Completeness for modal predicate logics

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Introduction

This talk is about first-order modal logics with expanding domains.

Overviews of the area were given earlier in my talks of 1990 (Chaika, Bulgaria) and 2006 (Noosa, Australia). All intersections are minimal.

Among others, we will make some references to the book [QNL] of 2009. It is getting old quickly.

Introduction

Kripke semantics fits almost well for modal propositional logics, but yet in the 1980s it became clear that too many first-order modal logics are Kripke incomplete. So three lines of research emerged:

- Identifying Kripke complete logics.
- Describing completions for incomplete logics.
- Studying completeness for more general semantics.

Further progress:

- The first examples of Kripke complete logics were quantified versions of the basic propositional logics: **QK**, **QS4**, **QS4** etc. But for other familiar systems incompleteness was discovered: **QGL**, **QGrz**, **QS4.1**. Moreover, unlike sporadic incompleteness in modal propositional logic, in incomplete first-order logics often appear in large (uncountable) families.
  Some other logics (**QS4.2**, **QS4.3**) are still Kripke complete, but the known completeness proofs are quite laborious.

Introduction

- As for finding completions of incomplete logics, in some cases the problem was solved. The first paper on this topic [Cresswell 2000] gave completions for **QS4.1**, **QS4.4**. But some logics turned out to be incompletable [Cresswell 1997], and for some others (like $\mathbf{QS4.2} + Ba$) the problem is still open.
- In this talk we focus on semantics of relational type:

Kripke frames (Kripke, 1960s) $\prec$ Kripke sheaves (1982) $\prec$ Kripke bundles (1982) $\prec$ Functor semantics (Ghilardi, 1989) $\prec$ Simplicial semantics (Skvortsov, 1990)

($\mathcal{S} \prec \mathcal{S}'$ means that semantics $\mathcal{S}'$ is strictly stronger than $\mathcal{S}$, i.e. $\mathcal{S}'$-completeness implies $\mathcal{S}$-completeness, but not the other way round.)
However, as we shall see, we still do not get a full solution of completeness problem by these generalizations.

Modal propositional logics

Modal propositional formulas are constructed from a countable set of proposition letters PL and $\to, \bot, \Box$.

Derived connectives are $\wedge, \vee, \Diamond, \top, \leftrightarrow$

Formulas without proposition letters are closed.

A (normal) modal logic is a set of formulas containing the minimal logic **K** and closed under the rules

- (MP) $A, A \to B \,/\, A$;
- (Nec) $A \,/\, \Box A$;
- (Sub) $A \,/\, SA$, where S is a propositional substitution.

All the logics are supposed consistent.

Modal predicate logics

Modal predicate formulas are built from the countable set of individual variables Var, predicate letters P^n_k $(n, k \geq 0)$, the connectives $\rightarrow, \bot, \Box$, and the quantifier $\forall$. ($\exists$ is derived)

Notation: $\overline{\forall} A$ denotes the universal closure of a formula A.

A modal predicate logic is a set of formulas containing

- the classical predicate tautologies;
- the axiom of **K**: $\Box(P^0_1 \rightarrow P^0_2) \rightarrow (\Box P^0_1 \rightarrow \Box P^0_2)$,

and closed under the rules (MP), (Nec),

- (Gen) $A \,/\, \forall x A$;
- (Sub) $A \,/\, SA$, where S is a predicate substitution (replacing every occurrence of $P(x_1, \ldots, x_n)$ with $B(x_1, \ldots, x_n)$).

$\mathbf{Q\Lambda}$ is the minimal predicate extension of a propositional logic $\mathbf{\Lambda}$.

$L + \Gamma$ is the minimal modal predicate logic containing a logic L and a set of sentences Γ.

Propositional Kripke semantics

A propositional Kripke frame is $F = (W, R)$, where $W \neq \varnothing$, $R \subseteq W^2$.

A Kripke model over F is $M = (F, \theta)$, where $\theta : PL \longrightarrow 2^W$ is a valuation.

The truth definition $M, u \vDash A$ and the validity definition $(F \vDash A)$ are standard.

$\mathbf{L}(F) := \{A \mid F \vDash A\}$ is the logic of a frame F.

$\mathbf{L}(\mathcal{C}) := \bigcap\{\mathbf{L}(F) \mid F \in \mathcal{C}\}$ is the logic of a class of frames $\mathcal{C}$.

Logics $\mathbf{L}(\mathcal{C})$ are (Kripke) complete.
If $\mathcal{C}$ consists of finite frames, $\mathbf{L}(\mathcal{C})$ has the finite model property.
If F is a finite frame, $\mathbf{L}(F)$ is tabular.

Completeness of modal propositional logics: a brief summary

(1) Complete logics

- Inductive (Goranko & Vakarelov, 2006) $\supset$ Sahlqvist (Sahlqvist, 1975) $\supset$ Horn type $\supset$ PTC
- Transitive logics of finite width (Fine, 1974)

(2) Logics with the fmp

- Subframe extensions of **K4** (Fine, 1985)
- Uniform extensions of **KD** (Fine, 1975)
- Logics of finite depth (Segerberg, 1971; Gabbay & S., 1998)

(3) Incomplete logics

Artificial examples by different authors from the 1970s (S. Thomason, Fine, Van Benthem, Sambin, V. Rybakov, S.).

Predicate Kripke semantics-1

Definition

A predicate Kripke frame over a propositional frame $F = (W, R)$ is a pair $\mathbf{F} = (F, D)$, where $D = (D_u)_{u \in W}$, $D_u \neq \varnothing$ is an expanding system of domains:

$$uRv \Rightarrow D_u \subseteq D_v.$$

Definition

A valuation on a Kripke frame $\mathbf{F}$ is a family of local valuations: $\xi = (\xi_u)_{u \in W}$, where $\xi_u(P_k^n) \subseteq D_u^n$ ($D_u^0 = \{()\}$ consists of an empty tuple).

A predicate Kripke model is $(\mathbf{F}, \xi)$

A variable assignment on $\mathbf{F}$ at $u \in W$ is a pair $(\mathbf{x}, \mathbf{a})$ (notation: $\mathbf{a}/\mathbf{x}$), where $\mathbf{x}$ is a list of n different variables, $\mathbf{a} \in D_u^n$.

Predicate Kripke semantics-2

We define the truth of a formula A at world u in a model M under a variable assignment $\mathbf{a}/\mathbf{x}$, where $\mathbf{x}$ contains all parameters of A.

Notation For an n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ and a map (an index transformation) $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$ let

$\mathbf{a} \cdot \sigma := (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$.

(For $n = 0$ we have $\mathbf{a} = ()$, σ is an empty map, $\mathbf{a} \cdot \sigma = ()$.)

Definition

- $M, u, \mathbf{a}/\mathbf{x} \vDash P(\mathbf{x} \cdot \sigma)$ iff $(\mathbf{a} \cdot \sigma) \in \xi_u(P)$
- $M, u, \mathbf{a}/\mathbf{x} \vDash \Box A$ iff $\forall v \in R(u)\ M, v, \mathbf{a}/\mathbf{x} \vDash A$
- $M, u, \mathbf{a}/\mathbf{x} \vDash \forall y A$ iff $\forall b \in D_u\ M, u, \mathbf{a}b/\mathbf{x}y \vDash A$ $(y \notin \mathbf{x})$.

Other cases are clear.

Predicate Kripke semantics-3

A modal formula A is true in M (in symbols, $M \vDash A$) if $M, u \vDash \overline{\forall} A$ for every $u \in W$.

A modal formula A is valid on a frame $\mathbf{F}$ (in symbols, $\mathbf{F} \vDash A$) if it is true in every Kripke model over $\mathbf{F}$.

$\mathbf{L}(\mathbf{F}) := \{A \mid \mathbf{F} \vDash A\}$ is the modal logic of $\mathbf{F}$.
The modal logic of a class of frames $\mathcal{C}$ is $\mathbf{L}(\mathcal{C}) := \bigcap\{\mathbf{L}(\mathbf{F}) \mid \mathbf{F} \in \mathcal{C}\}$.
Logics of this form are called Kripke complete.

A frame validating a modal predicate logic L is called an L-frame.
A formula A is a logical consequence of a logic L in Kripke semantics ($L \vDash_{\mathcal{K}} A$) if A is valid on all L-frames.

$\widehat{L} := \{A \mid L \vDash_{\mathcal{K}} A\}$ is the smallest Kripke complete extension of L, the Kripke completion of L.

Strong Kripke completeness

Definition

A (modal predicate) theory is a set of formulas. A theory Γ is L-consistent if $\Gamma \nvdash_L \bot$ (i.e. $\bot$ is not derivable from $L \cup \Gamma$ using MP).

Definition

A theory Γ is satisfiable in a Kripke model M at point u if $M, u, \delta \vDash \Gamma$ for some variable assignment δ.

Definition

A modal predicate logic L is strongly Kripke complete if every L-consistent theory is satisfiable in some Kripke model over an L-frame.

So strong completeness implies completeness.

Strong Kripke completeness

Proposition

If a modal predicate logic L is strongly Kripke complete and Γ is a set of closed propositional formulas, then $L + \Gamma$ is also strongly Kripke complete.

Canonical models

For any modal predicate logic L there exists a canonical frame F_L and a canonical model M_L over F_L. The worlds of F_L are L-places, i.e. maximal L-consistent Henkin theories with extra constants taken from a fixed set S (so M_L depends on S); the domain D_Γ of a world Γ is the set of constants appearing in Γ.
(A technical detail: usually S is countable; $S - D_\Gamma$ should be infinite.)

Canonical model theorem

For any formula $A(x_1, \dots, x_n)$ and $c_1, \dots, c_n \in D_\Gamma$

$$M_L, \Gamma, c_1, \dots, c_n/x_1, \dots, x_n \models A(x_1, \dots, x_n) \text{ iff } A(c_1, \dots, c_n) \in \Gamma,$$

thus

$$M_L \models A \text{ iff } L \vdash A.$$

Canonical logics-1

Definition

A logic L is canonical if $F_L \models L$.

Corollary (from Canonical model theorem)

Every canonical logic is strongly Kripke complete.

Canonical logics are rare. The well-known Kripke complete logics **QS5**, **QTriv** are not canonical.

Definition

A one-way pseudotransitive propositional formula is of the form $\Box p \to \Box^n p$. A one-way PTC-logic is a modal propositional logic axiomatized by one-way pseudotransitive and closed formulas.

$\Box p \to \Box^n p$ corresponds to the condition $R^n \subseteq R$ (R^0 is the diagonal).

Canonical logics-2

Well-known examples: $\mathbf{T} = \mathbf{K} + \Box p \to p$, $\mathbf{K4} = \mathbf{K} + \Box p \to \Box\Box p$, $\mathbf{S4} = \mathbf{T} + \mathbf{K4}$, $\mathbf{KD} = \mathbf{K} + \Diamond\top$.

Theorem 1 [QNL]

If $\mathbf{\Lambda}$ is a one-way PTC-logic, then $\mathbf{Q\Lambda}$ is canonical.

Theorem 2

If

$$\mathbf{\Lambda} = \mathbf{K} + B \vee \bigvee_i A_i,$$

where A_i are one-way pseudotransitive formulas without common proposition letters, B is closed, then $\mathbf{Q\Lambda}$ is canonical.

Theorem 3

The sum of canonical logics is canonical.

Kripke completeness of $Q\Lambda + Ba$

The Barcan formula $Ba := \forall x \Box P(x) \to \Box \forall x P(x)$ is valid on rooted Kripke frames exactly with constant domains.

Logics containing Ba are usually not canonical, but many of them are still complete; the proof is by a version of canonicity for constant domains.

Definition

A propositional modal logic $\mathbf{\Lambda}$ is universal if the class of frames validating $\mathbf{\Lambda}$ is universal (in the classical sense).

Theorem 4 [Tanaka & Ono 2000]

If $\mathbf{\Lambda}$ is universal and complete, then $\mathbf{Q\Lambda} + Ba$ is strongly Kripke complete.

Theorem 5 [QNL]

If $\mathbf{\Lambda}$ is tabular, then $\mathbf{Q\Lambda} + Ba$ is strongly Kripke complete.

Kripke completeness of $Q\mathbf{\Lambda}$: selection

Definition

A Kripke model M' over a frame $\mathbf{F} = (W', R', D)$ is a selective submodel of a Kripke model M over $\mathbf{F} = (W, R, D)$ if $W' \subseteq W$, $R' \subseteq R$, and for every $w \in W'$,

- $M, w, \delta \vDash A \Leftrightarrow M', w, \delta \vDash A$ for any assignment δ and atomic A,
- for every formula A and assignment δ,

$$M, w, \delta \models \Diamond A \Longrightarrow \exists u \in R'(w)\, M, u, \delta \vDash A.$$

Lemma

If M' is a selective submodel of M, then for any $w \in W'$, formula A and assignment δ $\quad M, w, \delta \vDash A$ iff $M', w, \delta \vDash A$.

Kripke completeness of $Q\mathbf{\Lambda}$: selection

Definition

A logic L is quasi-canonical if for any $\Gamma \in W_L$ there exists a selective submodel M' of M_L containing Γ over a frame validating L.

Lemma

Every quasi-canonical logic is strongly Kripke complete.

$$alt_n := \neg \bigwedge_{0 \leqslant i \leqslant n} \Diamond(p_i \wedge \bigwedge_{j \neq i} \neg p_j),$$

$$\mathbf{Alt}_n := \mathbf{K} + alt_n.$$

Condition on frames: $\forall w \, |R(w)| \leq n$. So $\mathbf{Alt}_n$ is universal and it is known to be canonical; the proof is rather easy.

However, $\mathbf{QAlt}_n$ is not canonical.

Kripke completeness of $Q\mathbf{\Lambda}$: selection

Theorem 6

The logics $\mathbf{QAlt}_n$, $\mathbf{QT}{+}\mathbf{QAlt}_n$ are quasi-canonical.

Theorem 7

If

$$\mathbf{\Lambda} = \mathbf{K} + alt_n \vee B \vee C,$$

where C is closed, B is one-way pseudo-transitive and does not have common proposition letters with alt_n, then $\mathbf{Q\Lambda}$ is quasi-canonical.

Kripke completeness of $Q\Lambda$: uniform trees

Definition

A path of length n is a frame (W, R) is a finite sequence of R-related worlds: $w_0 R w_1 \dots R w_n$. The depth of a world u in a frame F (notation: $d(u)$) is the maximal length of a path from u in F if it exists and ∞ otherwise.

Lemma

- $d(u) \leq n$ iff $F, u \models \Box^{n+1}\bot$,
- $d(u) = n$ iff $F, u \models \Box^{n+1}\bot \wedge \Diamond^{n}\top$.

We denote the latter formula by dep_n.

Theorem 8

If $\mathbf{\Lambda}$ axiomatized by formulas of the form $dep_n \to alt_m$, then $\mathbf{Q\Lambda}$ is quasi-canonical.

Kripke completeness of $Q\Lambda$: uniform trees

Definition

A tree is a propositional frame with a root u_0, in which for any world w, there exists a unique path from u_0 to w.

A uniform tree of type $(n_1, \ldots, n_k)$ (in symblols, $UT(n_1, \ldots, n_k)$) is a finite tree of depth k, in which

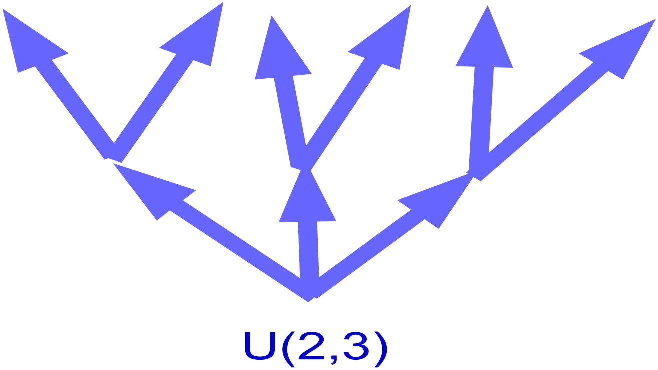
$$|R(w)| = n_i \text{ iff } d(w) = i.$$

Theorem 9

Let $\mathcal{K}UT(n_1, \ldots, n_k)$ be the class of all predicate frames of the form $(UT(n_1, \ldots, n_k), D)$. Then

$$\mathbf{L}(\mathcal{K}UT(n_1, \ldots, n_k)) = \mathbf{QK} + \{dep_i \to alt_{n_i} \mid 1 \leq i \leq k\} + \Box^{k+1}\bot.$$

Kripke completeness of $Q\mathbf{\Lambda}$: uniform trees



Kripke completeness of $Q\mathbf{\Lambda}$: uniform trees

Remarks

1. The previous theorem contrasts to the intuitionistic case, in which some of the corresponding logics are not finitely axiomatizable (Skvortsov).

2. We can also include the cases when $n_i = \infty$ for some i. In the axiomatization these i should be skipped.

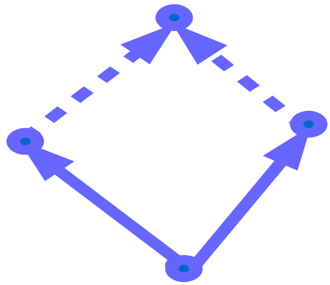
Variations on confluence

$$A2 := \Diamond\Box p \to \Box\Diamond p.$$

corresponds to confluence:

$$R^{-1} \circ R \subseteq R \circ R^{-1}.$$

$\mathbf{K2} := \mathbf{K} + A2$ is the logic of confluent frames.



Variations on confluence

Conjecture: **QK2** is incomplete.

Theorem 10

$\mathbf{QK} + dep_n \rightarrow A2$ is quasi-canonical for $n \geq 2$.

Thick frames

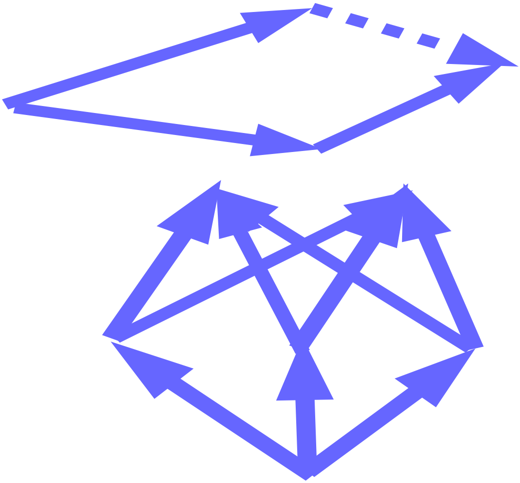
$$Ath := \Diamond^2 p \to \Box\Diamond p.$$

$$\mathbf{K05} := \mathbf{K} + Ath.$$

The frame condition for **K05** is thickness:

$$xRy \ \& \ xRz \Rightarrow R(y) = R(z).$$

Thick frames



Variations on thickness

Theorem 11

1. Every logic $\mathbf{QK} + (dep_n \to Ath) + \Box^{n+1}\bot$ is quasi-canonical for $n \geq 2$.
2. Every logic $\mathbf{QK} + dep_n \to Ath$ is strongly Kripke complete for $n \geq 2$.

Corollary

$\mathbf{QK05} + \Box^3\bot$ is quasi-canonical.

Remarks

1. Note that we cannot combine several axioms in the previous theorem.

2. Completeness of **QK05** is unknown. Conjecture: even $\mathbf{QK05} + \Box^4\bot$ is Kripke incomplete.

Kripke completeness of $Q\Lambda$: constructing models by games

There is a group of completeness results proved by a more delicate method. Instead of finding a selection directly, it uses step-by-step construction by playing a game.

These cases include logics with linearity, confluence, and density axioms:

- $\mathbf{QS4.2}$, $\mathbf{QK4.2}$ are strongly Kripke complete [Corsi & Ghilardi, 1989]
- $\mathbf{QK4.2} + Ad$, $\mathbf{QK4} + Ad$, $\mathbf{QK} + Ad$, $\mathbf{QK4.2} + Ad_2$, $\mathbf{QK4} + Ad_2$, $\mathbf{QK} + Ad_n$ are strongly Kripke complete [AiML18]
- $\mathbf{QS4.3}$ is strongly Kripke complete [Corsi, 1989]
- $\mathbf{QD4.3} + Ad = \mathbf{L}(\mathcal{K}(\mathbf{Q}, <))$ [Corsi, 1993]

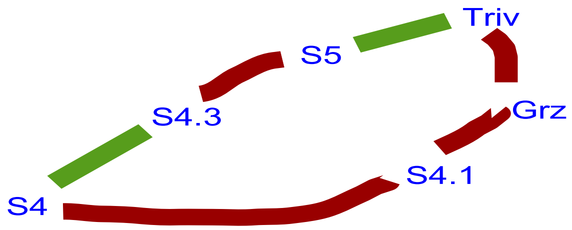
Examples of Kripke incompleteness

$\mathbf{S4.3} = \mathbf{S4} + \Box(\Box p \to q) \vee \Box(\Box q \to p)$ is the logic of non-branching $\mathbf{S4}$-frames.

Theorem 12 [Ghilardi 1991]

If $\mathbf{\Lambda} \supseteq \mathbf{S4}$ and $\mathbf{Q\Lambda}$ is complete, then either $\mathbf{\Lambda} \subseteq \mathbf{S4.3}$ or $\mathbf{\Lambda} \supseteq \mathbf{S5}$.

Logics $\mathbf{\Lambda} \supseteq \mathbf{S4}$, with incomplete / possibly complete $\mathbf{Q\Lambda}$:



Examples of Kripke incompleteness

Adding the Barcan formula may lead to incompleteness.

Theorem 13 [Talk90]

$\mathbf{QS4.2} + Ba$ is Kripke incomplete.

Kripke completeness can be restored by adding

$$ML := \forall x \, \Diamond\Box P(x) \to \Diamond\Box\forall x \, P(x).$$

Theorem 14 [Corsi & Ghilardi 1989]

$\mathbf{QS4.2} + Ba + ML = \mathbf{L}(\mathcal{C})$, where $\mathcal{C}$ is the class of **QS4**-frames with constant domains having the final cluster.

However, this logic is larger than (still unknown) Kripke completion of $\mathbf{QS4.2} + Ba$.

Examples of Kripke incompleteness

Theorem 15 [Gasquet, 1995]

$\mathbf{QKD} + Ad + Ba$ is Kripke incomplete.

Kripke completion in this case is also unknown.

On Kripke completions

Definition

A modal predicate logic L is R-elementary, if the class of predicate Kripke frames $\{F \mid F \vDash L\}$ is first-order recursively axiomatizable (in the 2-sorted language).

In particular, $\mathbf{Q\Lambda}$ is R-elementary, whenever $\mathbf{\Lambda}$ is elementary (i.e. the class of $\mathbf{\Lambda}$-frames is first-order finitely axiomatizable).

Theorem 16 [QNL]

If L is R-elementary, then $\widehat{L}$ is RE.

Theorem 17 on incompletable logics [Cresswell 97]

Let $\mathbf{\Lambda}$ be one of the logics **GL**, **Grz**, **D4.3Z** ($= \mathbf{L}(\mathbf{N}, <)$), **S4.3.1** ($= \mathbf{L}(\mathbf{N}, \leq)$). Then $\widehat{\mathbf{Q\Lambda}}$, $\widehat{\mathbf{Q\Lambda} + Ba}$ are not RE.

Boxing predicate logics-1

Definition

For a set of modal formulas Γ, put

$$\Box\Gamma := \{\Box A \mid A \in \Gamma\}.$$

For a modal propositional logic $\mathbf{\Lambda}$ put

$$\Box\cdot\mathbf{\Lambda} := \mathbf{K} + \Box\mathbf{\Lambda}.$$

For a modal predicate logic L put $\Box\bullet L := \mathbf{QK} + \Box L$.

For modal propositional logics boxing is easily axiomatized:

Lemma 1

$$\Box\cdot(\mathbf{K} + \Gamma) = \mathbf{K} + \Box\Gamma.$$

Boxing predicate logics-2

Lemma 2

For a set of sentences Γ

$$\mathbf{QK} + \Box\Gamma \subseteq \Box\bullet(\mathbf{QK} + \Gamma).$$

This inclusion may be strict, since $L \vdash \Box A \Rightarrow L \vdash \Box\forall x A$ is not always true.

The problem disappears for logics with the Barcan formula:

Proposition

$$\Box \bullet (\mathbf{QK} + \Gamma) + Ba = \mathbf{QK} + \Box\Gamma + Ba.$$

In particular, for a modal propositional logic $\mathbf{\Lambda}$,

$$\Box \bullet \mathbf{Q\Lambda} + Ba = \mathbf{Q}(\Box \cdot \mathbf{\Lambda}) + Ba.$$

Boxing predicate logics-3

Recall that $\mathbf{T} := \mathbf{K} + \Box p \to p$ is the logic of reflexive Kripke frames.

Theorem 18 [QK5]

If $\mathbf{QT} \subseteq \mathbf{QK} + \Gamma$, then $\Box\bullet(\mathbf{QK} + \Gamma) = \mathbf{QK} + \Box\Gamma + \Box\forall ref$, where

$$\Box\forall ref := \Box\forall x(\Box P(x) \to P(x)).$$

In particular, for a modal propositional logic $\mathbf{\Lambda} \supseteq \mathbf{T}$

$$\Box\bullet\mathbf{Q\Lambda} = \mathbf{Q}(\Box \cdot \mathbf{\Lambda}) + \Box\forall ref.$$

Transfer theorems for boxing-1

Theorem 19 [QK5]

1. Predicate boxing preserves canonicity.
2. If $\mathbf{Q\Lambda}$ is strongly Kripke complete, then $\Box\bullet(\mathbf{Q\Lambda})$ is strongly Kripke complete.
3. Predicate boxing preserves strong Kripke sheaf completeness.

Some counterexamples

In general predicate boxing does not preserve Kripke completeness (neither weak, nor strong). Consider the logics

$$\mathbf{Q\Lambda U_1} := \mathbf{Q\Lambda} + AU_1,$$

where $\mathbf{\Lambda}$ is a propositional modal logic,

$$AU_1 := \exists x P(x) \rightarrow \forall x P(x)$$

is the axiom of singleton domains.

Proposition [QK5]

Let $\mathbf{\Lambda}$ be a strongly complete consistent modal propositional logic. Then

- $\mathbf{Q\Lambda U_1}$ is strongly Kripke complete.
- $\Box \bullet \mathbf{Q\Lambda U_1} \models_{\mathcal{K}} AU_1$, but $\Box \bullet \mathbf{Q\Lambda U_1} \not\vdash AU_1$.

  Thus $\Box \bullet \mathbf{Q\Lambda U_1}$ is Kripke incomplete.

Incompleteness and completions for boxing

Theorem 20 [QK5]

Let $\mathbf{\Lambda}$ be a consistent propositional logic containing $\mathbf{T}$. Then

1. $\mathbf{Q}(\Box \cdot \mathbf{\Lambda})$ is Kripke incomplete.
2. If $\mathbf{Q\Lambda}$ is strongly Kripke complete, then

$$\mathbf{Q}\widehat{(\Box \cdot \mathbf{\Lambda})} = \Box \bullet \mathbf{Q\Lambda} = \mathbf{Q}(\Box \cdot \mathbf{\Lambda}) + \Box\forall ref.$$

The propositional $(\Box \cdot \mathbf{\Lambda})$-frames are of the form $1 + \bigsqcup_{i \in I} F_i$, where $F_i \vDash \mathbf{\Lambda}$.

Here $\bigsqcup$ denotes the disjoint union of frames, $1 + \dots$ is the operation of adding an irreflexive root.

Thus $\Box \bullet \mathbf{Q\Lambda}$ is complete w.r.t. predicate frames over propositional frames of this form.

Extensions of $\Box \cdot \mathbf{T}$

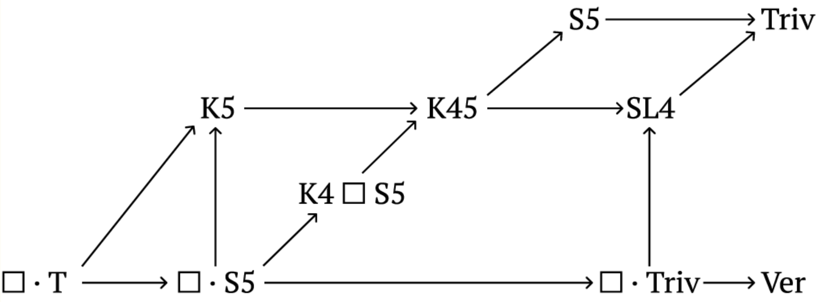
Theorem 21 [QK5]

If $\Box \cdot \mathbf{T} \subseteq \mathbf{\Lambda} \subseteq \mathbf{SL4}$, then $\mathbf{Q\Lambda}$ is Kripke incomplete.

Here the crucial formula is the same $\Box\forall ref$.

$$\Box \cdot \mathbf{T} = \mathbf{K} + \Box(\Box p \to p),\ \ \mathbf{K5} = \mathbf{K} + \Diamond\Box p \to \Box p,$$

$$\mathbf{SL4} = \mathbf{K} + \Box p \to \Box^2 p + \Diamond p \leftrightarrow \Box p = \mathbf{L}(\{0,1\},\{(0,1),(1,1)\}).$$



Axiomatization of boxing

Definition

The *n-shift* A^n of a formula A is obtained by replacing every atomic subformula $P_i^k(\mathbf{y})$ with $P_i^{k+n}(\mathbf{y}, \mathbf{x})$, where $\mathbf{x}$ is a fixed list of n new variables.

Theorem 22 [QK5]

$$\Box \bullet (\mathbf{QK} + \Gamma) = \mathbf{QK} + \{\Box \overline{\forall} A^n \mid A \in \Gamma,\ n \geq 0\}.$$

A priori this theorem gives an infinite axiomatization, but in special cases only $n = 0, 1$ are sufficient (Theorem 12).

Problem. Are all n really necessary for axiomatization of the logics $\Box \bullet \mathbf{Q\Lambda}$?

Generalizations of Kripke semantics

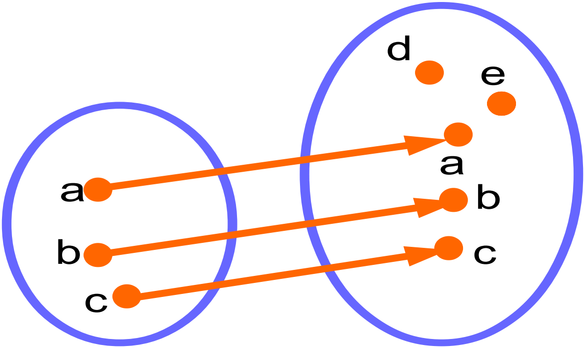
- Kripke frames: $uRv \Rightarrow D_u \subseteq D_v$ (inclusions).
- Kripke sheaves: $uR^*v \Rightarrow \rho_{uv} : D_u \longrightarrow D_v$ (transition functions).
  Conditions: $\rho_{uu} = id_{D_u}$; $uR^*vR^*w \Rightarrow \rho_{uw} = \rho_{vw} \cdot \rho_{uv}$.
- Kripke bundles: $uRv \Rightarrow \rho_{uv} \subseteq D_u \times D_v$ (inheritance relations).
  Condition: $a \in D_u \Rightarrow \rho_{uv}(a) \neq \varnothing$.
- Ghilardi's functor (functional) frames:
  $uR^*v \Rightarrow \mathcal{C}(u,v) \subseteq (D_v)^{D_u}$ (sets of transition functions).
  Conditions: $id_{D_u} \in \mathcal{C}(u,u)$;
  $uR^*vR^*w \& f \in \mathcal{C}(u,v) \& g \in \mathcal{C}(v,w) \Rightarrow g \cdot f \in \mathcal{C}(u,w)$.

Generalizations of Kripke semantics

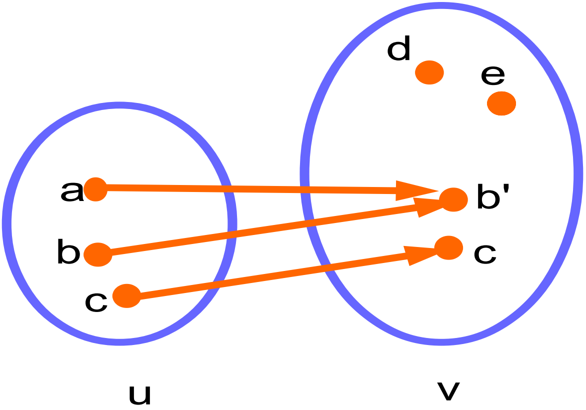
- Kripke metaframes: $uRv \Rightarrow (R_n)_{uv} \subseteq D_u^n \times D_v^n$ (inheritance relation between tuples).
- Simplicial frames: $uRv \Rightarrow (R_n)_{uv} \subseteq D_u^n \times D_v^n$ (inheritance relation between abstract tuples; the sets D_u^n may not be Cartesian powers).

In the last two cases it is convenient to consider global domains D^n and global relations $R_n \subseteq (D^n \times D^n)$; then $(R_n)_{uv} = R_n \cap (D_u^n \times D_v^n)$.

Generalizations of Kripke semantics

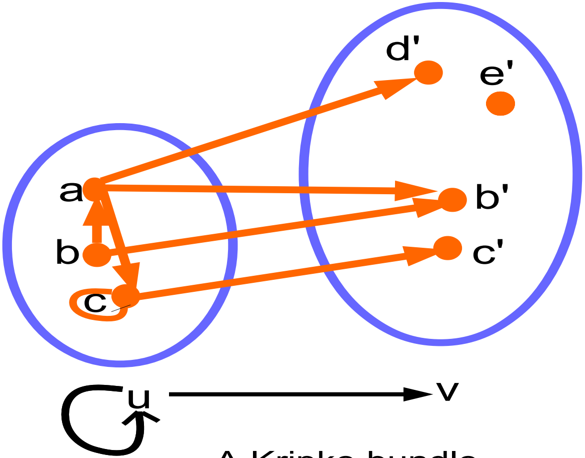


A Kripke frame



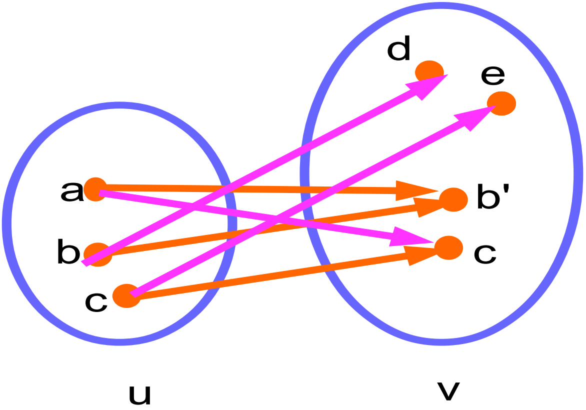
A Kripke sheaf

Generalizations of Kripke semantics



A Kripke bundle

Generalizations of Kripke semantics



A Ghilardi's frame

Kripke metaframes-1

Metaframe semantics is a generalization of predicate Kripke semantics. Its particular case is functor semantics introduced by Silvio Ghilardi.

A frame morphism $(W, R) \longrightarrow (W', R')$ is a map $f : W \longrightarrow W'$ such that $f(R(u)) = R'(f(u))$ for any $u \in W$.

Definition

Let $F_0 = (W, R)$ be a propositional Kripke frame. A metaframe over F_0 is a collection $\mathbf{F}$ of Kripke frames $F_n = (D^n, R^n)$, $n \geq 0$, such that

- $D^0 = W,\ R^0 = R$,
- $D^1 = \bigcup_{u \in W} D_u$, with all D_u nonempty and disjoint ('individual domains'); $D^n = \bigcup_{u \in W} D_u^n$ for $n > 1$, $D_u^n = (D_u)^n$.
- the map $(D^1, R^1) \longrightarrow (W, R)$ sending every individual to its world and all the maps $\mathbf{a} \mapsto \mathbf{a} \cdot \sigma$ are frame morphisms.

Kripke metaframes-2

Definition

A valuation on a metaframe $\mathbf{F}$ is a family $\xi = (\xi_u)_{u \in W}$, where $\xi_u(P_k^n) \subseteq D_u^n$. A metaframe model is $(\mathbf{F}, \xi)$.
A variable assignment on $\mathbf{F}$ at $u \in W$ is $\mathbf{a}/\mathbf{x}$, where (for some n) $\mathbf{x}$ is a list of n different variables, $\mathbf{a} \in D_u^n$.

We define the truth of a formula A in $M = (\mathbf{F}, \xi)$ at u under $\mathbf{a}/\mathbf{x}$, where x contains all parameters of A. The cases of $\to$, $\bot$ are clear.

Definition

- $M, u, \mathbf{a}/\mathbf{x} \vDash P_k^n(\mathbf{x} \cdot \sigma)$ iff $(\mathbf{a} \cdot \sigma) \in \xi_u(P_k^n)$.
- $M, u, \mathbf{a}/\mathbf{x} \vDash \Box A$ iff $\forall v \in R(u)\, \forall \mathbf{b} \in D_v^n\, (\mathbf{a} R^n \mathbf{b} \Rightarrow M, v, \mathbf{b}/\mathbf{x} \vDash A)$.
- $M, u, \mathbf{a}/\mathbf{x} \vDash \forall y A$ iff $\forall b \in D_u\ M, u, \mathbf{a}b/\mathbf{x}y \vDash A$ $(y \notin \mathbf{x})$.
- $M, u, \mathbf{a}/\mathbf{x} \vDash \forall x_i B$ iff $M, u, (\mathbf{a} - a_i)/(\mathbf{x} - x_i) \vDash \exists x_i B$.

Kripke metaframes-3

Here $(\mathbf{a} - a_i)$ is obtained by eliminating a_i from $\mathbf{a}$; similarly for $(\mathbf{x} - x_i)$.

We can also present $(\mathbf{a} - a_i)$ as $\mathbf{a} \cdot \delta_i^n$ for a specific transformation

$$\delta_i^n = \begin{pmatrix} 1 & \dots i-1 & i & \dots & n-1 \\ 1 & \dots i-1 & i+1 & \dots & n \end{pmatrix}$$

Definition

A formula A is valid on $\mathbf{F}$ ($\mathbf{F} \vDash A$) if $\overline{\forall} A$ is true at every world in every model over $\mathbf{F}$.

A formula A is strongly valid on $\mathbf{F}$ ($\mathbf{F} \vDash^+ A$) if all its substitution instances are valid.

Kripke metaframes-4

Definition

The n-shift A^n of a formula A is obtained by replacing every atomic subformula $P_i^k(\mathbf{y})$ with $P_i^{k+n}(\mathbf{y}, \mathbf{x})$, where $\mathbf{x}$ is a fixed list of n new variables.

Soundness theorem (D.P. Skvortsov, 1993; cf. [SS93], [QNL])

1. $\mathbf{F} \models^+ A$ iff $\forall n\, \mathbf{F} \models A^n$
2. $\mathbf{L}(\mathbf{F}) := \{A \mid \mathbf{F} \models^+ A\}$ is a modal predicate logic.

As in Kripke semantics we can define *logical consequence* for metaframes:

$$L \models_{\mathcal{MF}} A := \forall \mathbf{F}\, (\mathbf{F} \models^+ L \Rightarrow \mathbf{F} \models^+ A).$$

Kripke metaframes-5

Ghilardi's frames correspond to metaframes (with the same logic):

$$\mathbf{a}(R_n)_{uv}\mathbf{b} \;\text{ iff }\; \exists\gamma \in \mathcal{C}(u,v)\,\gamma\cdot\mathbf{a} = \mathbf{b},$$

where $\gamma\cdot(a_1,\ldots,a_n) = (\gamma(a_1),\ldots,\gamma(a_n))$.

Kripke bundles correspond to metaframes:

$$\mathbf{a}(R_n)_{uv}\mathbf{b} \;\text{ iff }\; \forall i\, a_i\rho_{uv}b_i \;\&\; \forall i,j(a_i = a_j \Rightarrow b_i = b_j).$$

Kripke bundles also correspond to Ghilardi's frames.

More incompleteness results

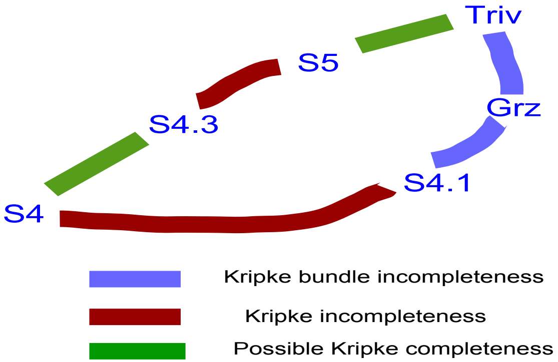
$$\mathbf{S4.1} := \mathbf{S4} + \Box\Diamond p \to \Diamond\Box p.$$

Theorem 23 [Isoda, 1997]

If $\mathbf{S4.1} \subseteq \mathbf{\Lambda} \subset \mathbf{Triv}$, then $\mathbf{Q\Lambda}$ is Kripke bundle incomplete.

Conjecture: **QS4.1** is metaframe incomplete. However, it is complete in simplicial semantics (discussed later), by Theorem 24, since **S4.1** is d-persistent.

More incompleteness results



Simplicial frames-1

Let Σ_{mn} be the set of all transformations $I_m \longrightarrow I_n$. Now we also consider $I_0 = \varnothing$. So $\Sigma_{0n} = \{\varnothing_n\}$, where $\varnothing_n$ is empty; and we can ignore $\Sigma_{m0} = \varnothing$ for $m > 0$. Thus we obtain a full subcategory $\mathbf{\Sigma} \subset \mathbf{SET}$ with objects $\{I_n \mid n \geq 0\}$.

Definition

A simplicial frame over a propositional Kripke frame $F_0 = (W, R)$ is $\mathbf{F} = ((F_n)_{n \geq 0}, \pi)$, where

- each $F_n = (D^n, R^n)$ is a propositional frame, $D^0 = W$, $R^0 = R$,
- $\pi = (\pi_\sigma)_{\sigma \in \Sigma}$ is a family of maps $\pi_\sigma : D^n \longrightarrow D^m$ for $\sigma \in \Sigma_{mn}$ (jections, co-transformations),
- $\pi_{\varnothing_n} : D^n \longrightarrow D^0$ is a surjection for $n > 0$, $\pi_{\varnothing_0} = id_{D^0}$.

A priori there are no special conditions on jections, but we will need them for soundness.

Simplicial frames-2

D^0 consists of possible worlds, D^1 is the set of individuals, D^n is the set of 'abstract n-tuples', or individuals of level n

The map $\pi_{\varnothing_n}$ sends every individual of level n to its world. We define the domain of level n at world u as $D^n_u := \pi^{-1}_{\varnothing_n}(u)$ and get the partition $D^n = \bigcup_{u \in W} D^n_u$.

Metaframes are a special kind of simplicial frames, where D^n is the set of 'real' n-tuples and $\pi_\sigma(\mathbf{a}) = \mathbf{a} \cdot \sigma$ for $\mathbf{a} \in D^n$, $\sigma \in \Sigma_{mn}$, $m \neq 0$.

Definition

A valuation on $\mathbf{F}$ is a a function ξ sending every predicate letter P^n_k to a subset $\xi(P^n_k) \subseteq D^n$. A simplicial model is $(\mathbf{F}, \xi)$.

A variable assignment on $\mathbf{F}$ at $u \in W$ is $\mathbf{a}/\mathbf{x}$, where (for some n) $\mathbf{x}$ is a list of n different variables, $\mathbf{a} \in D^n_u$.

Simplicial frames-3

As in metaframe semantics, we define the truth of a formula A at world u in a model M under a variable assignment $\mathbf{a}/\mathbf{x}$, where $\mathbf{x}$ contains all parameters of A

Definition

- $M, u, \mathbf{a}/\mathbf{x} \models P(\mathbf{x} \cdot \sigma)$ iff $\pi_\sigma \mathbf{a} \in \xi_u(P)$
- $M, u, \mathbf{a}/\mathbf{x} \models \Box A$ iff $\forall v \in R(u)\, \forall \mathbf{b} \in D_v^n\, (\mathbf{a} R^n \mathbf{b} \Rightarrow M, v, \mathbf{b}/\mathbf{x} \models A)$
- $M, u, \mathbf{a}/\mathbf{x} \models \exists y A$ iff $\exists \mathbf{b} \in D_u^{n+1} (\pi_{\delta_{n+1}^{n+1}} \mathbf{b} = \mathbf{a}\ \&\ M, u, \mathbf{b}/\mathbf{x}y \models A)$, for $y \notin \mathbf{x}$;
- $M, u, \mathbf{a}/\mathbf{x} \models \exists x_i B$ iff $M, u, \pi_{\delta_i^n} \mathbf{a}/(\mathbf{x} - x_i) \models \exists x_i B$.

Other cases are clear.

Simplicial frames-4

Definition

A formula A is valid on $\mathbf{F}$ ($\mathbf{F} \models A$) if $\overline{\forall} A$ is true at every world in every model over $\mathbf{F}$.
A formula A is strongly valid on $\mathbf{F}$ ($\mathbf{F} \models^{+} A$) if all its substitution instances are valid.

Proposition (D.P. Skvortsov, 1990; cf. [SS93])

1. $\mathbf{F} \models^{+} A$ iff $\forall n\, \mathbf{F} \models A^{n}$
2. For a propositional A, $\mathbf{F} \models A^{n}$ iff $F_n \models A$. Thus

$$\mathbf{F} \models^{+} A \text{ iff } \forall n\, F_n \models A.$$

Simplicial frames-5

Soundness theorem (D.P. Skvortsov, 1990; cf. [SS93])

Let $\mathbf{F} = ((F_n)_{n\geq 0}, \pi)$ be a simplicial frame such that

1. every π_σ for $\sigma \in \Sigma_{mn}$ is a morphism from F_n to F_m (perhaps, not surjective);
2. $\pi_{\sigma\cdot\tau} = \pi_\tau \cdot \pi_\sigma$, $\pi_{id_{I_n}} = id_{D^n}$;
3. if $\pi_{\delta^{m+1}_{m+1}}(\mathbf{b}) = \pi_\sigma(\mathbf{a})$, $\sigma \in \Sigma_{mn}$, then there exists $\mathbf{c} \in D^{n+1}$ such that $\pi_{\sigma^+}(\mathbf{c}) = \mathbf{b}$, $\pi_{\delta^{n+1}_{n+1}}(\mathbf{c}) = \mathbf{a}$.

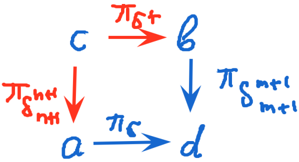
(Such a simplicial frame is called sound). Then $\mathbf{L}(\mathbf{F}) := \{A \mid \mathbf{F} \vDash^+ A\}$ is a modal predicate logic.

Here

$$\sigma^+ := \begin{pmatrix} 1 & \dots & m & m+1 \\ \sigma(1) & \dots & \sigma(m) & n+1 \end{pmatrix}$$

Simplicial frames-6

Soundness conditions 1, 2 mean that π is a contravariant functor from $\mathbf{\Sigma}$ to category of propositional frames and morphisms. The condition 3 is the 'Beck – Chevalley' property:



Simplicial frames-7

Lemma

Every metaframe is a sound simplicial frame.

The condition 2 follows from the equality $(\mathbf{a} \cdot \sigma) \cdot \tau = \mathbf{a} \cdot (\sigma \cdot \tau)$.

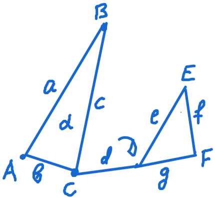
Beck – Chevalley is easily checked: given

$$\mathbf{a} = a_1 \dots a_m,\ \mathbf{d} = a_{\sigma(1)} \dots a_{\sigma(m)},\ \mathbf{b} = \mathbf{d}b_{m+1}$$

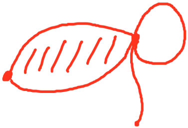
we can construct

$$\mathbf{c} = \mathbf{a}b_{m+1}.$$

Simplicial frames are based on a version of simplicial sets. Simplicial sets appeared in algebraic topology as a convenient generalization of simplicial complexes. Informally:



A simplicial complex



A simplicial set

d-persistence-1

Definition

A general propositional Kripke frame is $\mathcal{F} = (W, R, \mathcal{A})$, where $\mathcal{A}$ is a subalgebra of the modal algebra $MA(W, R)$ (the Boolean algebra 2^W with the operation $X \mapsto R^{-1}X$).
A modal formula B is *valid* on $\mathcal{F}$ if $(W, R, \theta) \vDash B$ whenever a valuation θ takes values in $\mathcal{A}$.

Definition

A general frame $\mathcal{F} = (W, R, \mathcal{A})$ is descriptive if

- $\forall u \forall v \neq u \, \exists V \in \mathcal{A} \, (u \in V \ \& \ v \notin V)$,
- $\forall u, v \, (\forall V \in \mathcal{A} \, (v \in V \Rightarrow u \in R^{-1}V) \Rightarrow uRv)$,
- $\mathcal{A}$ has the finite intersection property.

d-persistence-2

Definition

A propositional modal logic $\mathbf{\Lambda}$ is d-persistent if for any descriptive frame $(W, R, \mathcal{A})$, $(W, R, \mathcal{A}) \vDash \mathbf{\Lambda}$ implies $(W, R) \vDash \mathbf{\Lambda}$.

Theorem [Sambin, Vaccaro, 1989]

Every propositional logic axiomatized by Sahlqvist formulas is d-persistent.

A completeness theorem for simplicial semantics

Theorem 24 [SS 1993]

If $\mathbf{\Lambda}$ is a d-persistent propositional logic, then the predicate logics $\mathbf{Q\Lambda}$ and $\mathbf{Q\Lambda} + Ba$ are complete in simplicial semantics.

The proof is by a canonical model construction: its points of level n are n-types in the model-theoretic sense.

Theorem [Ghilardi 1992]

If $\mathbf{\Lambda}$ is a d-persistent intermediate propositional logic, then the predicate logic $\mathbf{Q\Lambda}$ is complete in Ghilardi's semantics.

Conjecture: Ghilardi's theorem does not transfer to modal logics, $\mathbf{QS4.1}$ may be a counterexample.

Case study: on axiomatization of $\Box \bullet \mathbf{QAlt_1}$

Consider the propositional logics

$$\mathbf{Alt_1} := \mathbf{K} + alt_1, \text{ where } alt_1 := \Diamond p \to \Box p,$$

$$\mathbf{SL} := \mathbf{K} + \Diamond p \leftrightarrow \Box p.$$

It well known that $\mathbf{Alt_1}$ is complete w.r.t. functional frames, $\mathbf{SL} = \mathbf{L}(\mathrm{N}, S)$, where $xSy \Leftrightarrow y = x + 1$.

Theorem 25

$$\mathbf{Q}(\Box \cdot \mathbf{Alt}_1) + \Box\overline{\forall} alt_1^1 \models_{\mathcal{MF}} \Box\overline{\forall} alt_1^2$$

Therefore we cannot prove non-finite axiomatizability of $\Box \bullet \mathbf{QAlt_1}$ by applying metaframes.

Conjecture (very likely).

$$\mathbf{Q}(\Box \cdot \mathbf{SL}) + \Box\overline{\forall} alt_1^1 \nvdash \Box\overline{\forall} alt_1^2.$$

Conclusion

We are still at the beginning of the research, there are too many open questions about particular systems.

It is not clear how to construct a satisfactory semantics for incompletable logics like **QGL**.

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THANK YOU!