Towards finite gap theory of Hitchin systems

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50 years to finite-gap integration
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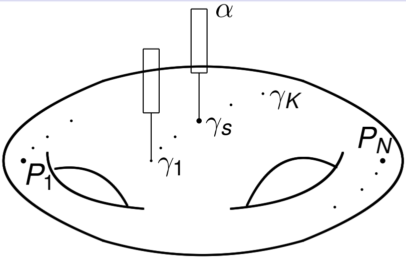
Lax operator due to Krichever: geometrical data

- Genus g Riemann surface Σ
- Marked points of two kinds: $P_1, \dots, P_N \in \Sigma$ and $\gamma_1, \dots, \gamma_K \in \Sigma$
- Vectors $\alpha_1, \dots, \alpha_K \in \mathbb{C}^n$ associated with γ's.



Pairs γ, α are referred to as Tyurin parameters, due to the

Theorem (*A.N.Tyurin):* Let $g = \textit{genus}\ \Sigma$, $n \in \mathbb{Z}_+$. Then there is a 1–1 correspondence between the pairs of sets

$$\gamma_1, \dots, \gamma_{ng} \in \Sigma; \quad \alpha_1, \dots, \alpha_{ng} \in \mathbb{C}P^{n-1} ,$$

and the equivalence classes of the semi-stable holomorphic rank n vector bundles on Σ.

Lax equations are overdetermined without the Tyurin parameters.

Lax equations on Riemann surfaces

$L, M : \Sigma \to \mathfrak{gl}(n)$ – meromorphic functions, with the pole divisor $D = m_1 P_1 + \ldots + m_N P_N + \gamma_1 + \ldots + \gamma_K$.

$$L(z) = \frac{L_{-1,s}}{(z - z_s)} + L_{0,s} + O((z - z_s)), \quad M = \frac{M_{-1,s}}{z - z_s} + M_{0,s} + O((z - z_s))$$

$$\boxed{L_{-1,s} = \beta_s \alpha_s^T, \quad \beta_s^T \alpha_s = 0, \quad \alpha_s^T L_{0,s} = \alpha_s^T \varkappa_s, \quad M_{-1,s} = \mu_s \alpha_s^T}$$

$(\alpha_s, \beta_s, \mu_s \in \mathbb{C}^n,\ \varkappa_s \in \mathbb{C}$, z – local coord., $z_s = z(\gamma_s))$

Lax equation is defined as

$$\dot{L} = [M, L]$$

Evolution equations of Tyurin data:

$$\dot{z}_s = -\mu_s^T \alpha_s, \quad \dot{\alpha}_s = -M_{0,s}\alpha_s + k_s \alpha_s.$$

Hierarchy of commuting flows

THEOREM (KRICHEVER): Given a generic L, there is a family of M-operators $M_a = M_a(L)$ $(a = (P_i, k, m), k > 0,\ m > -m_i)$ uniquely defined up to normalization, such that outside the γ-points M_a has pole at the point P_i only, and in the neighborhood of P_i

$$M_a(w_i) = w_i^{-m} L^k(w_i) + O(1),$$

The equations

$$\partial_a L = [M_a, L],\ \partial_a = \partial/\partial t_a \tag{1}$$

define a family of commuting flows on an open set of $\mathcal{L}^D$.

$(\mathfrak{g} = \mathfrak{gl}(n))$ — *Krichever, 2001; $SO(2n)$ — Sh., 2008* , for $SO(2n)$ k is odd).

Hamiltonian theory

Assume, $\exists \omega \in \Gamma_0(T^*\Sigma)$, s.t. $\sum_{i=1}^{N} m_i P_i \geq (\omega)$.

For $a = (i, k, m)$ let

$$H_a = \frac{1}{k+1} \operatorname{res}_{P_i} w_i^{-m} \operatorname{tr} L(w_i)^{k+1} \omega.$$

THEOREM (KRICHEVER): The ∂_a is a Hamiltonian vect. field w.r.t. the Krichever–Phong symplectic structure, with the Hamiltonian H_a.

Krichever–Phong 2-form:

$$\Omega(L) = \sum \operatorname{res} \operatorname{tr} \left(\Psi^{-1} dL \wedge d\Psi - \Psi^{-1} d\Psi \wedge dK \right)$$

where $L = \Psi K \Psi^{-1}$, K is diagonal.

Finite gap integration after Krichever

Consider the eigenvalue problem

$$(L(q,t)-\lambda)\psi(Q,t)=0,\ q\in\Sigma,\ Q=(q,\lambda). \tag{2}$$

Define the spectral curve $\widehat{\Sigma}$ of L by the equation

$$\det(L(q,t)-\lambda)=0,\ q\in\Sigma. \tag{3}$$

Normalize ψ by the condition

$$\sum_{i=1}^{n}\psi_i=1. \tag{4}$$

Then $\psi(Q)$ becomes a meromorphic function on $\widehat{\Sigma}$. Its pole divisor plays a special role and is referred to as a dynamical divisor of the problem. It depends on time. We denote it by $D(t)$.

Important step: to get rid of dynamical divisor !

ψ is L-eigenvector $\Longrightarrow$ ψ is $(\partial_a - M_a)$-eigenvector:

$$(\partial_a - M_a(q,t))\psi(Q,t) = f_a(Q,t)\psi(Q,t)$$

where $f_a(Q,t)$ is scalar. The vector-function

$$\widehat{\psi}(Q,t) = \phi(Q,t)\psi(Q,t); \quad \phi(Q,t) = \exp\left(-\int_0^{t_a} f_a(Q,\tau)d\tau\right)$$

satisfies the equations

$$L(q,t)\widehat{\psi}(Q,t) = \lambda(Q,t)\widehat{\psi}(Q,t); \quad (\partial_a - M_a(Q,t))\widehat{\psi}(Q,t) = 0.$$

Under transform $\psi \mapsto \widehat{\psi}$ the pole divisor $D(t)$ gets transformed to a time-independent divisor D of poles of $\widehat{\psi}$ (the explanation will follow). The time dependence of $\widehat{\psi}$ gets encoded in its essential singularities acquired at the constant poles of f_a.

Baker–Akhieser function and complition of integration

$\widehat{\psi}$ is what is called a vector-valued Baker–Akhieser function. It is parameterized by its pole divisor and its essential singularities.

Essential singularities are located at preimages of $P_1, \ldots, P_N$. For the exponent at $(w_i, \lambda_j) \sim P_i^{(j)}$ we have

$$\exp\Big(\sum_{m \geq -m_i,\ n>0} t_{(i,m,n)} w_i^{-m} \lambda_j^n \Big).$$

Given a vector-valued Baker–Akhieser function $\widehat{\psi}$ on $\widehat{\Sigma}$ we push it forward to Σ as a matrix-valued function $\widehat{\Psi}$. Then

$$L = \widehat{\Psi} \Lambda \widehat{\Psi}^{-1}, \quad M_a = \partial_a \widehat{\Psi} \cdot \widehat{\Psi}^{-1}$$

where $\Lambda(q,t)$ is the diagonal form of $L(q,t)$.

Lax operator: $\mathfrak{so}(2n)$ case

$$L(z) = \frac{L_{-1}}{(z - z_\gamma)} + L_0 + O(z - z_\gamma) \ \in \mathfrak{so}(2n)$$

$$\boxed{L_{-1} = \beta\alpha^T - \alpha\beta^T, \quad \beta^T\alpha = 0, \quad \alpha^T L_0 = \alpha^T \varkappa, \quad \alpha^T\alpha = 0}$$

Example: $SO(4)$. $\alpha^T = (1, \mathrm{i}, 0, 0)$, $\beta^T = (\beta_1, \mathrm{i}\beta_1, \beta_3, \beta_4)$

$$L(z) = \begin{pmatrix} 0 & \varkappa & \delta - \frac{\beta_3}{z} & \gamma - \frac{\beta_4}{z} \\ -\varkappa & 0 & \mathrm{i}(\delta - \frac{\beta_3}{z}) & \mathrm{i}(\gamma - \frac{\beta_4}{z}) \\ \frac{\beta_3}{z} - \delta & \mathrm{i}(\frac{\beta_3}{z} - \delta) & 0 & \sigma \\ \frac{\beta_4}{z} - \gamma & \mathrm{i}(\frac{\beta_4}{z} - \gamma) & -\sigma & 0 \end{pmatrix} + O(z).$$

Characteristic polynomial: $\lambda^4 + p\lambda^2 + q^2$,

$p = \varkappa^2 + \sigma^2 - 2a_{13}\beta_3 - 2\mathrm{i}a_{23}\beta_3 - 2a_{14}\beta_4 - 2\mathrm{i}a_{24}\beta_4 + O(z)$,

$q = \varkappa\sigma - \mathrm{i}a_{14}\beta_3 + a_{24}\beta_3 + \mathrm{i}a_{13}\beta_4 - a_{23}\beta_4 + O(z)$, a_{ij} come from L_1.

Corollary: Eigenvalues are holomorphic. Important !!!

LEMMA: In the leading term, any eigenfunction is collinear to α:

$$\psi(z-z_s) = X(z-z_s)^m + O(z-z_s)^{m+1}, \quad X = \nu\alpha, \quad \nu \in \mathbb{C}.$$

Proof (for $SO(4)$). $X^T = (x_1, x_2, x_3, x_4)$.

$$\begin{cases} \lambda x_1 + \varkappa x_2 + (\delta - \beta_3/z)x_3 + (\gamma - \beta_4/z)x_4 & = O(z) \\ -\varkappa x_1 + \lambda x_2 + \mathrm{i}(\delta - \beta_3/z)x_3 + \mathrm{i}(\gamma - \beta_4/z)x_4 & = O(z) \\ (\beta_3/z - \delta)x_1 + \mathrm{i}(\beta_3/z - \delta)x_2 + \lambda x_3 + \sigma x_4 & = O(z) \\ (\beta_4/z - \gamma)x_1 + \mathrm{i}(\beta_4/z - \gamma)x_2 - \sigma x_3 + \lambda x_4 & = O(z) \end{cases}$$

(Eq.1+i$*$Eq.2)$\Longrightarrow$ $x_1 + \mathrm{i}x_2 = 0$. Then x_1 and x_2 disappear in Eqs. 3,4, and $\lambda x_3 + \sigma x_4 = -\sigma x_3 + \lambda x_4 = 0$. But from Eq.1 $\beta_3 x_3 + \beta_4 x_4 = 0$ (by holomorphy of λ), hence generically $x_3 = x_4 = 0$, and

$$X^T = (x_1, \mathrm{i}x_1, 0, 0) = x_1\alpha^T.$$

To get rid of the dynamical divisor – 1

$$\underbrace{\left(\partial_a - \frac{\mu_s\alpha_s^T - \alpha_s\mu_s^T}{z - z_s} + \dots\right)}_{(\partial_a - M_a)} \overbrace{(\alpha_s(z - z_s)^m + \dots)}^{\psi} = f_a\,(\alpha_s(z - z_s)^m + \dots)$$

The left hand side is $O(z - z_s)^{m-1}$ while the right hand side is $O(z - z_s)^m$. Hence $f_a = \frac{f_{-1}}{z - z_s} + O(1)$, and we find out $f_{-1} = -(m+1)(\partial_a z_s)$ (using $-\partial_a z_s = \mu_s^T \alpha_s$). Hence

$$f_a(z) = (m+1)\frac{-\partial_a z_s}{z - z_s} + O(1) = (m+1)\partial_a \log(z - z_s) + O(1) \Longrightarrow$$

$$e^{-\int_0^{t_a} f_a dt} = O\left((z - z_s)^{-(m+1)}\right) \Longrightarrow \widehat{\psi} = e^{-\int_0^{t_a} f_a dt}\psi = \frac{\alpha_s}{z - z_s} + O(1)$$

So far we failed: $\widehat{\psi}$ still has a pole at z_s.

To get rid of the dynamical divisor – 2

It is the term $\mu_s^T \alpha_s/(z - z_s)$ which is responsible for this failure.

Let $\phi(P)$ be a function having the poles $\mu_s^T \alpha_s/(z - z_s)$ at z_s's, and an order g pole at some P_ϕ. Define f_a from the equation

$$(\partial_a - M_a - \phi E)\psi = f_a \psi.$$

Then

$$f_{-1} = m(-\partial_a z_s)$$

Hence $e^{-\int_0^{t_a} f_a dt} = O(z - z_s)^{-m}$, and $\widehat{\psi}$ is holomorphic at z_s.

To encode ϕ, we require that the Baker–Akhiezer funct. has the exponent $\exp(w^{-g})$ at $P_\phi^{(j)}$, $\forall j$.

To get rid of the dynamical divisor – 3

The corresponding matrix $\widehat{\Psi}$ satisfies

$$(\partial_a - M_a - \phi E)\widehat{\Psi} = 0,$$

hence

$$M_a = \partial_a \widehat{\Psi} \cdot \widehat{\Psi}^{-1} - \phi E.$$

After we prove that $\partial_a \widehat{\Psi} \cdot \widehat{\Psi}^{-1}$ is the sum of a skew-symmetric matrix, and a scalar matrix, we are able to retrieve ϕ as

$$\phi = \frac{1}{n} \cdot \operatorname{tr}(\partial_a \widehat{\Psi} \cdot \widehat{\Psi}^{-1}),$$

since $\operatorname{tr} M_a = 0$.

Spectral data for $SO(2n)$ Hitchin system

Hitchin system: $\sum m_i P_i = (\omega)$, $\omega \in \Omega_{hol}(\Sigma)$

Description of spectral curves:

$$\widehat{\Sigma}: \ \lambda^{2n} + \sum_{j=1}^{n-1} \lambda^{2(n-j)} r_j + r_n^2 = 0,$$

$r_j \in \mathcal{O}(2j(\omega), \Sigma)$, $(j < n)$, $r_n \in \mathcal{O}(n(\omega), \Sigma)$. $\sigma : \lambda \to -\lambda$ – involution on $\widehat{\Sigma}$. Singular points: $\lambda = 0$. Normalization map: $\pi : \widehat{\Sigma}_n \to \widehat{\Sigma}$. Normalization point $P_0 \in \Sigma$.

Divisor: $\deg D_h = \dim Prym(\widehat{\Sigma}_n)$ $(= h)$ s.t. $\exists(!)$ Prym differential Ω with zeroes on D_h and simple poles $Q_1, Q_1^\sigma \in \pi^{-1}(P_0)$. Let $(\Omega) = D + D^\sigma - Q_1 - Q_1^\sigma$, $\deg D = \widehat{g}$, $D > D_h$.

Skew-symmetry of L from spectral data

Let ψ, ψ^+ be the pair of dual Baker–Akhieser functions on $\widehat{\Sigma}$ with pole divisors $D + Q_2 + Q_3 + \ldots + Q_{2n}$, $D^\sigma + Q_1 + Q_3 + \ldots$ $\ldots + Q_{2n}$ where $Q_2 = Q_1^\sigma$, resp. Let exponents of ψ, ψ^+ be $\pm \sum_m t_m \lambda^{2m-1}$. Let $\widehat{\Psi}$, $\widehat{\Psi}^+$ be the corresponding matrices. Then

$$\widehat{\Psi}^+ \widehat{\Psi} = \xi \quad \text{where} \quad \xi \quad \text{is diagonal..}$$

$\widehat{\psi}^+ = (\widehat{\psi}^\sigma)^T$ by uniqueness of the Baker–Akhieser funct., and $(\widehat{\psi}^\sigma)^T = (\mathfrak{s}\widehat{\psi})^T$ where $\mathfrak{s} = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$, hence $\widehat{\Psi}^+ = \mathfrak{s}\widehat{\Psi}^T\mathfrak{s}$. We plug $\widehat{\Psi}^{-1} = \xi^{-1}\mathfrak{s}\widehat{\Psi}^T\mathfrak{s}$ into $L = \widehat{\Psi}\Lambda\widehat{\Psi}^{-1}$ where Λ comes from the spectral curve. By $\mathfrak{s}^2 = E$, $\mathfrak{s}\Lambda\mathfrak{s} = -\Lambda$, $\mathfrak{s}\xi\mathfrak{s} = \xi$ we obtain

$$L = \widehat{\Psi}\Lambda\xi^{-1}\mathfrak{s}\widehat{\Psi}^T\mathfrak{s}, \quad L^T = \mathfrak{s}\Psi\mathfrak{s}\Lambda\xi^{-1}\widehat{\Psi}^T = -\mathfrak{s}L\mathfrak{s}^{-1}.$$

Thank you!