Global KP integrability

(on a joint work with A.Alexandrov, B.Bychkov, P.Dunin-Barkowski, and S.Shadrin)

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KP hierarchy and its formal solutions

Kadomtsev-Petviashvili (KP) hierarchy is an infinite system of PDE's

$$F_{2,2} = F_{1,3} - \tfrac{1}{2}F_{1,1}^2 - \tfrac{1}{12}F_{1,1,1,1},$$
$$F_{2,3} = F_{1,4} - F_{1,1}F_{2,1} - \tfrac{1}{6}F_{2,1,1,1},$$
$$\dots$$

Formal solution as a generating series for the quantities (correlators) $f_{k_1,\dots,k_n}$:

$$F(p_1,p_2,\dots) = \sum_{n\geq 1}\frac{1}{n!}\sum_{k_1,\dots,k_n} f_{k_1,\dots,k_n}p_{k_1}\dots p_{k_n}.$$

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Examples:

- Hurwitz numbers (simple, double, monotone, weighted etc.);
- enumeration of maps (hypermaps, fully simple, weighted etc.);
- correlators of matrix models;
- correlators of CohFT's (GW invariants);
- WP volumes, MV volumes, etc.

n-point function

n-point functions serve as an alternative way to encode solutions

$$F \longleftrightarrow \{W_n\}_{n\geq 1}$$

$$F(p_1, p_2, \dots) = \sum_{n\geq 1} \frac{1}{n!} \sum_{k_1,\dots,k_n\geq 1} f_{k_1,\dots,k_n} p_{k_1} \dots p_{k_n},$$

$$W_n(x_1, \dots, x_n) = \sum_{k_1,\dots,k_n} k_1 \dots k_n \, f_{k_1,\dots,k_n} \, x_1^{k_1-1} \dots x_n^{k_n-1}, \quad n = 1, 2, \dots.$$

KP integrability in terms of n-point functions

Baker-Akhieser kernel

$$K(x_1,x_2) = \frac{e^{F|_{p_k=x_1^k-x_2^k}}}{x_1-x_2} = \frac{1}{x_1-x_2} + (\text{regular series in } x_1,x_2).$$

Theorem ([J.Zhou '15])

KP hierarchy is equivalent to the collection of determinantal identities

$$W_n(x_1,\dots,x_n) = (-1)^{n-1}\sum_{\sigma\in\mathrm{cycl}(n)}\prod_{i=1}^{n}K(x_i,x_{\sigma(i)}) - \frac{\delta_{n,2}}{(x_1-x_2)^2},\qquad n\geq 2.$$

$$W_2(x_1,x_2) = -K(x_1,x_2)K(x_2,x_1) - \frac{1}{(x_1-x_2)^2},$$

$$W_3(x_1,x_2,x_3) = K(x_1,x_2)K(x_2,x_3)K(x_3,x_1) + K(x_1,x_3)K(x_3,x_2)K(x_2,x_1).$$

KP integrability in terms of n-point functions

Baker-Akhieser kernel

$$K(x_1,x_2)=\frac{e^{F|_{p_k=x_1^k-x_2^k}}}{x_1-x_2}=\frac{1}{x_1-x_2}e^{\sum_{n=1}^{\infty}\frac{1}{n!}\int_{x_2}^{x_1}\cdots\int_{x_2}^{x_1}W_n(\tilde{x}_1,\ldots,\tilde{x}_n)d\tilde{x}_1\ldots d\tilde{x}_n}$$

$$=\frac{1}{x_1-x_2}+(\text{regular series in } x_1,x_2).$$

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$$W_2(x_1,x_2)=-K(x_1,x_2)K(x_2,x_1)-\frac{1}{(x_1-x_2)^2},$$

$$W_3(x_1,x_2,x_3)=K(x_1,x_2)K(x_2,x_3)K(x_3,x_1)+K(x_1,x_3)K(x_3,x_2)K(x_2,x_1).$$

n-point differentials

$$\omega_n(x_1,\dots,x_n) = W_n(x_1,\dots,x_n)dx_1\dots dx_n$$

$$= \sum_{k_1,\dots,k_n} f_{k_1,\dots,k_n} \prod_{i=1}^{n} d(x_i^{k_i})$$

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Determinantal formulas in terms of n-point differentials:

$$\mathbb{K}(x_1,x_2) = K(x_1,x_2)\sqrt{dx_1\,dx_2}$$

$$\omega_n(x_1,\dots,x_n) = (-1)^{n-1}\sum_{\sigma\in\mathrm{cycl}(n)}\prod_{i=1}^{n}\mathbb{K}(x_i,x_{\sigma(i)}),\quad n\geq 2.$$

Genus expansion

$$f_{k_1,\dots,k_n} = \sum_{g\geq 0} \hbar^{2g-2+n} f^{(g)}_{k_1,\dots,k_n}$$

$$F = \sum_{g\geq 0, n\geq 1} \hbar^{2g-2+n} F^{(g)}_n$$

$$\omega_n = \sum_{g\geq 0} \hbar^{2g-2+n} \omega^{(g)}_n$$

Determinantal formulas for KP solutions hold with the genus expansion taken into account

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Example (Hurwitz numbers)

$$d = \sum_{i=1}^n k_i, \quad m = 2g-2+n+d,$$

$$f^{(g)}_{k_1,\dots,k_n} = \frac{|\mathrm{Aut}(\mathbf{k})|}{m!d!} \# \left\{ (\tau_1,\dots,\tau_m) \;\middle|\; \begin{array}{l} \text{1) } \tau_i \in S(d) \text{ a transposition} \\ \text{2) } \tau_1 \circ \dots \circ \tau_m \text{ has cyclic type } (k_1,\dots,k_n) \\ \text{3) connectness condition} \end{array} \right\}$$

Rationality

Consider a system of multidifferentials $\omega_n^{(g)}$, $g \geq 0$, $n \geq 1$ such that $\omega_n^{(g)} - \delta_{g,0}\delta_{n,2}\frac{dx_1 dx_2}{(x_1-x_2)^2}$ is non-singular.

Definition

We say that $\{\omega_n^{(g)}\}$ possesses a *rational spectral curve* if there is a change of coordinates $x = x(z)$, $x_i = x(z_i)$ such that $\omega_n^{(g)}$ becomes *rational* in z-coordinates.

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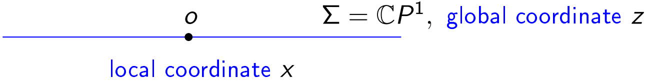
Example (Hurwitz numbers)

$$x(z) = z\,e^{-z} = z - z^2 + \frac{z^2}{2} - \dots \quad\Rightarrow\quad \begin{aligned} \omega_2^{(0)} &= \frac{dz_1\,dz_2}{(z_1-z_2)^2}, \\ \omega_3^{(0)} &= \frac{dz_1\,dz_2\,dz_3}{(1-z_1)^2(1-z_2)^2(1-z_3)^2}, \\ \omega_1^{(1)} &= \frac{(4-z_1)\,z_1}{24\,(1-z_1)^4}\,dz_1. \end{aligned}$$

Spectral curve

$\Sigma = \mathbb{C}P^1$ *spectral curve*.



$\omega_n^{(g)}$ is global meromorphic on Σ^n.

The potential F is associated with the power expansion of $\omega_n^{(g)}$'s at a given point $o \in \Sigma$ in a given local coordinate x.

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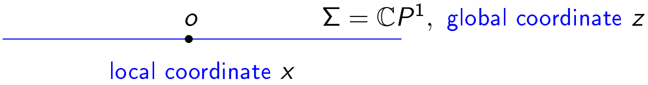

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$$(o, x) \rightsquigarrow F = F_{o,x}$$

A different choice of the expansion point o and a local coordinate x leads to a different potential

$$(\tilde{o}, \tilde{x}) \rightsquigarrow \tilde{F} = F_{\tilde{o},\tilde{x}}$$

Global KP integrability

Consider a system of symmetric meromorphic multidifferentials $\{\omega_n^{(g)}\}_{g\geq 0, n\geq 1}$ such that $\omega_n^{(g)} - \delta_{g,0}\delta_{m,2}\frac{dz_1 dz_2}{(z_1-z_2)}$ is regular at a generic point of the diagonal.

Theorem ([ABDKS '23])

KP integrability is an internal property of a system of differentials: if $F = F_{o,x}$ is a KP solution for some choice of o and x, then the same is true for any other choice.

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Proof. The differential $\mathbb{K}$ extends as a meromorphic differential on Σ^2 and the determinant identities extend to an equality of global meromorphic differentials

$$\omega_n(z_1,\dots,z_n) = (-1)^{n-1} \sum_{\sigma\in \mathrm{cycl}(n)} \prod_{i=1}^{n} \mathbb{K}(z_i, z_{\sigma(i)}).$$

Moreover, $\mathbb{K}$ has a universal invariant meaning: it is independent of a choice of the expansion point and a local coordinate. □

Topological recursion: an overview

Topological recursion is an inductive procedure to compute $\omega_n^{(g)}$ in a closed form inductively in g and n
Chekhov-Eynard-Orantin '06
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$$\omega_n^{(g)} = \frac{\begin{pmatrix}\text{symmetric polynomial}\\ \text{in } z_1,\dots,z_n\end{pmatrix}}{\prod_{i=1}^n \prod_{j=1}^N (z_i - q_j)^{2(3g-3+n)+2}} dz_1 \dots dz_n, \quad 2g - 2 + n > 0.$$

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- the recursion studies the behaviour of $\omega_n^{(g)}$ near the poles

Topological recursion: initial data

Initial data: $(\Sigma, dx, dy, B, \mathcal{P})$

- $\Sigma = \mathbb{C}P^1$ (generalization: a smooth algebraic complex curve);
- $B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$ (generalization: a bidifferential on Σ^2 with similar singularity on the diagonal)
- dx, dy meromorphic differentials on Σ
- $\mathcal{P} = \{q_1, \dots, q_N\}$ a set of simple zeroes of dx such that $dy|_{q_j} \neq 0$

Initial differentials:

$$\omega_1^{(0)}(z_1) = y(z_1)\, dx(z_1), \qquad \omega_2^{(0)}(z_1, z_2) = B(z_1, z_2)$$

Topological recursion: two step induction

$\mathbf{2g-2+n>0}$: $\quad K=\{2,\dots,n\},\ z_K=(z_2,\dots,z_n),$

First step: $z\approx q_j\in\mathcal{P}$, $x(z)=x(\sigma(z))$

$$\tilde\omega^{(g)}_n(z,z_K)=\frac{\omega^{(g-1)}_{n+1}(z,\sigma(z),z_K)+\sum\limits_{\substack{g_1+g_2=g,\ J_1\sqcup J_2=K\\(g_i,|J_i|+1)\neq(0,1)}}\omega^{(g_1)}_{|J_1|+1}(z,z_{J_1})\omega^{(g_2)}_{|J_2|+1}(\sigma(z),z_{J_2})}{(y(z)-y(\sigma(z)))\,dx(z)}.$$

Second step:

$$\omega^{(g)}_n(z,z_K)=\tilde\omega^{(g)}_n(z,z_K)+(\text{holomorphic}),\quad z\to q_j,\quad j=1,\dots,N.$$

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Equivalently:

$$\omega^{(g)}_n(z_1,z_K)=\sum_{j=1}^N\operatorname*{res}_{z=q_j}\tilde\omega^{(g)}_n(z,z_K)\int^z B(\cdot,z_1).$$

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Idea of the proof: variation of the functions x, y of the initial data leads to a variation of the TR differentials, and one shows that these variations are infinitesimal KP symmetries.

Thank you!