Krichever tau-function and Verlinde formula

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History I

Important (personal!) key points:

- Periodic problem in integrable systems (starting from KdV);
- Krichever construction $[\hat{X}, \hat{Y}] = 0$, embedding $F(X, Y) = 0$ of a curve $\Sigma \subset \mathbb{C} \times \mathbb{C}^\times$ with extra data. Example: affine Toda chains, $\hat{Y} = \mathcal{L}$, $\hat{X} = T_N$

$$w + \frac{\Lambda^{2N}}{w} = P_N(z) = z^N + \sum_{k=0}^{N-2} u_k z^k, \qquad dx = \frac{dw}{w}, \quad dy = dz$$

- Deformation of integrable systems: Whitham equations: what happens in the vicinity of finite-gap solutions ...

History II

- Riemann surfaces (and integrable systems) in string theory;
  - CFT on 2d string's world-sheets, free fermions and tau-functions;
  - Riemann surfaces as (part of) target-space geometry.
- Both Novikov and Krichever: string equation

$$\left[\hat{X}, \hat{Y}\right] = \hbar$$

  as quantization of finite-gap systems. Solution of Painlevé-I equation as first case of isomonodromic problem in this context;
- 2d quantum gravity, matrix models and $\hbar$-expansion: KdV or Whitham?
- Krichever's tau-function: *The τ-function of the universal Whitham hierarchy, matrix models and topological field theories*, `arXiv:hep-th/9205110`
  - Actually $\mathcal{F} = \log \tau$, residue formula, $\tau \sim \tau_{IM}$;
  - Recent: back to 2d gravity and Verlinde formula from 2d CFT.

Definition: prepotential

- Krichever data: g-parametric family of Riemann surfaces Σ_g, endowed with a pair of differentials or *generating* differential and *connection* $\nabla_{\rm mod}$ on moduli space

$$dS \propto ydx, \qquad \nabla_{\rm mod} dS = \text{holomorphic}$$

where $y(P) = \int^P dy$, $P \in \Sigma_g$.

- Prepotential (a particular case of Krichever tau-function)

$$\mathsf{a} = \frac{1}{2\pi i}\oint_{\mathsf{A}} dS, \qquad \mathsf{a}_D = \oint_{\mathsf{B}} dS := \frac{\partial \mathcal{F}}{\partial \mathsf{a}}$$

where A and B are dual cycles in $H_1(\Sigma_g)$.

- Integrability from *Riemann bilinear relations*

$$\frac{\partial a^D_\alpha}{\partial a_\beta} = \underbrace{T_{\alpha\beta} = T_{\beta\alpha}}_{RBI} = \frac{\partial a^D_\beta}{\partial a_\alpha}$$

Prepotential: proof

- $\nabla_{\rm mod}$: connection via covariantly constant coordinate on Σ_g, e.g. $x = 0$ (peculiarities at $dx = 0$). Then

$$\nabla_{\rm mod} dS = (\nabla_{\rm mod} y) dx$$

  and $\nabla_{\rm mod} y$ is defined from equation of $\Sigma_g \subset \mathbb{C}^2$.

- Then

$$\delta_{\alpha\beta} = \frac{\partial a_\alpha}{\partial a_\beta} = \frac{1}{2\pi i}\oint_{A_\alpha} \frac{\partial dS}{\partial a_\beta}$$

  so that $\frac{\partial dS}{\partial a_\beta} = d\omega_\beta$ is *normalized* holomorphic differential.

- Hence

$$\frac{\partial a^D_\alpha}{\partial a_\beta} = \oint_{B_\alpha} \frac{\partial dS}{\partial a_\beta} = \oint_{B_\alpha} d\omega_\beta = T_{\alpha\beta}$$

Consequence: for the second derivatives

$$\frac{\partial^2 \mathcal{F}}{\partial a_\alpha \partial a_\beta} = T_{\alpha\beta}$$

Prepotential: SW example

- (Σ_{N-1}, dx, dy) given by

$$w + \frac{\Lambda^{2N}}{w} = P_N(z) = z^N + \sum_{k=0}^{N-2} u_k z^k, \qquad dx = \frac{dw}{w}, \quad dy = dz$$

since obviously

$$\oint_{(A,B)} dz = 0, \qquad \oint_{(A,B)} \frac{dw}{w} \in 2\pi i\mathbb{Z}$$

- From $\nabla_{\rm mod} w = 0$ and $\nabla_{\rm mod} z \; P'_N(z) = \sum_{k=0}^{N-2} \delta u_k z^k$

$$\nabla_{\rm mod} dS = \nabla_{\rm mod} z \; \frac{dw}{w} = \sum_{k=0}^{N-2} \delta u_k \frac{z^k}{P'_N(z)} \frac{dw}{w}$$

holomorphic on Σ_{N-1}.

Definition: (extended) Krichever tau-function

- To complete definition by the time-variables associated with the second-kind Abelian differentials with singularities at a point P_0

$$t_k = \frac{1}{k} \mathrm{res}_{P_0} \xi^{-k} dS, \quad k > 0$$
$$\frac{\partial \mathcal{F}}{\partial t_k} := \mathrm{res}_{P_0} \xi^k dS, \quad k > 0$$

where ξ is an *inverse* local co-ordinate at P_0: $\xi(P_0) = \infty$.

- The consistency condition for (7) is ensured by

$$\frac{\partial^2 \mathcal{F}}{\partial t_n \partial t_k} = \mathrm{res}_{P_0} (\xi^k d\Omega_n)$$

symmetric due to $(\Omega_n)_+ = \xi^n$, for the main, singular at P_0, part.

- Also

$$\frac{\partial^2 \mathcal{F}}{\partial t_n \partial a_\alpha} = \oint_{B_\alpha} d\Omega_n = \mathrm{res}_{P_0} \xi^n d\omega_\alpha$$

which again follows from RBI;

Remarks:

- Definition from RBI;
- Can be defined for any set of Abelian differentials $\{dH_I\} = \{d\omega_\alpha, d\Omega_n, d\Omega_0, \ldots\}$ and corresponding flat-coordinates $\{T_I\} = \{a_\alpha, t_n, t_0, \ldots\}$.
- *pq*-duality: $dx \leftrightarrow dy$ generally a nontrivial tiny point:
    - Prepotentials: $\nabla^x_{\rm mod} \leftrightarrow \nabla^y_{\rm mod}$;
    - dKP: a non-trivial relation (e.g. a Fourier-transform for a matrix integral);
    - A nontrivial relation for residue formulas ...
- Starting point for the "topological recursion" ...

Residue formula: statement

Theorem

$$\frac{\partial^3 \mathcal{F}}{\partial T_I \partial T_J \partial T_K} = \operatorname{res}_{dx=0} \left(\frac{dH_I dH_J dH_K}{dx dy} \right)$$

- Idea of proof: to take one mode derivative of a second-derivative formula ...
- Prepotential case:

$$\frac{\partial T_{\alpha\beta}}{\partial a_\gamma} \equiv \partial_\gamma T_{\alpha\beta} = \int_{B_\beta} \partial_\gamma d\omega_\alpha = - \int_{\partial\Sigma} \omega_\beta \partial_\gamma d\omega_\alpha$$

- Further

$$\partial_\gamma T_{\alpha\beta} = - \int_{\partial\Sigma} \omega_\beta \partial_\gamma d\omega_\alpha = \int_{\partial\Sigma} \partial_\gamma \omega_\beta d\omega_\alpha = \sum \operatorname{res}_{dx=0} \left(\partial_\gamma \omega_\beta d\omega_\alpha \right)$$

since the expression acquires poles at $dx = 0$.

Residue formula: proof

- Use expansions where $dx = 0$

$$\omega_\beta(x) \underset{x\to x_\alpha}{=} \omega_{\beta a} + c_{\beta a}\sqrt{x-x_a} + \ldots, \qquad d\omega_\beta \underset{x\to x_a}{=} \frac{c_{\beta a}}{2\sqrt{x-x_a}}\, dx + \ldots$$

$$\nabla_{\rm mod}: \quad \partial_\gamma\omega_\beta \equiv \partial_\gamma\omega_\beta\big|_{x={\rm const}} \underset{x\to x_a}{=} -\frac{c_{\beta a}}{2\sqrt{x-x_a}}\partial_\gamma x_a + {\rm regular}$$

- Then

$$\mathrm{res}\,\left(\partial_\gamma\omega_\beta d\omega_\alpha\right) = \sum_a \mathrm{res}\left(\frac{c_{\beta a}\partial_\gamma x_a}{2\sqrt{x-x_a}}d\omega_\alpha\right) = \sum_a \mathrm{res}\left(\frac{d\omega_\beta}{dx}d\omega_\alpha\partial_\gamma x_a\right) =$$

$$= \sum_a \mathrm{res}\left(\frac{d\omega_\alpha d\omega_\beta d\omega_\gamma}{dxdy}\right)$$

where last equality similarly follows from

$$y(x) \underset{x\to x_a}{=} y_a\sqrt{x-x_a} + \ldots, \qquad dy \underset{x\to x_a}{=} \frac{y_a}{2\sqrt{x-x_a}}\, dx + {\rm regular}$$

$$d\omega_\gamma = \partial_\gamma dS \underset{x\to x_a}{=} -\frac{y_a\partial_\gamma x_a}{2\sqrt{x-x_a}}\, dx + {\rm regular}$$

Landau-Ginzburg topological theories

The polynomial superpotential (generally of several complex variables)

$$W(\lambda) = \lambda^N + \sum_{k=0}^{N-2} u_k \lambda^k$$

The primaries (dKP equation)

$$\phi_k(\lambda) := \frac{\partial W}{\partial t_k} = \left(\frac{d}{d\lambda} W^{k/N} \right)_+$$

where flat times

$$t_k = \frac{1}{k} \operatorname{res}_{P_0} \xi^{-k} dS = -\frac{N}{k(N-k)} \operatorname{res}_{\infty} \left(W^{1-k/N} d\lambda \right)$$

for $(\Sigma_g, dx, dy) = (\Sigma_0, dW, d\lambda)$ with $\xi = W(\lambda)^{1/N}$, reduced dKP.

Landau-Ginzburg topological theories

- The derivatives of Krichever tau-function

$$\frac{\partial\mathcal{F}}{\partial t_k} = \mathrm{res}_{P_0}\xi^k dS = \frac{N}{N+k}\,\mathrm{res}_\infty\left(W^{1+\frac{k}{N}}d\lambda\right)$$

$$\mathcal{F}_{ik} = \frac{\partial^2\mathcal{F}}{\partial t_i\partial t_k} = \mathrm{res}_\infty\left(W^{k/N}\frac{\partial W}{\partial t_i}\right) = \mathrm{res}_\infty\left(W^{k/N}\partial_\lambda W_+^{i/N}\right)$$

$$\mathcal{F}_{ijk} = \mathop{\mathrm{res}}_{W'=0}\frac{\partial_\lambda W_+^{i/N}\partial_\lambda W_+^{j/N}\partial_\lambda W_+^{k/N}}{W'} = -\,\mathrm{res}_\infty\frac{\partial_\lambda W_+^{i/N}\partial_\lambda W_+^{j/N}\partial_\lambda W_+^{k/N}}{W'}$$

- WDVV equations

$$\mathsf{F}_i\mathsf{F}_j^{-1}\mathsf{F}_k = \mathsf{F}_k\mathsf{F}_j^{-1}\mathsf{F}_i \qquad \text{for all } \; i,j,k.$$

where $\|\mathsf{F}_i\|_{jk} := \mathcal{F}_{ijk}$.

- Follow from residue formula if $\#\{T\} = \#(dx=0)$.

2d minimal gravity

- For each (p,q)-th point take a pair of polynomials

$$X = \lambda^p + \dots, \qquad Y = \lambda^q + \dots$$

of degrees p and q respectively. Landau-Ginzburg $(p,q) = (N,1)$.

- A dispersionless version of the Lax and Orlov-Shulman operators from KP theory

$$\left[\hat{X},\hat{Y}\right] = \hbar, \qquad \hat{X} = \partial^p + \dots, \qquad \hat{Y} = \partial^q + \dots$$

- An invariant way: an algebraic equation

$$Y^p - X^q - \sum f_{ij} X^i Y^j = 0$$

with some $\{f_{ij}\}$. Generally, this is a smooth curve of genus

$$g = \frac{(p-1)(q-1)}{2} = \# \text{ primaries}$$

2d (minimal) gravity

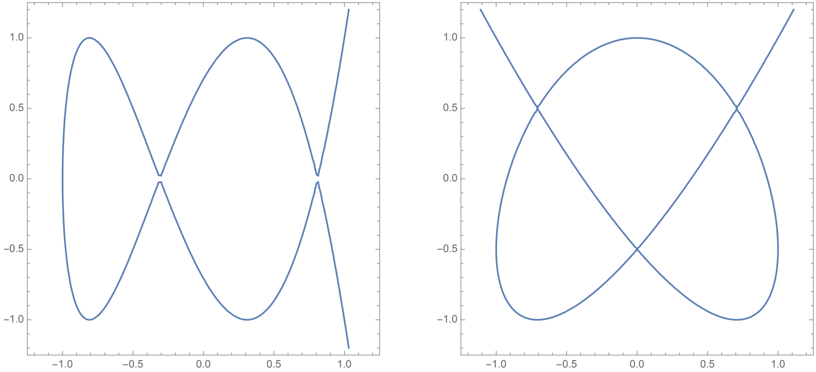

Figure: Degenerate curves of Yang-Lee and Ising models of $g = 2$ and $g = 3$.

Chebyshev curves

- At only cosmological constant $\mu \neq 0$

$$T_p(Y) = T_q(X)$$

parameterized by $z \in \mathbb{P}^1$

$$X = T_p(z), \qquad Y = T_q(z)$$

Degenerate at

$$U_{p-1}(Y) = 0, \qquad U_{q-1}(X) = 0$$

- For $(p, q) = (2K+1, 2)$ degenerate hyperelliptic curve with nodal singularities at

$$\zeta_n^{\pm} = \pm \cos \frac{\pi(2n-1)}{2(2K+1)}, \qquad n = 1, 2, \ldots, K$$

$$Y_n = T_2(\zeta_n^{\pm}) = \cos \frac{\pi(2n-1)}{2K+1}, \quad X_n = T_{2K+1}(\zeta_n^{\pm}) = \pm \cos \pi \left(n - \frac{1}{2} \right) = 0$$

Chebyshev curves

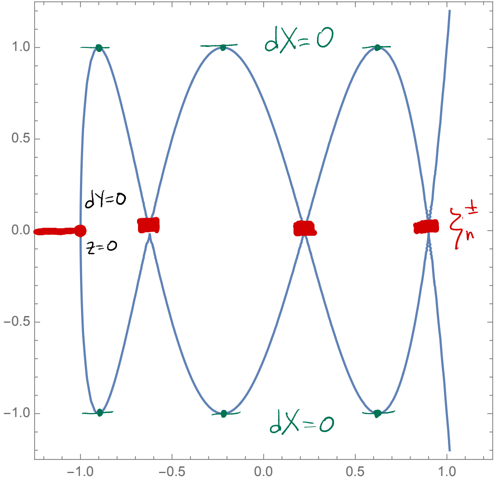

Figure: Chebyshev curve for $(2K+1,2)$-series, with (degenerate) cuts $\{\zeta_n^\pm\}$, critical points $\{z_m\}$ where $dX=0$, and the point $z=0$ where $dY=0$.

Residue formula

Residue formula on a Chebyshev curve

$$\mathcal{F}_{ijk} = \frac{pq}{ijk}\ \mathrm{res}_{\,dX=0} \frac{dH_i dH_j dH_k}{dXdY} = \sum_{U_{p-1}(z)=0} \frac{U_{i-1}(z)U_{j-1}(z)U_{k-1}(z)}{U_{p-1}(z)U_{q-1}(z)} =$$

$$= \sum_{m=1}^{p-1} \frac{U_{i-1}(z_m)U_{j-1}(z_m)U_{k-1}(z_m)}{U'_{p-1}(z_m)U_{q-1}(z_m)} = \sum_{m=1}^{p-1} \frac{\sin\frac{i\pi m}{p}\sin\frac{j\pi m}{p}\sin\frac{k\pi m}{p}}{\sin\frac{q\pi m}{p}\sin^2\frac{\pi m}{p}U'_{p-1}(z_m)} =$$

$$= \frac{1}{p}\sum_{m=1}^{p-1}(-1)^m \frac{\sin\frac{i\pi m}{p}\sin\frac{j\pi m}{p}\sin\frac{k\pi m}{p}}{\sin\frac{q\pi m}{p}}$$

where we have used that $U_{p-1}(z_m) = 0$ at $\{z_m = \cos\frac{\pi m}{p} | m = 1, \ldots, p-1\}$ and

$$U_n(x)' = \frac{nxU_n(x) - (n+1)U_{n-1}(x)}{x^2 - 1}$$

i.e.

$$U_{p-1}(z_m)' = \frac{pU_{p-2}(z_m)}{1 - z_m^2} = \frac{p(-1)^m}{\sin^2\frac{\pi m}{p}}$$

Verlinde formula: basics

- S-matrix: $\chi_a(-1/\tau) = \sum_b \mathcal{S}_a^b \chi_b(\tau)$, unitarity $\mathcal{S}^\dagger \mathcal{S} = 1$
- Verlinde formula: relation with fusion algebra:

$$\mathcal{N}_{ab}^c = \sum_m \frac{\mathcal{S}_a^m \mathcal{S}_b^m (\mathcal{S}^\dagger)_m^c}{\mathcal{S}_1^m}$$

- Minimal (p, q)-model $\mathcal{S}^2 = 1$:

$$\mathcal{S}_{rs,\rho\sigma} = 2\sqrt{\frac{2}{pq}}(-1)^{1+s\rho+r\sigma} \sin \pi \frac{p}{q} r\rho \;\; \sin \pi \frac{q}{p} s\sigma$$

- Upon substitution $k_{r,s} = rq - sp$,

$$S_{\rho\sigma}(k_{r,s}) = 2\sqrt{\frac{2}{pq}} \sin \frac{\pi k_{r,s}\rho}{p} \sin \frac{\pi k_{r,s}\sigma}{q}$$

- $\{T_{k_{r,s}}\} \in \Pi_{p,q} = \{r = 1, \ldots, p-1, s = 1, \ldots, q-1 | k_{r,s} > 0\}$ parameterization by KP times!

Verlinde formula and Krichever tau-function

Factorization:

$$\mathcal{N}_{k_1 k_2 k_3} = \sum_{(\rho,\sigma)\in\Pi_{p,q}} \frac{S_{\rho\sigma}(k_1)S_{\rho\sigma}(k_2)S_{\rho\sigma}(k_3)}{S^{(0)}_{\rho\sigma}} = \frac{1}{2}\sum_{\rho=1}^{p-1}\sum_{\sigma=1}^{q-1} \frac{S_{\rho\sigma}(k_1)S_{\rho\sigma}(k_2)S_{\rho\sigma}(k_3)}{S_{\rho\sigma}(p-q)} =$$

$$= -\frac{4}{pq}\sum_{\rho=1}^{p-1}(-1)^\rho \frac{\sin\frac{\pi k_1\rho}{p}\sin\frac{\pi k_2\rho}{p}\sin\frac{\pi k_3\rho}{p}}{\sin\frac{\pi q\rho}{p}} \sum_{\sigma=1}^{q-1}(-1)^\sigma \frac{\sin\frac{\pi k_1\sigma}{q}\sin\frac{\pi k_2\sigma}{q}\sin\frac{\pi k_3\sigma}{q}}{\sin\frac{\pi p\sigma}{q}}$$

or just

$$\frac{1}{4}\mathcal{N}_{ijk} = -\mathcal{F}_{ijk}\mathcal{F}^D_{ijk}$$

for any admissible triples $\{ijk\} \in \Pi^{\otimes 3}_{p,q}$, with also "pq-dual" 3-point function

$$\mathcal{F}^D_{ijk} = \frac{pq}{ijk}\ \mathrm{res}_{\,dY=0}\frac{dH_i dH_j dH_k}{dXdY}$$

Proof for $(p,q) = (2K+1, 2)$

If $q = 2$ and therefore $\sigma = 1$

$$\sum_{\sigma=1}^{q-1} (-1)^\sigma \frac{\sin\frac{\pi k_1 \sigma}{q} \sin\frac{\pi k_2 \sigma}{q} \sin\frac{\pi k_3 \sigma}{q}}{\sin\frac{\pi p \sigma}{q}} \underset{\{k_i = 2l_i+1\}}{=} (-1)^{1+K+l_1+l_2+l_3}$$

so that

$$\mathcal{N}_{ijk} = 2(-1)^{K+l_i+l_j+l_k} \mathcal{F}_{ijk}$$

and Krichever's residue formula proves Verlinde's one: for $t_{\tilde{i}} = t_{2l_i+1} = t_{2(K-\tilde{i})+1}$ with $\tilde{i} = 1, \dots, K$

$$\mathcal{N}_{\tilde{i}\tilde{j}\tilde{k}} = 2(-1)^{\tilde{i}+\tilde{j}+\tilde{k}} \mathcal{F}_{\tilde{i}\tilde{j}\tilde{k}} = (-1)^{1+\tilde{i}+\tilde{j}+\tilde{k}} \frac{4}{2K+1} \sum_{m=1}^{K} \frac{\sin\frac{2\pi m\tilde{i}}{2K+1} \sin\frac{2\pi m\tilde{j}}{2K+1} \sin\frac{2\pi m\tilde{k}}{2K+1}}{\sin\frac{2\pi m}{2K+1}}$$

is computed just by moving contour.

Proof for $(p,q) = (2K+1, 2)$

The residue formula gives

$$(-1)^{1+\tilde{i}+\tilde{j}+\tilde{k}}\mathcal{N}_{\tilde{i}\tilde{j}\tilde{k}} = \frac{4}{2K+1}\sum_{m=1}^{K}\frac{\sin\frac{2\pi m\tilde{i}}{2K+1}\sin\frac{2\pi m\tilde{j}}{2K+1}\sin\frac{2\pi m\tilde{k}}{2K+1}}{\sin\frac{2\pi m}{2K+1}} =$$

$$= -\operatorname{res}_{U_{2K}(z)=0}\frac{U_{2l_i}(z)U_{2l_j}(z)U_{2l_k}(z)dz}{zU_{2K}(z)} = \sum \operatorname{res}_{z=0,\infty}\frac{U_{2l_i}(z)U_{2l_j}(z)U_{2l_k}(z)dz}{zU_{2K}(z)} =$$

$$= \sum \operatorname{res}_{w=i,0}\frac{\left(w^{2l_i+1}-w^{-2l_i-1}\right)\left(w^{2l_j+1}-w^{-2l_j-1}\right)\left(w^{2l_k+1}-w^{-2l_k-1}\right)}{\left(w^2-w^{-2}\right)\left(w^{2K+1}-w^{-2K-1}\right)}\frac{dw}{w}$$

and results in

$$\mathcal{N}_{\tilde{i}\tilde{j}\tilde{k}} = \sum_{l=0}^{\min(\tilde{i},\tilde{j})-1}\delta_{|\tilde{i}-\tilde{j}|+2l+1,\tilde{k}} + \sum_{l=0}^{\left[\frac{K-2}{2}\right]}\delta_{\tilde{i}+\tilde{j}+\tilde{k},2(K+l+1)}$$

Beyond finite-gap

"Continuous" theory

$$\mathcal{N}(p_1, p_2, p_3) = 2b \sum_{m=1}^{\infty} (-1)^m \frac{\sin 2\pi m b p_1 \sin 2\pi m b p_2 \sin 2\pi m b p_3}{\sin \pi m b^2}$$

$b^2 \in i\mathbb{R}$ instead of $b^2 = \frac{p}{q}$ in minimal theory. Note also

$$(-1)^m \sin \pi m b^2 = \sin(\pi m b^2 + \pi m) = \sin 2\pi m b p_0 = \mathcal{S}_0^m$$

where $p_0 = (1/b + b)/2$ corresponds to $h_0 = 0$ or "unity" operator.

Comes from residue formula for a non-algebraic curve

$$X(\mathrm{z}) = \cos \pi b^{-1} \mathrm{z}, \qquad Y(\mathrm{z}) = \cos \pi b \mathrm{z}$$

Non-algebraic residue formula

Indeed

$$\mathcal{N}(p_1,p_2,p_3)=\sum_{dX=0}\frac{dH_{p_1}dH_{p_2}dH_{p_3}}{dX\ dY}=\sum_{X'(\mathsf{z})=0}\frac{\phi(p_1\mathsf{z})\phi(p_2\mathsf{z})\phi(p_3\mathsf{z})}{X''(\mathsf{z})Y'(\mathsf{z})}$$

since $X'(\mathsf{z})\sim\sin\pi b^{-1}\mathsf{z}=0$ at $\mathsf{z}_m=bm,\ m\in\mathbb{Z}$, where

$$Y'(\mathsf{z}_m)\sim b\sin\pi b\mathsf{z}_m=b\sin\pi b^2 m,\qquad X''(\mathsf{z}_m)\sim b^{-2}\cos\pi b^{-1}\mathsf{z}_m=b^{-2}(-1)^m$$

and the rest comes from identification $\phi(p\mathsf{z})=\sin 2\pi p\mathsf{z}$, just as in the case of Chebyshev U-polynomials.

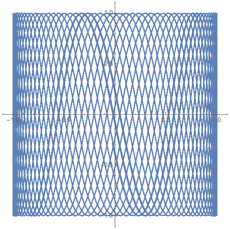

Figure: Non-algebraic curve gives rise to an infinite sum over $dX=0$.

Many other developments

Just one particular application of Krichever's tau-function constructed for (degenerate) finite-gap solution ...

- Instanton partition functions;
- 2d conformal field theories and their (qt)-deformations;
- Cluster algebras, integrable systems on cluster varieties;
- Isomonodromic deformations;
- ...