Lax Triads, Finite-Gap Integration and Pseudo-Differential Operators

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The Korteweg–de Vries equation. Lax Representation

The remarkable theory of finite-gap integration was established in 1974 by S.P. Novikov and developed by him with his colleagues B.A. Dubrovin, V.B. Matveev and A.R. Its. The review was presented in 1976. The main and simplest object was the Korteweg–de Vries equation (KdV)

$$u_t = uu_x + \frac{1}{12}u_{xxx},$$

whose Lax pair has the form

$$\psi_t = \hat{L}_3\psi, \quad \lambda\psi = \hat{L}_2\psi,$$

where λ is the so-called spectral parameter and (the functions u and v depend on x and t)

$$\hat{L}_2 = \frac{1}{2}\partial_x^2 + u, \quad \hat{L}_3 = \frac{1}{3}\partial_x^3 + u\partial_x + v.$$

In this particular case $v = u_x/2$.

The construction of finite-gap integration is based on the existence of the Lax triad $\hat{L}_2, \hat{L}_3$ and $\hat{L}_k$, where $\hat{L}_k$ are differential operators (u and other functions $u_{m,N}$ depend on x and t)

$$\hat{L}_M = \frac{1}{M}\partial_x^M + u\partial_x^{M-2} + \sum_{m=0}^{M-3} u_{m,M}\partial_x^m, \quad M = 4, 5, \ldots$$

So, N-gap solution of the KdV equation

$$u_t = uu_x + \frac{1}{12}u_{xxx},$$

is determined by the Lax triad

$$\psi_t = \hat{L}_3\psi, \quad \lambda\psi = \hat{L}_2\psi, \quad \mu\psi = \hat{L}_{2N+1}\psi.$$

The Boussinesq Equation. Lax Representation

Another remarkable example of integrable system is the Boussinesq equation

$$u_{yy} + (uu_x)_x + \frac{1}{12} u_{xxxx} = 0,$$

determined by the Lax pair (the functions u and v depend on x and y)

$$\psi_y = \hat{L}_2 \psi, \quad \mu\psi = \hat{L}_3 \psi.$$

Here μ plays the role of a spectral parameter, while the function v is determined by its first order derivatives

$$v_x = u_y + \frac{1}{2} u_{xx}, \quad v_y = \frac{1}{2} u_{xy} - uu_x - \frac{1}{12} u_{xxx}.$$

Again, here

$$\hat{L}_2 = \frac{1}{2} \partial_x^2 + u, \quad \hat{L}_3 = \frac{1}{3} \partial_x^3 + u\partial_x + v.$$

The Finite-Gap Integration

Multi-gap solutions of the Boussinesq equation are determined by another Lax triad (N again is a natural number)

$$\psi_y = \hat{L}_2\psi, \quad \mu\psi = \hat{L}_3\psi, \quad \lambda\psi = \hat{L}_{3N\pm1}\psi,$$

where

$$\hat{L}_M = \frac{1}{M}\partial_x^M + u\partial_x^{M-2} + \sum_{m=0}^{M-3} u_{m,M}\partial_x^m, \quad M = 4, 5, ...$$

The standard theory of finite-gap integration suggests the Lax triad

$$\psi_y = \hat{L}_2\psi, \quad \mu\psi = \hat{L}_M\psi, \quad \lambda\psi = \hat{L}_K\psi, \quad K, M = 2, 3, 4, \ldots$$

Pseudo-Differential Lax Triads

The standard theory of finite-gap integration suggests the Lax triad

$$\psi_y = \hat{L}_2\psi, \quad \mu\psi = \hat{L}_M\psi, \quad \lambda\psi = \hat{L}_K\psi, \quad K, M = 2, 3, 4, \ldots$$

Here we extend this construction to a class of pseudo-differential operators (the functions p_m and q_m depend on x and y)

$$\hat{L}_{N,M} = \hat{L}_N + \sum_{m=1}^{M} p_m \partial_x^{-1} q_m, \quad M = 1, 2, \ldots$$

The Boussinesq Equation. Pseudo-Differential Lax Triad

For instance, the Boussinesq equation has a particular finite-gap solution selected by the Lax triad

$$\psi_y = \hat{L}_2\psi, \quad \mu\psi = \hat{L}_3\psi, \quad \lambda\psi = \left(\partial_x + \sum_{m=1}^{M} p_m \partial_x^{-1} q_m\right)\psi,$$

whose compatibility conditions yield the system (ξ_k are arbitrary constants) containing also vector (Manakov) NLS equation

$$p_{k,y} = \frac{1}{2}p_{k,xx} + up_k, \quad -q_{k,y} = \frac{1}{2}q_{k,xx} + uq_k, \quad u = \sum_{m=1}^{M} p_m q_m,$$

together with a stationary version of its higher commuting flow known as a vector complex MKdV equation

$$\frac{1}{3}p_{k,xxx}+up_{k,x}+vp_k=\xi_k p_k, \; \frac{1}{3}q_{k,xxx}+uq_{k,x}+(u_x\text{-}v)q_k=\text{-}\xi_k q_k, \; v=\sum_{m=1}^{M} q_m p_{m,x}$$

This overdetermined system can be solved. The last step is a computation of the Baker–Akhiezer function and of the corresponding Riemann surface.

Without loss of generality we restrict our consideration to the first nontrivial case

$$\psi_y = \hat{L}_2\psi, \quad \mu\psi = \hat{L}_M\psi, \quad \lambda\psi = (\partial_x + p\partial_x^{-1}q)\psi, \quad M = 3, 4, \ldots$$

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First Example: $M = 3$ (the Boussinesq equation).
In this case the Lax triad is

$$\psi_y = \left(\frac{1}{2}\partial_x^2 + u\right)\psi, \quad \mu\psi = \left(\frac{1}{3}\partial_x^3 + u\partial_x + v\right)\psi, \quad \lambda\psi = (\partial_x + p\partial_x^{-1}q)\psi.$$

Gelfand–Dickey Reduction. Pseudo-Differential Lax Triad

In the Lax triad

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the third above equation can be rewritten in the equivalent form

$$\lambda(\partial_x - s)\psi = (\partial_x^2 - s\partial_x + u)\psi,$$

where $u = pq, \quad s = (\ln p)_x$ and $v = us$.

Gelfand–Dickey Reduction. Pseudo-Differential Lax Triad

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$$\lambda(\partial_x - s)\psi = (\partial_x^2 - s\partial_x + u)\psi,$$

where $u = pq$, $s = (\ln p)_x$ and $v = us$. Then under the substitution $\psi = a\varphi$, the above Lax triad becomes

$$\varphi_y = \frac{1}{2}(\lambda + s)\varphi_x - \frac{1}{4}s_x\varphi, \quad \varphi_{xx} = \left(\frac{1}{4}\lambda^2 - \frac{1}{2}\lambda s + \frac{1}{4}s^2 - \frac{1}{2}s_x - u\right)\varphi,$$

$$\zeta\varphi = (\lambda^2 + \lambda s + 2u + s^2 + s_x)\varphi_x - \frac{1}{2}(\lambda s_x + 2u_x + 2ss_x + s_{xx})\varphi,$$

where

$$\zeta = 3\mu - \frac{1}{2}\lambda^3 - \frac{1}{2}c_1, \quad \frac{a_x}{a} = \frac{1}{2}(\lambda + s), \quad \frac{a_y}{a} = \frac{1}{4}(\lambda^2 + 2u + s^2 + s_x).$$