer Polynomial x Space Technique,* Nucl. Phys. B 174 (1980) 345-377

K. G. Chetyrkin, F. V. Tkachov, *Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops,* Nucl. Phys. B 192 (1981) 159-204

The constant $Z(5) = 4418\zeta_7$ calculated by D.Kazakov in 1983

D. I. Kazakov, *The method of uniqueness, a new powerful technique for multiloop calculation,* Phys. Lett. B 133 (1983) 406-410

D. I. Kazakov, *Multiloop Calculations: Method of Uniqueness and Functional Equations,* Teor. Mat. Fiz. 62 (1984) 127-135

The 5-loop contribution $Z(6) = 168\zeta_9$ to the β-function (in 6-loop order) was found by D.Broadhurst in 1985

D. J. Broadhurst, *Massless scalar feynman diagrams: five loops and beyond,* Report of The Open University, Milton Keynes, England (UK), 1985, OUT-4102-18 , arXiv:1604.08027 (2016).

D.Broadhurst and D.Kreimer conjecture

D.Broadhurst and D.Kreimer in 1995

D. J. Broadhurst, D. Kreimer, *Knots and numbers in ϕ^4 theory to 7 loops and beyond,* Int. J. Mod. Phys. C 6 (1995) 519-524

evaluated $Z(n)$ numerically up to $n=(M+1)=10$ loops and based on these data they formulated a remarkable conjecture that the constant $Z(M+1)$ is given by the following expression

$$Z(M+1) = 4C_M \sum_{p=1}^{\infty} \frac{(-1)^{(p-1)(M+1)}}{p^{2(M+1)-3}} =$$

$$= \begin{cases} 4\, C_M\, \zeta_{2M-1} \ \text{ for } \ M = 2N+1\,, \\ 4\, C_M\, (1 - 2^{2(1-M)})\, \zeta_{2M-1} \ \text{ for } \ M = 2N\,, \end{cases}$$

where M is the number of loops in the zig-zag diagrams and $C_M = \frac{1}{(M+1)}\binom{2M}{M}$ is the Catalan number.

The proof of the Broadhurst-Kreimer conjecture was found by F. Brown and O. Schnetz in 2012

F. Brown, O. Schnetz, *Single-valued polylogarithms and a proof of the zig-zag conjecture,* J. Number Theory 148 (2015) 478-506 , e-Print: arXiv:1208.1890

Alternative proof was found in

S. Derkachov, A.P. Isaev, L. Shumilov, *Conformal triangles and zig-zag diagrams,* Phys.Lett.B 830 (2022) 137150 , e-Print: 2201.12232 [hep-th]

Q-operator

$$[Q\Psi](x_1, x_2) = \int d^4y_1 d^4y_2 \frac{\delta^4(x_1 - y_2)}{(x_2 - y_1)^2(y_1 - y_2)^2} \Psi(y_1, y_2) = \int d^4y \frac{1}{(x_1 - y)^2(x_2 - y)^2} \Psi(y, x_1)$$

Conformal transformations

- translations $\vec{x} \to \vec{x}' = \vec{x} + \vec{a}$
- rotations $\vec{x} \to \vec{x}' = \Lambda\vec{x}$
- dilatations $\vec{x} \to \vec{x}' = \lambda\vec{x}$
- inversion $\vec{x} \to \vec{x}' = \frac{\vec{x}}{x^2}$

Representation of conformal group in the space of functions

$$[T_\Delta \Psi](x) = \left| \frac{Dx'}{Dx} \right|^{\frac{\Delta}{4}} \Psi(x')$$

Tensor product of two representations

$$[T_{\Delta_1, \Delta_2} \Psi](x_1, x_2) = \left| \frac{Dx_1'}{Dx_1} \right|^{\frac{\Delta_1}{4}} \left| \frac{Dx_2'}{Dx_2} \right|^{\frac{\Delta_2}{4}} \Psi(x_1', x_2')$$

Operator Q is conformal invariant $\Delta_1 = 2, \Delta_2 = 1$

$$Q\, T_{\Delta_1, \Delta_2} = T_{\Delta_1, \Delta_2}\, Q$$

Q-operator

$$[Q\Psi](x_1,x_2) = \int d^4y \frac{1}{(x_1-y)^2(x_2-y)^2}\Psi(y,x_1)$$

Inversion $\vec{x}' = \frac{\vec{x}}{x^2}$; $QR = RQ$

$$\left|\frac{Dx'}{Dx}\right| = \left(\frac{1}{x^2}\right)^4 \quad ; \quad [R\Psi](x_1,x_2) = \frac{1}{(x_1^2)^2}\frac{1}{x_2^2}\Psi\left(\frac{x_1}{x_1^2},\frac{x_2}{x_2^2}\right)$$

The simplest scalar eigenfunction $\Psi_\lambda(x_1,x_2) = (x_1-x_2)^{-2\lambda}$

$$[Q\Psi_\lambda](x_1,x_2) = \int d^4y \frac{1}{(x_1-y)^2(x_2-y)^2}\frac{1}{(y-x_1)^{2\lambda}} = \frac{\pi^2}{\lambda(1-\lambda)}\frac{1}{(x_1-x_2)^{2\lambda}}$$

Scalar chain rule $(\alpha = 1, \beta = 1+\lambda)$

$$\int d^4z \frac{1}{(x-z)^{2\alpha}(z-y)^{2\beta}} = \pi^2\frac{\Gamma(2-\alpha)\Gamma(2-\beta)\Gamma(\alpha+\beta-2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(4-\alpha-\beta)}\frac{1}{(x-y)^{2(\alpha+\beta-2)}}$$

Q-operator

$$[Q\Psi](x_1,x_2) = \int d^4y \, \frac{1}{(x_1-y)^2(x_2-y)^2} \Psi(y,x_1)$$

The tensor eigenfunction $\Psi_\lambda^{\mu_1\cdots\mu_n}(x_1\,,x_2) = \frac{(x_1-x_2)^{\mu_1\cdots\mu_n}}{(x_1-x_2)^{2\lambda}}$

$$[Q\Psi_\lambda^{\mu_1\cdots\mu_n}](x_1,x_2) = \int d^4y \, \frac{1}{(x_1-y)^2(x_2-y)^2} \, \frac{(y-x_1)^{\mu_1\cdots\mu_n}}{(y-x_1)^{2\lambda}} = \\ = (-1)^n \frac{\pi^2}{\lambda(1-\lambda+n)} \, \frac{(x_1-x_2)^{\mu_1\cdots\mu_n}}{(x_1-x_2)^{2\lambda}}$$

Tensor chain rule $(\alpha = 1\,, \beta = 1+\lambda)$ $\quad$ $x^{\mu_1\cdots\mu_n} = x^{\mu_1}\cdots x^{\mu_n} - \mathit{traces}$

$$\int d^4z \, \frac{1}{(x-z)^{2\alpha}} \, \frac{(z-y)^{\mu_1\cdots\mu_n}}{(z-y)^{2\beta}} = \\ \pi^2 \frac{\Gamma(2-\alpha)\Gamma(2-\beta+n)\Gamma(\alpha+\beta-2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(4-\alpha-\beta+n)} \, \frac{(x-y)^{\mu_1\cdots\mu_n}}{(x-y)^{2(\alpha+\beta-2)}} \,,$$

Q-operator

Conformal invariance

$$\Psi_{\lambda}^{\mu_1\cdots\mu_n}(x_1\,,x_2) = \frac{(x_1-x_2)^{\mu_1\cdots\mu_n}}{(x_1-x_2)^{2\lambda}} \to \frac{1}{(x_1^2)^{2-\lambda}}\,\frac{1}{(x_2^2)^{1-\lambda}}\,\frac{\left(\frac{x_1}{x_1^2}-\frac{x_2}{x_2^2}\right)^{\mu_1\cdots\mu_n}}{(x_1-x_2)^{2\lambda}} \to$$

$$\Psi_{\lambda\,,x_0}^{\mu_1\cdots\mu_n}(x_1\,,x_2) = \frac{\left(\frac{x_1-x_0}{(x_1-x_0))^2}-\frac{x_2-x_0}{(x_2-x_0)^2}\right)^{\mu_1\cdots\mu_n}}{(x_1-x_0)^{2(2-\lambda)}(x_2-x_0)^{2(1-\lambda)}\,(x_1-x_2)^{2\lambda}}$$

Rich set of eigenfunctions

$$Q\,\Psi_{\lambda\,,x_0}^{\mu_1\cdots\mu_n} = (-1)^n\,\frac{\pi^2}{\lambda(1-\lambda+n)}\,\Psi_{\lambda\,,x_0}^{\mu_1\cdots\mu_n}$$

$$\langle\Phi_{\Delta_1}(x_1)\Phi_{\Delta_2}(x_2)\Phi_{\Delta}^{\mu_1\cdots\mu_n}(x_0)\rangle =$$

$$\frac{\left(\frac{x_0-x_1}{(x_0-x_1)^2}-\frac{x_0-x_2}{(x_0-x_2)^2}\right)^{\mu_1\cdots\mu_n}}{(x_1-x_2)^{\Delta_1+\Delta_2-\Delta+n}(x_0-x_1)^{\Delta_1+\Delta-\Delta_2-n}(x_0-x_2)^{\Delta_2+\Delta-\Delta_1-n}}$$

A.M. Polyakov, *Conformal symmetry of critical fluctuations,* JETP Lett. 12 (1970) 381-383, Pisma Zh.Eksp.Teor.Fiz. 12 (1970) 538-541.

A.M. Polyakov, *Nonhamiltonian approach to conformal quantum field theory,* Zh.Eksp.Teor.Fiz. 66 (1974) 23-42, Sov.Phys.JETP 39 (1974) 9-18.

Scalar product

With respect to the standard scalar product

$$(\Psi|\Phi) = \int d^4x_1 d^4x_2\, \Psi^*(x_1,x_2)\,\Phi(x_1,x_2)$$

the operator Q is Hermitian up to the equivalence transformation:

$$Q^\dagger = U\,Q\,U^{-1} \quad ; \quad U = (x_1-x_2)^{-2}\,\mathcal{P}_{12} \quad ; \quad \mathcal{P}_{12}\,\Phi(x_1,x_2) = \Phi(x_2,x_1)$$

We modify the scalar product

$$\langle\Psi|\Phi\rangle = (\Psi|U|\Phi) = \int d^4x_1 d^4x_2 \frac{\Psi^*(x_1\,,x_2)\,\Phi(x_2\,,x_1)}{(x_1-x_2)^2}$$

and with respect to this scalar product the operator Q is Hermitian.
Finally, the orthogonal system of eigenfunctions of the operator Q is given by $\lambda = \frac{1+n}{2} - i\nu\,, \nu \in R$

$$\Psi^{\mu_1\cdots\mu_n}_{\nu,x_0}(x_1\,,x_2) = \frac{\left(\frac{x_1-x_0}{(x_1-x_0)^2} - \frac{x_2-x_0}{(x_2-x_0)^2}\right)^{\mu_1\cdots\mu_n}}{(x_1-x_2)^{1+n-2i\nu}(x_1-x_0)^{3-n+2i\nu}(x_2-x_0)^{1-n+2i\nu}}$$

$$Q\,\Psi^{\mu_1\cdots\mu_n}_{\nu\,,x_0} = \frac{(-1)^n(2\pi)^2}{(1+n)^2+4\nu^2}\,\Psi^{\mu_1\cdots\mu_n}_{\nu\,,x_0}$$

Orthogonality and completeness

Orthogonality relation $\nu, \lambda \geq 0$

$$\langle \Psi^{\nu_1 \dots \nu_m}_{\lambda, y_0} | \Psi^{\mu_1 \dots \mu_n}_{\nu, x_0} \rangle = C(n\,,\nu)\,\delta_{nm}\,\delta(\nu - \lambda)\,\delta^4(x_0 - y_0)\,P^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_n}$$

$P^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_n}$ is the projector on the symmetric traceless tensors

$$C(n\,,\nu) = \frac{\pi^5}{2^{n+3}(1+n)\,\nu^2}\,\tau(\nu, n) \quad ; \quad \tau(\nu, n) = \frac{(-1)^n (2\pi)^2}{(1+n)^2 + 4\nu^2}$$

Completeness relation

$$I = \sum_{n=0}^{\infty} \int\limits_0^{\infty} \frac{d\nu}{C(n\,,\nu)} \int d^4 x_0\, |\Psi^{\mu_1 \cdots \mu_n}_{\nu, x_0}\rangle \langle \Psi^{\mu_1 \cdots \mu_n}_{\nu, x_0}| U.$$

$$\delta^4(x_1 - y_1)\,\delta^4(x_2 - y_2) = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\nu}{C(n\,,\nu)} \int d^4 x_0\, \frac{\Psi^{\mu_1 \dots \mu_n}_{-\nu, x_0}(x_1\,, x_2)\,\Psi^{\mu_1 \dots \mu_n}_{\nu, x_0}(y_2\,, y_1)}{(y_1 - y_2)^2}$$

V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova, I.T. Todorov, *Harmonic analysis on the n-dimensional Lorentz group and its application to conformal quantum field theory,* Lect.Notes Phys. 63 (1977) pp. 294.

Back to diagrams

$$G_2(x_2,y_1)=\frac{\pi^{2M}}{(x_2-y_1)^2}\,Z(M+1)=\int d^4x_1d^4y_2\,\frac{\langle x_1,x_2|Q^M|y_1,y_2\rangle}{(x_1-x_2)^2}$$

$$\langle x_1,x_2|Q^M|y_1,y_2\rangle=$$

$$\sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\nu}{C(n,\nu)}\int d^4x_0\,\langle x_1,x_2|Q^M|\Psi^{\mu_1\cdots\mu_n}_{\nu,x_0}\rangle\langle\Psi^{\mu_1\cdots\mu_n}_{\nu,x_0}|\,U|y_1,y_2\rangle=$$

$$\sum_{n=0}^{\infty}\int_0^{\infty}d\nu\,\frac{\tau^M(\nu,n)}{C(n,\nu)}\int d^4x_0\,\frac{\langle x_1,x_2|\Psi^{\mu_1\cdots\mu_n}_{\nu,x_0}\rangle\langle\Psi^{\mu_1\cdots\mu_n}_{\nu,x_0}|y_2,y_1\rangle}{(y_1-y_2)^2}=$$

$$\sum_{n=0}^{\infty}\int_0^{\infty}d\nu\,\frac{\tau^M(\nu,n)}{C(n,\nu)}\int d^4x_0\,\frac{\Psi^{\mu_1\cdots\mu_n}_{-\nu,x_0}(x_1,x_2)\,\Psi^{\mu_1\cdots\mu_n}_{\nu,x_0}(y_2,y_1)}{(y_1-y_2)^2}$$

Back to diagrams

$$G_2(x_2,y_1) = \frac{\pi^{2M}}{(x_2-y_1)^2}\, Z(M+1) =$$

$$\sum_{n=0}^{\infty}\int_0^{\infty} d\nu \frac{\tau^M(\nu,n)}{C(n\,,\nu)} \int d^4x_1 d^4y_2\, d^4x_0\, \frac{\Psi^{\mu_1\cdots\mu_n}_{-\nu,x_0}(x_1,x_2)\,\Psi^{\mu_1\cdots\mu_n}_{\nu,x_0}(y_2,y_1)}{(x_1-x_2)^2(y_1-y_2)^2} =$$

$$\frac{1}{(x_2-y_1)^2}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{2^n}\int_0^{\infty} d\nu\, \frac{\tau^{M+3}(\nu,n)}{C(n\,,\nu)}$$

where we use the integral

$$\int d^4x_1 d^4y_2\, d^4x_0\, \frac{\Psi^{\mu_1\cdots\mu_n}_{-\nu,x_0}(x_1,x_2)\,\Psi^{\mu_1\cdots\mu_n}_{\nu,x_0}(y_2,y_1)}{(x_1-x_2)^2(y_1-y_2)^2} = \frac{(-1)^n(n+1)}{2^n}\,\frac{\tau^3(\nu,n)}{(x_2-y_1)^2}$$

E.S. Fradkin, M.Y. Palchik, *Recent developments in conformal invariant quantum field theory,* Physics Reports, 44(5) (1978) 249-349.

S.D., A.P. Isaev, L. Shumilov, *Ladder and zig-zag Feynman diagrams, operator formalism and conformal triangles,* JHEP 06 (2023) 059 , e-Print: 2302.11238 [hep-th]

Back to diagrams

$$Z(M+1) = \pi^{-2M} \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2^n} \int\limits_0^{\infty} d\nu \, \frac{\tau^{M+3}(\nu, n)}{C(n\,,\nu)} =$$

$$\frac{2^{2M+7}}{\pi} \sum_{n=0}^{\infty} (-1)^{n(M+1)} (n+1)^2 \int\limits_0^{\infty} d\nu \, \frac{\nu^2}{\left((1+n)^2 + 4\nu^2\right)^{M+2}} =$$

$$4\, C_M \sum_{n=0}^{\infty} (-1)^{n(M+1)} \frac{1}{(n+1)^{2M-1}}$$

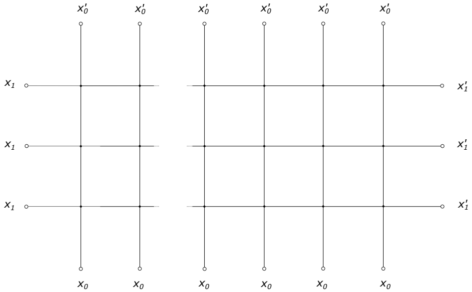
$C_M = \frac{1}{(M+1)} \binom{2M}{M}$ is the Catalan number

$$\int_0^{+\infty} d\nu \, \frac{\nu^2}{\left(4\nu^2 + (1+n)^2\right)^{M+2}} = \frac{1}{2^5 (n+1)^{2M+1}} \, \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(M + \frac{1}{2}\right)}{\Gamma\left(M+2\right)}$$

$$\frac{\Gamma(M+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(M+2)} = \frac{(2M)!\, \pi}{2^{2M} M!(M+1)!}$$

Basso-Dixon diagram

B.Basso, L.Dixon, *Gluing Ladder Feynman Diagrams into Fishnets*, Phys.Rev.Lett. 119 (2017) no.7, 071601.



N horizontal lines ($N = 3$ here), L vertical lines. Solid lines are the scalar propagators $1/(x-y)^2$ where x and y are the two endpoints of each segment. The boundary points are identified into four points (x_0, x_1, x_0', x_1').

$$I_{NL}(x_0, x_1, x_0', x_1') = \frac{(x_0 - x_0')^{-2(N+L)}}{(x_1 - x_0)^{2N}(x_1' - x_0)^{2N}} \left[\frac{(z\bar{z})^{\frac{1}{2}}}{z - \bar{z}}\right]^N I_{NL}(z, \bar{z})$$

$$u = \frac{x_{1'0}^2 x_{10'}^2}{x_{10}^2 x_{1'0'}^2} = z\bar{z} \quad ; \quad v = \frac{x_{11'}^2 x_{00'}^2}{x_{10}^2 x_{1'0'}^2} = (1-z)(1-\bar{z}) \,,$$

Basso-Dixon formula

Step one $\nu_k \in \mathbb{R}\,; \ell_k = 0, 1, 2, \ldots\,; k = 1, \ldots, N\;\; ; N < L + 1$

$$\boldsymbol{\nu} = (\nu_1\,, \nu_2, \ldots, \nu_N) \quad ; \quad \boldsymbol{\ell} = (\ell_1\,, \ell_2, \ldots, \ell_N)$$

$$I_{NL}(z, \bar{z}) = \sum_{\ell_1, \ldots, \ell_N} \int_{-\infty}^{+\infty} \frac{d\nu_1 \cdots d\nu_N}{(2\pi)^N N!}\, \mu(\boldsymbol{\nu}, \boldsymbol{\ell}) \prod_{k=1}^{N} (\ell_k + 1)\; \frac{z^{i\nu_k + \frac{\ell_k+1}{2}}\, \bar{z}^{i\nu_k - \frac{\ell_k+1}{2}}}{\left(\frac{(\ell_k+1)^2}{4} + \nu_k^2\right)^{N+L}}$$

$$\mu(\boldsymbol{\nu}, \boldsymbol{\ell}) \;=\; \prod_{1\le i<k\le N} \left((\nu_i - \nu_k)^2 + \frac{(\ell_i - \ell_k)^2}{4} \right) \left((\nu_i - \nu_k)^2 + \frac{(\ell_i + \ell_k + 2)^2}{4} \right)$$

Ladder diagramm $N = 1\,, L = p$

$$I_{1p}(z, \bar{z}) = \sum_{\ell} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi}\; (\ell + 1)\; \frac{z^{i\nu + \frac{\ell+1}{2}}\, \bar{z}^{i\nu - \frac{\ell+1}{2}}}{\left(\frac{(\ell+1)^2}{4} + \nu^2\right)^{p+1}}$$

Basso-Dixon formula

N. I. Usyukina and A. I. Davydychev, *Exact results for three and four point ladder diagrams with an arbitrary number of rungs,* Phys. Lett. B305, 136 (1993)

$$I_{1,p}(z,\bar{z}) = L_p(z,\bar{z}) = \sum_{j=0}^{p} \frac{(-1)^j(2p-j)!}{p!j!(p-j)!} \ln^j(z\bar{z}) \left(\mathrm{Li}_{2p-j}(z) - \mathrm{Li}_{2p-j}(\bar{z})\right)$$

with $\mathrm{Li}_k(z) = \sum_{\ell=1}^{+\infty} z^\ell/\ell^k$ – the polylogarithm.
Step two

$$I_{NL}(z,\bar{z}) = \frac{\det M}{\prod_{k=1}^{N}(L-N+2k-2)!(L-N+2k-1)!}$$

where M is a $N \times N$ Hankel matrix with ij element

$$M_{ij} = (L-N+i+j-2)!(L-N+i+j-1)! \times L_{L-N+i+j-1}(z,\bar{z})$$

B.Basso, L.Dixon, D.Kosower, A.Krajenbrink, De-liang Zhong, *Fishnet four-point integrals: integrable representations and thermodynamic limits,* JHEP 07 (2021) 168

Plan

- Graph building operator and commuting Q-operators
- Construction of eigenfunctions of the Q-operator

S.D., V.Kazakov, E.Olivucci, *Basso-Dixon Correlators in Two-Dimensional Fishnet CFT,* JHEP 1904 (2019) 032

S.D., E.Olivucci, *Exactly solvable magnet of conformal spins in four dimensions,* Phys.Rev.Lett. 125 (2020) 3, 031603

S.D., G.Ferrando, E.Olivucci, *Mirror channel eigenvectors of the d-dimensional fishnets,* e-Print: 2108.12620 [hep-th]

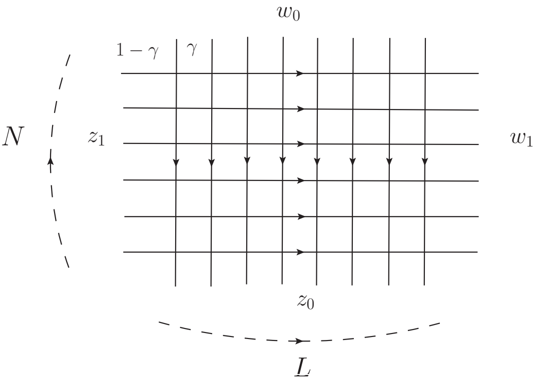
Based on the previous works S.D., G. Korchemsky, A. Manashov, *Noncompact Heisenberg spin magnets from high-energy QCD,* Nucl.Phys. B617 (2001) 375-440

S.D., A.Manashov, *Iterative construction of eigenfunctions of the monodromy matrix for SL(2,C) magnet,* J.Phys. A47 (2014) 305204

Heavily based on uniqueness method

A. N. Vasiliev, Y. M. Pismak, Y. R. Khonkonen, *$1/N$ Expansion: Calculation of the exponent eta in the order $1/N^3$ by the conformal bootstrap method,* Theor. Math. Phys. 50 (1982) 127-134

Two-dimensional Basso-Dixon diagram



$$\eta = \frac{z_0 - w_1}{z_0 - z_1} \frac{w_0 - z_1}{w_0 - w_1}$$

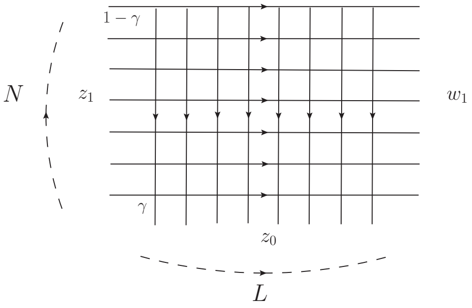
$$z \xrightarrow{\ \alpha\ } w \quad = \frac{1}{[w-z]^{\alpha}}$$

The propagator in $d = 2$ is given by the following expression $(\alpha - \bar{\alpha} \in Z)$

$$\frac{1}{[z-w]^{\alpha}} \equiv \frac{1}{(z-w)^{\alpha}(\overline{z}-\overline{w})^{\bar{\alpha}}} = \frac{(\overline{z}-\overline{w})^{\alpha-\bar{\alpha}}}{|z-w|^{2\alpha}}$$

$$I^{BD}_{L,N}(z_0, z_1, w_0, w_1) = \frac{[z_1 - z_0]^{(\gamma-1)N}[w_1 - w_0]^{(\gamma-1)N}}{[z_0 - w_0]^{(\gamma-1)N+\gamma L}} [\eta]^{\frac{\gamma-1}{2}N} B_{L,N}(\eta, \bar{\eta})$$

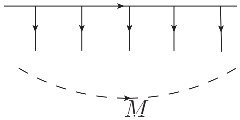
Two-dimensional Basso-Dixon diagram



$$\eta = \frac{z_0-w_1}{z_0-z_1}\frac{w_0-z_1}{w_0-w_1} \to \eta = \frac{z_0-w_1}{z_0-z_1}$$

$$w_0 \to \infty$$

$I_{M+1}(\eta, \bar{\eta})$



$$I^{BD}_{L,N}(z_0, z_1, w_0, w_1) = \frac{[z_1 - z_0]^{(\gamma-1)N}[w_1 - w_0]^{(\gamma-1)N}}{[z_0 - w_0]^{(\gamma-1)N+\gamma L}} [\eta]^{\frac{\gamma-1}{2}N} B_{L,N}(\eta, \bar{\eta})$$

$$B_{L,N}(\eta, \bar{\eta}) = (2\pi)^{-N}\pi^{-N^2} \underset{1\leq j,k\leq N}{Det} \left[(\eta\partial_\eta)^{i-1}(\bar{\eta}\partial_{\bar{\eta}})^{k-1} I_{N+L}(\eta, \bar{\eta})\right]$$

Two-dimensional Basso-Dixon diagram

Step one

$$B_{L,N}(\eta,\bar\eta) \leftrightarrow \int \mathcal{D}x_1 \cdots \mathcal{D}x_N \prod_{k<j}[x_k - x_j] \prod_{k=1}^{N} \left([\eta]^{ix_k} \lambda^{N+L}(x_k) \right)$$

where

$$x = \nu + i\frac{n}{2} \ , \ \bar{x} = \nu - i\frac{n}{2} \ , \ \int \mathcal{D}x = \sum_{n\in Z} \int_{-\infty}^{+\infty} d\nu$$

$$\prod_{k<j}[x_k - x_j] = \prod_{k<j}(x_k - x_j)(\bar{x}_k - \bar{x}_j) = \prod_{k<j}(x_k - x_j) \prod_{k<j}(\bar{x}_k - \bar{x}_j)$$

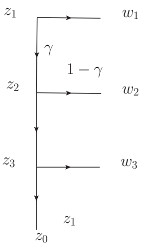
$$\mu(\boldsymbol{\nu},\boldsymbol{\ell}) = \prod_{1\le i<k\le N} \left((\nu_i - \nu_k)^2 + \tfrac{(\ell_i-\ell_k)^2}{4} \right) \left((\nu_i - \nu_k)^2 + \tfrac{(\ell_i+\ell_k+2)^2}{4} \right)$$

Step two – Determinant representation

$$\int \mathcal{D}x_1 \cdots \mathcal{D}x_N \prod_{k<j}[x_k - x_j] \prod_{k=1}^{N} [\eta]^{ix_k} \lambda^{N+L}(x_k) = N!\, Det\, M$$

$$M_{ik} = \int \mathcal{D}x\ x^{i-1}\bar{x}^{j-1}[\eta]^{ix} \lambda^{N+L}(x) = (\eta\partial_\eta)^{i-1}(\bar\eta\partial_{\bar\eta})^{k-1} \int \mathcal{D}x\,[\eta]^{ix} \lambda^{N+L}(x)$$
$$i,k = 1,\ldots,N$$

Basso-Dixon integrals



$$\mathbf{V} \leftrightarrow \mathbf{V}(\mathbf{z_1}, \mathbf{z_2}, \mathbf{z_3} | \mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3})$$

$\mathbf{V^4}$

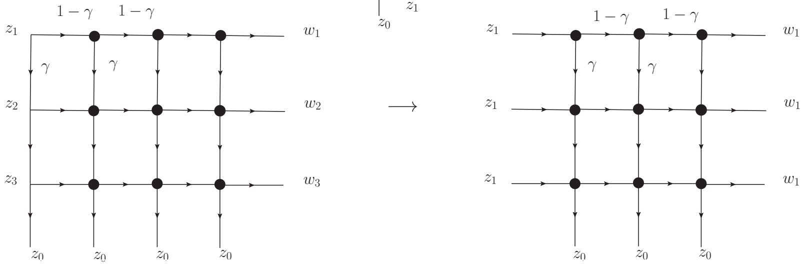
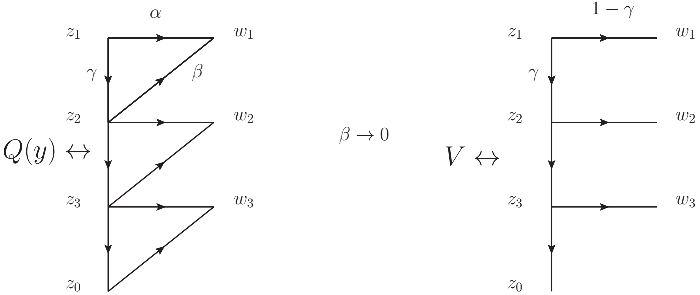


Figure: The reduction to the Basso-Dixon diagram: amputation of the left lines and gluing the end points $z_k \to z_1$, $w_k \to w_1$. There are three external points z_1, w_1 and z_0.

Q-operator $\to$ graph-building operator



- Commuting family of operators $[Q(y), Q(y')] = 0$.
  The diagram for the Q-operator $Q(y)$
  $\alpha = 1 - s - iy\,, \beta = 1 - s + iy\,, \gamma = 2s - 1.$
- Reduction of the diagram for $\beta \to 0$ or, equivalently, $iy \to s - 1$

$SL(2,\mathbb{C})$ spin magnet

$SL(2,\mathbb{C})$ spin magnet is a straightforward generalization of the XXX_s spin chain. $(2s+1)$-dimensional representations of the $SU(2)$ group $\longrightarrow$ unitary principal series representations of the $SL(2,\mathbb{C})$ group

$$\mathbb{V}_k = \mathbb{C}^{2s+1} \to \mathbb{V}_k = L_2(\mathbb{C})$$

unitary principal series representations of $\mathrm{SL}(2,\mathbb{C})$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \quad ; \quad g \to \mathrm{T}^{(s,\bar{s})}(g) : L_2(\mathbb{C}) \to L_2(\mathbb{C})$$

$$\left[\mathrm{T}^{(s,\bar{s})}(g)\,\phi\right](z,\bar{z}) = (d-bz)^{-2s}\,\left(\bar{d}-\bar{b}\bar{z}\right)^{-2\bar{s}}\,\phi\left(\frac{-c+az}{d-bz}, \frac{-\bar{c}+\bar{a}\bar{z}}{\bar{d}-\bar{b}\bar{z}}\right)$$

$$\langle \phi\,|\psi\rangle = \int d^2z\,\overline{\phi(z,\bar{z})}\,\psi(z,\bar{z}) \quad ; \quad \langle\, T^{(s,\bar{s})}(g)\phi\,|\,T^{(s,\bar{s})}(g)\psi\,\rangle = \langle \phi\,|\psi\rangle$$

The spins s and $\bar{s}$ are parameterized as follows $(n_s \in \mathbb{Z}\,, \nu_s \in \mathbb{R})$

$$s = \frac{1+n_s}{2} + i\nu_s \quad ; \quad \bar{s} = \frac{1-n_s}{2} + i\nu_s$$

generators

$$S_- = -\partial_z \quad , \quad S = z\partial_z + s \quad , \quad S_+ = z^2\partial_z + 2s\,z$$
$$\bar{S}_- = -\partial_{\bar{z}} \quad , \quad \bar{S} = \bar{z}\partial_{\bar{z}} + \bar{s} \quad , \quad \bar{S}_+ = \bar{z}^2\partial_{\bar{z}} + 2\bar{s}\,\bar{z}$$

$SL(2,\mathbb{C})$ spin magnet

The Hilbert space of the $SL(2,\mathbb{C})$ spin magnet is given by the tensor product of the unitary principal series representations of the $SL(2,\mathbb{C})$ group

$$\mathbb{H}_N = \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \cdots \otimes \mathbb{V}_N, \qquad \mathbb{V}_k = L_2(\mathbb{C}), \qquad k = 1, \dots, N.$$

The space $\mathbb{H}_N$ is the space of functions $\Psi(z_1, \bar{z}_1 \dots, z_N, \bar{z}_N)$.
To each site k we associate the pair of quantum L-operators with subscript k acting nontrivially on the $k-$th space in the tensor product

$$L_k(u) = \left(\begin{array}{cc} u + iS^{(k)} & iS_-^{(k)} \\ iS_+^{(k)} & u - iS^{(k)} \end{array}\right), \qquad \bar{L}_k(\bar{u}) = \left(\begin{array}{cc} \bar{u} + i\bar{S}^{(k)} & i\bar{S}_-^{(k)} \\ i\bar{S}_+^{(k)} & \bar{u} - i\bar{S}^{(k)} \end{array}\right),$$

$\bar{u}$ is complex conjugate to u

$$T(u) = L_1(u)L_2(u)\dots L_N(u) \qquad \bar{T}(\bar{u}) = \bar{L}_1(\bar{u})\bar{L}_2(\bar{u})\dots \bar{L}_N(\bar{u})$$

$$T(u) = \left(\begin{array}{cc} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{array}\right) \qquad \bar{T}(\bar{u}) = \left(\begin{array}{cc} \bar{A}_N(\bar{u}) & \bar{B}_N(\bar{u}) \\ \bar{C}_N(\bar{u}) & \bar{D}_N(\bar{u}) \end{array}\right)$$

R-operator and Q-operator

$$[R_{12}(x)\Phi]\,(z_1,z_2) = \int d^2w_1\, \frac{[z_1-z_2]^{1-2s}}{[w_1-z_1]^{1-s-ix}[w_1-z_2]^{1-s+ix}}\,\Phi(w_1,z_2)$$

$$x = \tfrac{in}{2}+\nu \quad ; \quad \bar{x} = -\tfrac{in}{2}+\nu \quad ; \quad s = \tfrac{1+n_s}{2}+i\nu_s \quad ; \quad \bar{s} = \tfrac{1-n_s}{2}+i\nu_s$$

Let us define operator

$$Q(x) = R_{12}(x)\,R_{23}(x)\cdots R_{N-1\,N}(x)\,R_{N0}(x)$$

Using the defining relation for $R_{12}(x)$ it is possible to prove the following relations which are characteristic properties of Q-operators

- commutation relation with operator $A_N(u)+z_0B_N(u)$

$$Q(x)\,\left(A_N(u)+z_0B_N(u)\right) = \left(A_N(u)+z_0B_N(u)\right)\,Q(x)$$

- mutual commutativity

$$Q(x)\,Q(y) = Q(y)\,Q(x)$$

- Baxter relations

$$\left(A_N(u)+z_0B_N(u)\right)\,Q(u,\bar{u}) = (u+is)^N\,Q(u+i,\bar{u})$$

Eigenfunctions of operator $A(u)+z_0B(u)$

Operator $A(u)+z_0B(u)$ is polynomial in u

$[A(u)+z_0B(u)]\ \Psi_{x_1\ldots x_N}(z_1,\ldots,z_N)=(u-x_1)\cdots(u-x_N)\,\Psi_{x_1\ldots x_N}(z_1,\ldots,z_N)$

Spectrum is continuous and simple.

Eigenfunctions are labelled by N zeroes
$x_k=\nu_k+i\frac{n_k}{2}$; $\nu_k\in\mathbb{R}$ $n_k\in\mathbb{Z}$; $k=1,\ldots,N$ of polynomial eigenvalue

Orthogonality relation

$$\int \overline{\Psi_{x_1\ldots x_N}(z)}\,\Psi_{x_1'\ldots x_N'}(z)\prod_{j=1}^{N}d^2z_j=\mu^{-1}(x)\,\delta_{sym}(x-x')$$

$$\delta_{sym}\left(x-x'\right)=\frac{1}{N!}\sum_{\sigma\in S_N}\prod_{k=1}^{N}\delta_{n_i n_k'}\delta(\nu_i-\nu_k')$$

Sklyanin measure

$$\boldsymbol{\mu}(\boldsymbol{x})=C_N\prod_{k<j\le N}[x_k-x_j]=C_N\prod_{k<j\le N}(x_k-x_j)\prod_{k<j\le N}(\bar{x}_k-\bar{x}_j)$$

$$C_N=\frac{(2\pi)^{-N}\pi^{-N^2}}{N!}$$

Completeness

Completeness relation

$$\int \mathcal{D}_N x\, \mu(x)\, \Psi_{x_1\ldots x_N}(z)\, \overline{\Psi_{x_1\ldots x_N}(z')} = \prod_{k=1}^{N} \delta^2(\vec{z}_k - \vec{z}_k')$$

$$\Downarrow$$

$$C_N \int \mathcal{D}_N x \prod_{k<j} [x_k - x_j]\, \Psi_{x_1\ldots x_N}(z)\, \overline{\Psi_{x_1\ldots x_N}(z')} = \prod_{k=1}^{N} \delta^2(\vec{z}_k - \vec{z}_k')$$

The symbol $\mathcal{D}_N x$ stands for

$$\int \mathcal{D}_N x = \prod_{k=1}^{N} \left(\sum_{n_k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu_k \right)$$

Alexander N. Manashov, *Unitarity of the SoV Transform for $SL(2, C)$ Spin Chains,* SIGMA 19 (2023) 086 , e-Print: 2303.11461 [math-ph]

Iterative construction of eigenfunctions $\Psi_{x_1\dots x_N}(z_1\,,\,\dots\,,z_N)$

- Common eigenfunction of two commuting operators

$$[A(u)+z_0B(u)]\;\Psi_{x_1\dots x_N}(z)=(u-x_1)\cdots(u-x_N)\,\Psi_{x_1\dots x_N}(z)$$
$$Q_N(u)\,\Psi_{x_1\dots x_N}(z)=q(u\,,x_1)\cdots q(u\,,x_N)\,\Psi_{x_1\dots x_N}(z)$$

- Iterative construction

$$\Psi_{x_1\,,\dots\,,x_N}(z_1\,,\,\dots\,,z_N)=\Lambda^{(N)}(x_N)\cdots\Lambda^{(2)}(x_2)\,\Psi_{x_1}$$

- Main commutation relation $n=2\,,\,\dots\,,N$

$$Q_n(u)\,\Lambda^{(n)}(x_n)=q(u\,,x_n)\,\Lambda^{(n)}(x_n)\,Q_{n-1}(u)$$

$$Q_N(u)\,\Lambda^{(N)}(x_N)\,\Lambda^{(N-1)}(x_{N-1})\,\cdots\,\Lambda^{(2)}(x_2)\,\Psi_{x_1}=$$
$$q(u,x_N)\,\Lambda^{(N)}(x_N)\,Q_{N-1}(u)\,\Lambda^{(N-1)}(x_{N-1})\,\cdots\,\Lambda^{(2)}(x_2)\,\Psi_{x_1}=$$
$$q(u,x_N)\,q(u,x_{N-1})\,\Lambda^{(N)}(x_N)\,\Lambda^{(N-1)}(x_{N-1})\,Q_{N-2}(u)\,\cdots\,\Lambda^{(2)}(x_2)\,\Psi_{x_1}$$

- Symmetry $x_i\leftrightarrow x_k$

$$\Lambda^{(n)}(x)\,\Lambda^{(n-1)}(x')=\Lambda^{(n)}(x')\,\Lambda^{(n-1)}(x)$$

- Calculation of the scalar product is based on relation $(x\neq x')$

$$\left(\Lambda^{(n)}(x)\right)^\dagger\;\Lambda^{(n)}(x')=\frac{1}{[x-x']}\,\Lambda^{(n-1)}(x')\;\left(\Lambda^{(n-1)}(x)\right)^\dagger$$

Basic example $N=1$. Diagonalisation of $A(u)+z_0B(u)$.

$$\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = i \begin{pmatrix} s-iu+z\partial & -\partial \\ z^2\partial+2sz & -s-iu-z\partial \end{pmatrix}$$

$$(A(u)+z_0\,B(u))\;\Psi_{x,\bar{x}}(z\,,\bar{z}) = (u+is+i(z-z_0)\partial)\,\Psi_{x,\bar{x}} = (u-x)\,\Psi_{x,\bar{x}}$$
$$\left(\bar{A}(\bar{u})+\bar{z}_0\,\bar{B}(u)\right)\;\Psi_{x,\bar{x}}(z\,,\bar{z}) = (\bar{u}+i\bar{s}+i(\bar{z}-\bar{z}_0)\bar{\partial})\,\Psi_{x,\bar{x}} = (\bar{u}-\bar{x})\,\Psi_{x,\bar{x}}$$

eigenfunction $x=\frac{in}{2}+\nu\;\;;\;\;\bar{x}=-\frac{in}{2}+\nu\;\;;\;\;s=\frac{1+n_s}{2}+i\nu_s\;\;;\;\;\bar{s}=\frac{1-n_s}{2}+i\nu_s$

$$\Psi_{x,\bar{x}}(z\,,\bar{z}) = [z-z_0]^{ix-s} = (z-z_0)^{ix-s}(\bar{z}-\bar{z}_0)^{i\bar{x}-\bar{s}}$$

orthogonality $x_1=\frac{in_1}{2}+\nu_1\,,x_2=\frac{in_2}{2}+\nu_2$

$$\int d^2z\,\overline{[z-z_0]^{ix_1-s}}[z-z_0]^{ix_2-s} = 2\pi^2\,\delta_{n_1n_2}\,\delta(\nu_1-\nu_2)$$

completeness $x=\frac{in}{2}+\nu$

$$\sum_{n\in\mathbb{Z}}\int_{-\infty}^{+\infty}\mathrm{d}\nu\,\overline{[z_1-z_0]^{ix-s}}\,[z_2-z_0]^{ix-s} = 2\pi^2\,\delta^2(\vec{z}_1-\vec{z}_2)$$

Q-operator for $N = 1$

$$[Q(u)\Phi](z) = \int d^2w \, \frac{[z - z_0]^{1-2s}}{[w - z]^{1-s-iu}[w - z_0]^{1-s+iu}} \, \Phi(w) \quad ; \quad u = \frac{in_u}{2} + \nu_u$$

$$Q(u)\,\Psi_{x,\bar{x}}(z,\bar{z}) = q(u\,,x)\,\Psi_{x,\bar{x}}(z,\bar{z})$$

$$\Psi_{x,\bar{x}}(z,\bar{z}) = [z - z_0]^{ix-s} \quad ; \quad q(u,x) = \pi\,(-1)^{n_u+n}\, a(1 - s - iu, s + ix, 1 + iu - ix)$$

Chain relation

$$\int d^2w \, \frac{1}{[z_1 - w]^{\alpha}[w - z_2]^{\beta}} = \pi\,(-1)^{\gamma-\bar{\gamma}}\, a(\alpha,\beta,\gamma)\, \frac{1}{[z_1 - z_2]^{\alpha+\beta-1}}$$

$$a(\alpha) = \frac{\Gamma(1-\bar{\alpha})}{\Gamma(\alpha)}\,, \quad a(\alpha,\beta,\gamma,\ldots) = a(\alpha)a(\beta)a(\gamma)\ldots.$$

Diagonalization of the operator $Q(u) \leftrightarrow$ Transition to the representation of separated variables

E.K. Sklyanin, *The Quantum Toda Chain*, Lect.Notes Phys. 226 (1985) 196-233

E.K. Sklyanin, *Separation of variables - new trends*, Prog.Theor.Phys.Suppl. 118 (1995) 35-60 , e-Print: solv-int/9504001 [nlin.SI]

Diagram technique: main rules

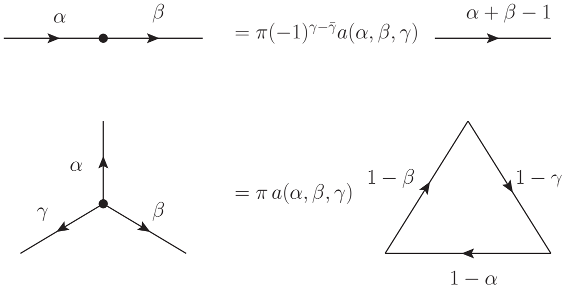


Figure: The chain and star–triangle relations, $\alpha+\beta+\gamma=2$.

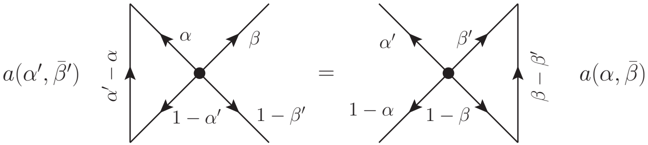
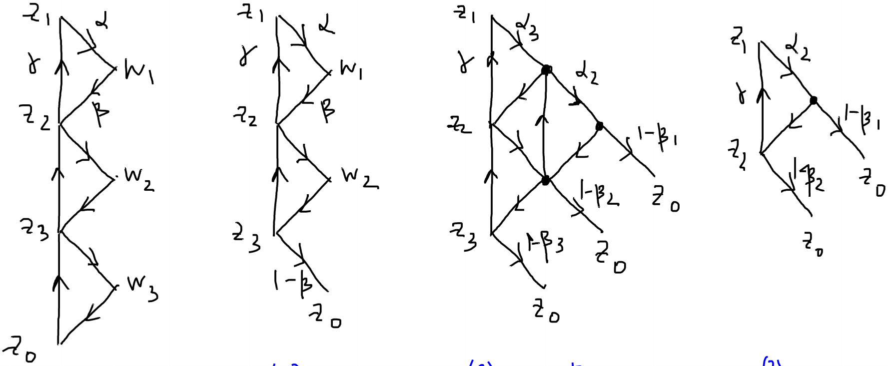


Figure: The cross relation, $\alpha+\beta=\alpha'+\beta'$.

Diagram technique: Q-operators and Λ-operators



$Q_3(x)$ $\quad$ $\Lambda^{(3)}(x)$ $\quad$ $\Lambda^{(3)}(x_3)\,\Lambda^{(2)}(x_2)\,\Psi_{x_1}$ $\quad$ $\Lambda^{(2)}(x_2)\,\Psi_{x_1}$

$\alpha = 1 - s - ix \qquad \beta = 1 - s + ix \qquad \gamma = 2s - 1$

Diagram technique: cross relation and commutativity

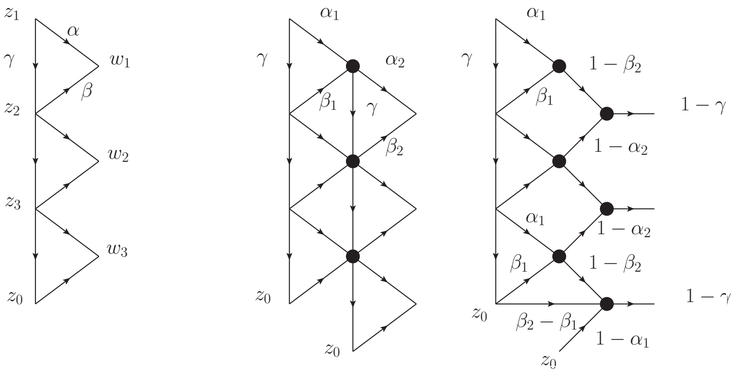
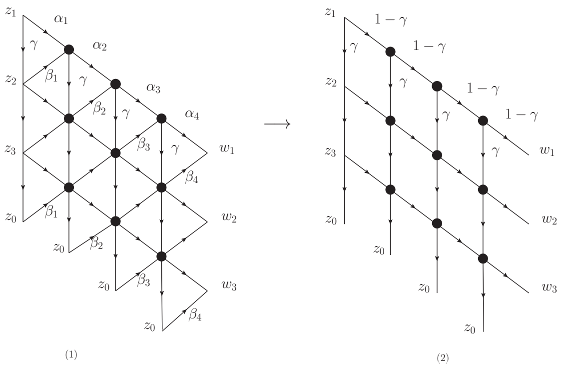


Figure: Diagrammatic proof of commutativity. $\frac{1}{[z-z_0]^{1-\alpha_2}} = \frac{1}{[z-z_0]^{\beta_2-\beta_1}} \frac{1}{[z-z_0]^{1-\alpha_1}}$

$$\alpha_1 = 1-s-ix_1\,,\beta_1 = 1-s+ix_1\,,\gamma = 2s-1$$
$$\alpha_2 = 1-s-ix_2\,,\beta_2 = 1-s+ix_2\,,\gamma = 2s-1$$
$$Q(x_1)\,Q(x_2) = Q(x_2)\,Q(x_1)$$

Back to Basso-Dixon integrals



Representation for the kernel of the operator $Q(y_1)\cdots Q(y_{L+1})$

$$[Q(y_1)\cdots Q(y_{L+1})]\,(z|w) =$$

$$= \frac{(2\pi)^{-N}\pi^{-N^2}}{N!}\int \mathcal{D}_N x\,\prod_{k<j}[x_k - x_j]\,\prod_{k=1}^{N}\prod_{l=1}^{L+1} q(y_l, x_k)\,\Psi_{x_1\ldots x_N}(z)\,\overline{\Psi_{x_1\ldots x_N}(w)}$$

Determinant representation for the Basso-Dixon integrals

$$Amp\ \ [Q(y_1)\cdots Q(y_{L+1})]\,(z_1\ldots z_N|w_1\ldots w_N)|_{\substack{z_k\to z_1\\ w_k\to w_1\\ iy_k\to s-1}} =$$

$$([z_0-z_1]\,[z_0-w_1])^{N(s-1)}$$

$$\frac{(2\pi)^{-N}\pi^{-N^2}}{N!}\ \int\mathcal{D}_N\boldsymbol{x}\ \prod_{k=1}^N\ \left([\eta]^{ix_k}\,\lambda^{N+L}(x_k)\right)\ \prod_{k<j}[x_k-x_j]$$

where

$$\lambda(x_k)=\pi a(2-2s,s+ix_k,s-ix_k)\,(-1)^{n_s+n_k}\quad;\quad\eta=\frac{z_0-w_1}{z_0-z_1}$$

$$s=\frac{1+n_s}{2}+i\nu_s\quad;\quad x_k=\frac{in_k}{2}+\nu_k$$

Determinant representation

$$\int\mathcal{D}_N\boldsymbol{x}\ \prod_{k<j}[x_k-x_j]\ \prod_{k=1}^N[\eta]^{ix_k}\,\lambda^{N+L}(x_k)=N!\ Det\ M$$

$$M_{ik}=\int\mathcal{D}x\ x^{i-1}\bar{x}^{j-1}[\eta]^{ix}\,q^{N+L}(x)=(\eta\partial_\eta)^{i-1}(\bar{\eta}\partial_{\bar{\eta}})^{k-1}\ \int\mathcal{D}x\,[\eta]^{ix}\,\lambda^{N+L}(x)$$

$$i,k=1\,,\ldots\,,N$$

4-dimensional Basso-Dixon diagram

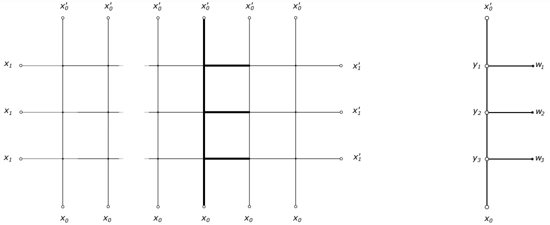
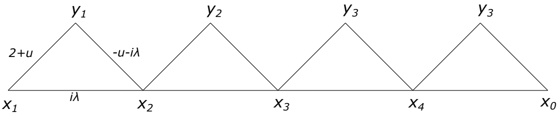
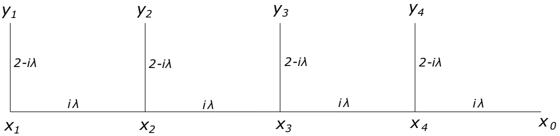


Diagram $\leftrightarrow V_N^{L+1}$





- The diagram for the Q-operator $Q_N(u)$: one-parametric family of commuting operators $[Q_N(u), Q_N(v)] = 0$.
- Reduction of the diagram: $u \to -i\lambda$ and $\lambda \to -i$ gives $Q_N(u) \to V_N$.

Diagonalization of the operator $V_N \leftrightarrow$ Transition to the representation of separated variables

- Spectrum of operator V_N is continuous
- Eigenfunctions $\nu_k \in \mathbb{R}\,, \ell_k\,, m_k \in \mathbb{Z}_+$

$$\Psi_{\nu_1,\ell_1,m_1\,,\dots,\nu_N,\ell_N,m_N}(x_1\,,\dots\,,x_N) = |\nu_1,\ell_1,m_1\,,\dots\,,\nu_N,\ell_N,m_N\rangle = |\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}\rangle$$

$$V_N\,|\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}\rangle = \lambda(\nu_1,\ell_1)\cdots\lambda(\nu_N,\ell_N)\,|\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}\rangle$$

One point example

$$[V_1\,\Phi]\,(x) \leftrightarrow (x-x_0)^{-2}\int d^4y\,(x-y)^{-2}\,\Phi(y)\,.$$

$$\Psi_{\nu,\ell,m}(x) = \frac{(x-x_0)^{\mu_1\cdots\mu_\ell}}{(x-x_0)^{2(1+i\nu+\frac{\ell}{2})}} \quad;\quad \lambda(\nu,\ell) = \frac{\pi^2}{\left(1-i\nu+\frac{\ell}{2}\right)\left(i\nu+\frac{\ell}{2}\right)}$$

$$\int d^4x\,\frac{x^{\alpha_1\dots\alpha_\ell}}{x^{2(1+\ell/2+i\nu)}}\,\frac{x^{\beta_1\dots\beta_{\ell'}}}{x^{2(1+\ell'/2-i\nu')}} = c_\ell\delta_{\ell\ell'}\delta\left(\nu-\nu'\right)P^{\alpha_1\dots\alpha_\ell}_{\beta_1\dots\beta_\ell}$$

$$\sum_{\ell\geq 0}\frac{1}{c_\ell}\int_{\mathbb{R}} d\nu\,\frac{x^{\mu_1\dots\mu_\ell}}{x^{2(1+\ell/2+i\nu)}}\,\frac{y^{\mu_1\dots\mu_\ell}}{y^{2(1+\ell/2-i\nu)}} = \delta^{(4)}\left(x-y\right) \quad;\quad c_\ell = \frac{\pi^3}{2^{\ell-1}\left(\ell+1\right)},$$

Eigenfunctions

$$V_N \, |\boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}\rangle = \lambda(\nu_1, \ell_1) \cdots \lambda(\nu_N, \ell_N) \, |\boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}\rangle$$

- Iterative construction

$$\Psi_{\nu_1, \ell_1, m_1, \ldots, \nu_N, \ell_N, m_N}(x_1, \ldots, x_N) = \Lambda^{(N)}_{\nu_N, \ell_N, m_N} \cdots \Lambda^{(2)}_{\nu_2, \ell_2, m_2} \Psi_{\nu_1, \ell_1, m_1}$$

- Main commutation relation

$$V_N \, \Lambda^{(N)}_{\nu_N, \ell_N, m_N} = \lambda(\nu_N, \ell_N) \, \Lambda^{(N)}_{\nu_N, \ell_N, m_N} \, V_{N-1}$$

$$V_N \, \Lambda^{(N)} \Lambda^{(N-1)} \cdots \Lambda^{(2)} \Psi_{\nu_1, \ell_1, m_1} = \lambda(\nu_N, \ell_N) \, \Lambda^{(N)} \, V_{N-1} \, \Lambda^{(N-1)} \cdots \Lambda^{(2)} \Psi_{\nu_1, \ell_1, m_1} =$$

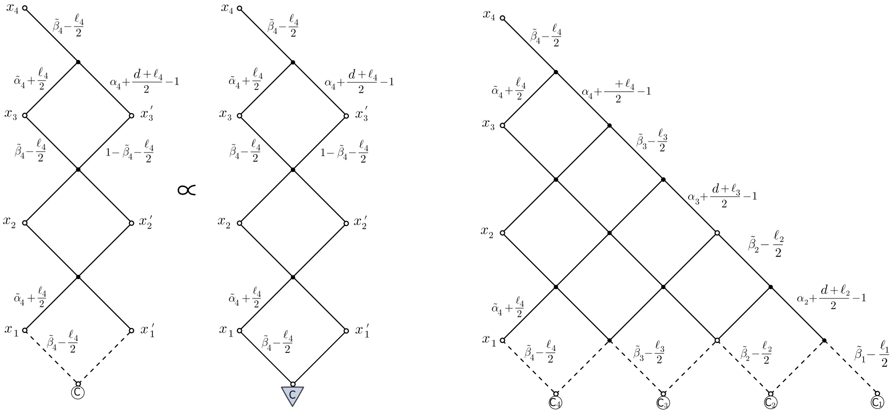
$$\lambda(\nu_N, \ell_N) \, \lambda(\nu_{N-1}, \ell_{N-1}) \, \Lambda^{(N)} \Lambda^{(N-1)} \, V_{N-1} \cdots \Lambda^{(2)}_{\nu_2, \ell_2, m_2} \Psi_{\nu_1, \ell_1, m_1} = \ldots$$

- Symmetry $\leftrightarrow$ Faddeev-Zamolodchikov algebra

$$\Lambda^{(N)}_{\nu, \ell, m} \, \Lambda^{(N-1)}_{\nu', \ell', m'} = R^{nn'}_{mm'}(\nu - \nu') \, \Lambda^{(N)}_{\nu', \ell', n'} \, \Lambda^{(N-1)}_{\nu, \ell, n}$$

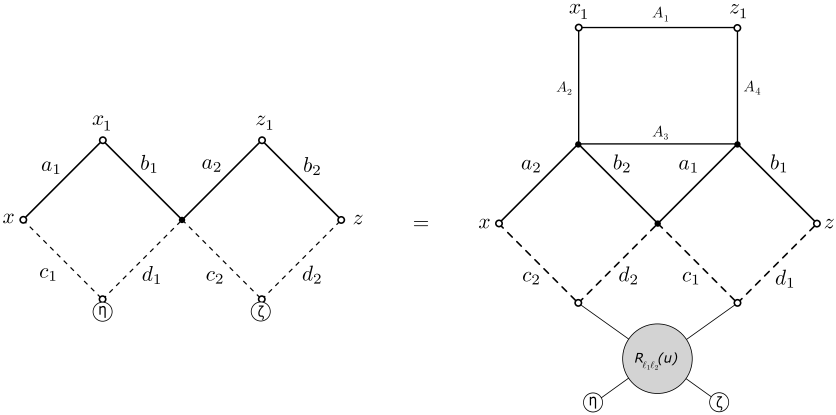
where $R(\nu - \nu')$ – R-matrix, solution of the Yang-Baxter equation

Eigenfunctions



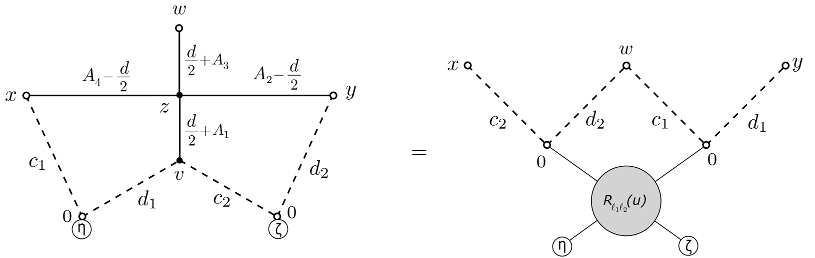
Layer operator $\Lambda^{(3)}$ and construction of eigenfunction.

Eigenfunctions



Interchange relation $\leftrightarrow$ symmetry of eigenfunctions $\leftrightarrow$ Faddeev-Zamolodchikov algebra

Eigenfunctions



For simplicity $x_0 \to 0$

$$\int d^d z d^d v \frac{(z-w)^{2\left(iu+\frac{\ell_1+\ell_2}{2}-1\right)} \left[\zeta\cdot\left(\frac{y}{y^2}-\frac{v}{v^2}\right)\right]^{\ell_1} \left[\eta\cdot\left(\frac{x}{x^2}-\frac{v}{v^2}\right)\right]^{\ell_2}}{(z-x)^{2\left(iu+\frac{\ell_{21}}{2}\right)}(z-y)^{2\left(iu+\frac{\ell_{12}}{2}\right)}(z-v)^{2\left(d-1+\frac{\ell_1+\ell_2}{2}-iu\right)} v^{2\left(1-\frac{\ell_1+\ell_2}{2}+iu\right)}}$$

$$\leftrightarrow \frac{w^{2\left(iu+\frac{\ell_1+\ell_2}{2}-1\right)}}{x^{2\left(iu+\frac{\ell_{21}}{2}\right)} y^{2\left(iu+\frac{\ell_{12}}{2}\right)}} \left[R_{\ell_1,\ell_2}(u)\zeta^{\otimes\ell_1}\otimes\eta^{\otimes\ell_2}\right]\cdot\left[\left(\frac{x}{x^2}-\frac{w}{w^2}\right)^{\otimes\ell_1}\otimes\left(\frac{y}{y^2}-\frac{w}{w^2}\right)^{\otimes\ell_2}\right]$$

Representation of separated variables

- Orthogonality

$$\langle \boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m} | \boldsymbol{\nu}', \boldsymbol{\ell}', \boldsymbol{m}' \rangle = \mu^{-1}(\boldsymbol{\nu}, \boldsymbol{\ell})\, \delta(\boldsymbol{\nu}, \boldsymbol{\ell} | \boldsymbol{\nu}', \boldsymbol{\ell}')\, P_{\boldsymbol{m}\boldsymbol{m}'}$$

$$\Lambda^{(N)\dagger}_{\nu',\ell',m'}\,\Lambda^{(N)}_{\nu,\ell,m} \quad = \quad \frac{R^{nm'}_{mn'}(\nu-\nu')\,\Lambda^{(N-1)}_{\nu,\ell,n}\,\Lambda^{(N-1)\dagger}_{\nu',\ell',n'}}{\left[(\nu-\nu')^2+\frac{(\ell-\ell')^2}{4}\right]\left[(\nu-\nu')^2+\frac{(d-2+\ell+\ell')^2}{4}\right]}$$

- Completeness

$$\sum_{\boldsymbol{\ell},\boldsymbol{m}} \int d^N\boldsymbol{\nu}\, \mu(\boldsymbol{\nu}, \boldsymbol{\ell})\, |\boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}\rangle\langle \boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}| = \mathbb{1}$$

$$\mu(\boldsymbol{\nu}, \boldsymbol{\ell}) \;=\; \prod_{1\le i<k\le N} \left((\nu_i-\nu_k)^2+\frac{(\ell_i-\ell_k)^2}{4}\right)\left((\nu_i-\nu_k)^2+\frac{(\ell_i+\ell_k+d-2)^2}{4}\right)$$

$$\mu(\boldsymbol{\nu}, \boldsymbol{\ell}) \;=\; \prod\nolimits_{1\le i<k\le N} \left((\nu_i-\nu_k)^2+\tfrac{(\ell_i-\ell_k)^2}{4}\right)\left((\nu_i-\nu_k)^2+\tfrac{(\ell_i+\ell_k+2)^2}{4}\right)$$

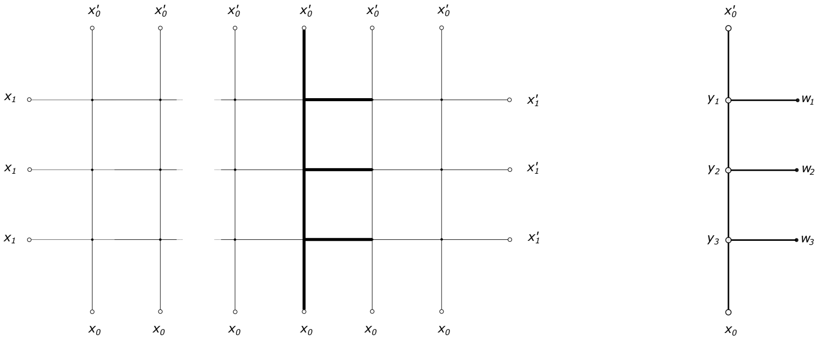
- Spectral decomposition

$$V^{L+1} = \sum_{\boldsymbol{\ell},\boldsymbol{m}} \int d^N\boldsymbol{\nu}\, \mu(\boldsymbol{\nu}, \boldsymbol{\ell})\, \lambda^{L+1}(\nu_1,\ell_1)\cdots\lambda^{L+1}(\nu_N,\ell_N)\, |\boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}\rangle\langle \boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}|$$

Spectral representation

$$V^{L+1}\left(x_1\,,\ldots\,,x_N\,,x_1'\,,\ldots\,,x_N'\right)=\sum_{\boldsymbol{\ell},\boldsymbol{m}}\int d^N\boldsymbol{\nu}\,\mu(\boldsymbol{\nu},\boldsymbol{\ell})\,\lambda^{L+1}(\nu_1,\ell_1)\cdots\lambda^{L+1}(\nu_N,\ell_N)$$

$$\overline{\Psi}_{\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}}\left(x_1\,,\ldots\,,x_N\right)\Psi_{\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}}\left(x_1'\,,\ldots\,,x_N'\right)$$



- For the functions $\overline{\Psi}_{\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}}\left(x_1\,,\ldots\,,x_N\right)\leftrightarrow$ amputation of the left vertical lines and then reduction $x_k\to x_1$
- For the functions $\Psi_{\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}}\left(x_1'\,,\ldots\,,x_N'\right)\leftrightarrow$ reduction $x_k'\to x_1'$