Complete interpolating sequences in Fock spaces and their applications to Gabor frames generated by a hyperbolic secant type function

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Spaces with a reproducing kernel

Let $\mathcal{H}$ be a reproducing kernel Hilbert space whose elements are functions analytic in some domain Ω. This means that for any point $w \in \Omega$ the functional $f \mapsto f(w)$ is continuous and, hence, there exists $k_w \in \mathcal{H}$ (reproducing kernel) such that

$$f(w) = (f, k_w).$$

Examples.

1. Hardy space $H^2 = H^2(\mathbb{D})$, $k_w(z) = \frac{1}{1-\bar{w}z}$ – the Cauchy kernel

2. Bergman space $A^2 = A^2(\mathbb{D})$, $k_w(z) = \frac{1}{(1-\bar{w}z)^2}$.

3. Paley–Wiener space

$$PW_a = \{F \in H(\mathbb{C}) : \text{эксп. тип} \le a, F \in L^2(\mathbb{R})\} = \{\hat{f} : \ f \in L^2(-a,a)\},$$

$k_w(z) = \frac{\sin a(z-\bar{w})}{z-\bar{w}}$ – sinc-функции.

4. Bargmann–Segal–Fock space $\mathcal{F}_\varphi$ with $\varphi(r) = ar^2$, $k_w(z) = e^{a\bar{w}z}$.

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Sampling and interpolation

A sequence $\{\lambda_n\} \subset \Omega$ is said to be a sampling sequence for $\mathcal{H}$ if there exist $A, B > 0$:

$$A\|f\|^2 \leq \sum_n |f(\lambda_n)|^2 \|k_{\lambda_n}\|^{-2} \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

A sequence $\{\lambda_n\}$ is said to be interpolating for $\mathcal{H}$ if for any $\{c_n\} \in \ell^2$ there exists $f \in \mathcal{H}$ such that $f(\lambda_n) = \|k_{\lambda_n}\| c_n$.

If such function f is unique we say that $\{\lambda_n\}$ is a complete interpolating sequence. Equivalent definition: complete IS =IS + sampling.

- $\{\lambda_n\}$ is a sampling sequence $\leftrightsquigarrow$ $\{k_{\lambda_n}/\|k_{\lambda_n}\|\}$ is a frame for $\mathcal{H}$ ($\{f_n\}$ is a frame if $A\|f\|^2 \leq \sum_n |(f, f_n)|^2 \leq B\|f\|^2$, $f \in \mathcal{H}$);
- $\{\lambda_n\}$ is an IS $\leftrightsquigarrow$ $\{k_{\lambda_n}/\|k_{\lambda_n}\|\}$ is a Riesz sequence;
- $\{\lambda_n\}$ is a complete IS $\leftrightsquigarrow$ $\{k_{\lambda_n}/\|k_{\lambda_n}\|\}$ is a Riesz basis in $\mathcal{H}$.

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Fock type spaces

Let $\varphi : [0,\infty) \to [0,\infty)$, $\log r = o(\varphi(r))$, $r \to \infty$.

$$\mathcal{F}_\varphi = \left\{ F \in H(\mathbb{C}) : \|F\|^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-2\varphi(|z|)} dm_2(z) < \infty \right\}.$$

$\mathcal{F}_\varphi$ is a reproducing kernel space, $\{z^n\}_{n\geq 0}$ is an orthogonal basis in $\mathcal{F}_\varphi$.

The classical Bargmann–Segal–Fock space: $\varphi(r) = \alpha r^2$, $k_w(z) = e^{2\alpha \bar{w} z}$.

Questions:

1. Do there exist complete interpolating sequences for $\mathcal{F}_\varphi$ (i.e, Riesz bases of normalized reproducing kernels)?

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Existence of Riesz bases of reproducing kernels

- K. Seip, R. Wallsten, Yu.I. Lyubarskii, 1992: sampling and interpolation in the classical Bargmann–Fock space. No complete IS.

  A separated set $\Lambda \subset \mathbb{C}$ (i.e., $\inf_{\lambda \neq \lambda' |\lambda - \lambda'| > 0}$) is a sampling set (for $\varphi(r) = \pi r^2$) iff

  $$D_2^-(\Lambda) = \liminf_{r \to \infty} \frac{\inf_{z \in \mathbb{C}} \operatorname{card}(\Lambda \cap B_r(z))}{\pi r^2} > 1.$$

- N. Marco, X. Massaneda, J. Ortega-Cerda, 2003: φ is subharmonic, $\Delta\varphi$ is a doubling measure.

- A. Borichev, R. Dhuez, K. Kellay, 2007: fast growing weights.

Existence of Riesz bases of reproducing kernels

- A. Borichev, Yu. Lyubarskii, 2010: first examples of Fock type spaces which have complete IS. Sharp “bound”.

Theorem. If $\varphi(r) = \alpha \log^\beta r$, $1 < \beta \le 2$, then in $\mathcal{F}_\varphi$ there exist complete IS (= Riesz bases of reproducing kernels).

$$\rho(z) = (\Delta\varphi(z))^{-1/2} = \left(\varphi''(r) + \frac{\varphi'(r)}{r}\right)^{-1/2}.$$

$\rho(z) = const$ for $\varphi(r) = \alpha r^2$ and $\rho(z) = const \cdot r$ for $\varphi(r) = \alpha \log^2 r$.

Theorem. If $\rho(r) = o(r)$ + some regularity then there are no complete IS in $\mathcal{F}_\varphi$.

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Existense of Riesz bases of reproducing kernels

A.B., Yu. Belov, A. Borichev, 2017: Let $\psi(t) = \varphi(e^t)$ ($\psi(t) = \alpha t^2$ for $\varphi(r) = \alpha(\log r)^2$). If $\psi'' \nearrow \infty$, then there are no complete IS. If ψ'' is nonincreasing + some regularity, then complete IS exist.

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Let $\varphi(r) = \frac{1}{2}(\log r)^2$,

$$\mathcal{F} = \left\{ F \text{ entire} : \quad \|F\|^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\log^2 |z|} dm_2(z) < \infty \right\}.$$

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Description of complete IS in the small Fock space

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A. Borichev, Yu. Lyubarskii: $\{e^n\}_{n\geq 1}$ is a complete IS for $\mathcal{F}$.

Let $\{\lambda_n\}_{n\geq 1}$ be given, $0 < |\lambda_n| \leq |\lambda_{n+1}|$ and

$$\lambda_n = e^n e^{\delta_n} e^{i\theta_n}, \qquad \delta_n \in \mathbb{R}.$$

Theorem (A.B., A. Dumont, A. Hartmann, K. Kellay). $\{\lambda_n\}_{n\geq 1}$ is a complete interpolating sequence for $\mathcal{F}$ iff

(i) $\{\lambda_n\}$ is $\mathcal{F}$-separated that is there exists $\gamma > 0$ such that $|\lambda_m - \lambda_n| \geq \gamma |\lambda_n|$, $m \neq n$;

(ii) $(\delta_n)_{n\geq 1} \in \ell^\infty$;

(iii) there exists $N \geq 1$ such that

$$\sup_{n\geq 1} \frac{1}{N} \Big| \sum_{k=n+1}^{n+N} \delta_k \Big| < \frac{1}{2}.$$

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Exponential bases and Avdonin's condition

Problem (R. Paley, N. Wiener). Describe exponential Riesz bases $e^{i\lambda_n x}$ in $L^2(-\pi, \pi)$.

This is equivalent to a description of the Riesz bases of sinc-functions or complete interpolating sequences in PW_π.

Kadets' 1/4 Theorem, 1965: if $\sup_{n\in\mathbb{Z}} |\lambda_n - n| < 1/4$, then $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(-\pi, \pi)$.

S.A. Avdonin, 1974: let $\lambda_n = n + \delta_n$, $n \in \mathbb{Z}$, be a separated sequence in $\mathbb{R}$, $(\delta_n) \in \ell^\infty$. If there exists N such that

$$\sup_{n\in\mathbb{Z}} \frac{1}{N} \Big| \sum_{k=n+1}^{n+N} \delta_k \Big| < \frac{1}{4},$$

then $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(-\pi, \pi)$.

B.S. Pavlov (1979), S.V. Khruschev, N.K. Nikolski, B.S. Pavlov, 1981: a complete description in terms of the Muchenhoupt (Helson–Szegö) condition.

Time-frequency analysis and Gabor systems

Let $g \in L^2(\mathbb{R})$. The short-time Fourier transform of a function $f \in L^2(\mathbb{R})$ with window g is the transform

$$F(x,y) = \int_{\mathbb{R}} f(t)g(t-x)e^{-2\pi iyt}dt.$$

The functions $M_y T_x g = e^{-2\pi iyt}g(t-x)$ are called time-frequency shifts of a function g. The window function g is assumed to be sufficiently well-localized (e.g., $g(t) = \chi_{[0,1]}(t)$ or $g(t) = e^{-at^2}$).

In applications one needs to work with discretizations of the short-time Fourier transform, therefore, one considers Gabor systems

$$\mathcal{G}(g, \Lambda \times M) = \{M_y T_x g : \ (x,y) \in \Lambda \times M\}.$$

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In 1946, D. Gabor considered the system $\mathcal{G}(g, \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\})$ for $g(t) = e^{-\pi t^2}$ and conjectured that any function $f \in L^2(\mathbb{R})$ can be represented uniquely as

$$f = \sum_{(x,y)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} c_{x,y} \cdot M_y T_x g$$

with ℓ^2-control of the coefficients $c_{x,y}$. However, a Gabor system corresponding to a Gaussian is never a Riesz basis in $L^2(\mathbb{R})$. But, it can be a frame.

A typical situation: $\alpha\mathbb{Z} \times \beta\mathbb{Z}$, $\alpha\beta < 1$. There exists a heuristic principle: for a "good" (i.e., smooth with a fast decay) function g a Gabor system $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a frame $\iff \alpha\beta < 1$. Necessity is known (Balian–Low theorem).

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What is known about frames of the form $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$?

- $g(x) = e^{-ax^2}$, $a > 0$: $\alpha\beta < 1$ (Seip, Wallsten, Lyubarskii).
- $g(x) = e^{-a|x|}$: $\alpha\beta < 1$ (Jannsen).
- $g(x) = \chi_{(0,\infty)}(x)e^{-ax}$: $\alpha\beta \le 1$ (Jannsen).
- Jannsen, Strohmer (2002) – $g(x) = (e^{ax} + e^{-ax})^{-1}$ (hyperbolic secant): $\alpha\beta < 1$.
- Gröchenig, Stockler (2011) – totally positive functions of finite type; Gröchenig, Romero, Stockler (2018) – totally positive functions of finite Gaussian type: $\alpha\beta < 1$.

$$\hat{g}(t) = \prod_{j=1}^{n}(1+i\delta_j t)^{-1} \quad \text{or} \quad \hat{g}(t) = e^{-ct^2}\prod_{j=1}^{n}(1+i\delta_j t)^{-1},\ c > 0,\ \delta_j \in \mathbb{R}.$$

- Belov, Kulikov, Lyubarskii (2023) – Herglotz functions of finite type,

$$g(x) = \sum_{k=1}^{N} \frac{a_k}{x - iw_k}, \qquad a_k,\ w_k > 0,$$

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Irregular frames

Almost nothing is known about description of irregular frames of the form $\mathcal{G}(g, \Lambda \times M)$, where Λ, M are discrete separated sets on the line (not lattices) or even $\mathcal{G}(g, S) = \{M_y T_x g : (x, y) \in S\}$, where $S \subset \mathbb{R}^2$. This problem is solved only for the case of the Gaussian.

Theorem (Seip, Wallsten, Lyubarskii, 1992). Let $S \subset \mathbb{R}^2$ be a separated set and let $g(x) = e^{-\pi x^2}$. Then $\mathcal{G}(g, S)$ is a frame in $L^2(\mathbb{R})$ if and only if

$$D_2^-(S) = \liminf_{r \to \infty} \frac{\inf_{z \in \mathbb{C}} \operatorname{card}(S \cap B_r(z))}{\pi r^2} > 1.$$

There exists a unitary operator (Bargmann transform) from $L^2(\mathbb{R})$ onto the Bargmann–Segal–Fock space:

$$\mathcal{F} = \left\{ F \in H(\mathbb{C}) : \|F\|^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi |z|^2} dm_2(z) < \infty \right\}.$$

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Semi-regular frames

Consider the frames of the form $\mathcal{G}(g, \Lambda \times \alpha\mathbb{Z})$.

Theorem. Let $\Lambda \subset \mathbb{R}$ be a separated set (i.e., $\inf_{\lambda \neq \lambda'} |\lambda - \lambda'| > 0$), and let f be a function of one of the following types:

- totally positive functions of finite type (Gröchenig, Stockler, 2011);
- totally positive functions of finite Gaussian type (Gröchenig, Romero, Stockler, 2018);
- hyperbolic secant $(e^{ax} + e^{-ax})^{-1}$, $a > 0$ (Gröchenig, Romero, Stockler 2020).

Then the following statements are equivalent:

1. $\mathcal{G}(g, \Lambda \times \alpha\mathbb{Z})$ is a frame in $L^2(\mathbb{R})$;
2. $\mathcal{G}(g, \alpha\mathbb{Z} \times \Lambda)$ is a frame in $L^2(\mathbb{R})$;
3. $D^-(\Lambda) > \alpha$, where

$$D^-(\Lambda) = \lim_{r \to \infty} \frac{\inf_{x \in \mathbb{R}} \operatorname{card}(\Lambda \cap [x, x+r])}{r}.$$

Shift-invariant spaces

For $g \in L^2(\mathbb{R})$ consider the space

$$V^2(g) = \left\{ f(z) = \sum_{n\in\mathbb{Z}} c_n g(z-n) : \ (c_n) \in \ell^2(\mathbb{Z}) \right\}$$

with the standard $L^2(\mathbb{R})$ norm. We assume that the function g is "well localized": $\|f\|_{L^2(\mathbb{R})} \asymp \|(c_n)\|_{\ell^2}$, $f \in V^2(g)$.

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Shift-invariant spaces

For $g \in L^2(\mathbb{R})$ consider the space

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Sampling in $V^2(g)$

Theorem (Gröchenig, Romero, Stockler, 2018). Let $g(x) = e^{-ax^2}$, $a > 0$, or a totally positive functions of finite Gaussian type. Then

(i) each separated sequence $\Lambda \subset \mathbb{R}$ such that

$$D^-(\Lambda) = \lim_{r\to\infty} \frac{\inf_{x\in\mathbb{R}} \operatorname{card}\left(\Lambda \cap [x, x+r]\right)}{r} > 1$$

is a sampling sequence for $V^2(g)$;

(ii) if Λ is a sampling sequence for $V^2(g)$, then Λ is a finite union of separated sequences and contains a separated sequence $\widetilde{\Lambda}$ such that $D^-(\widetilde{\Lambda}) \geq 1$.

Interpolation in $V^2(g)$

A sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is said to be interpolating for $V^2(g)$ if for any sequence $(a_n) \in \ell^2(\mathbb{Z})$ there exists a function $f \in V^2(g)$ such that $f(\lambda_n) = a_n$.

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is interpolating for V^2;

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Description of complete IS for the Gaussian

We say that a sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is a complete interpolating sequence for $V^2(g)$, if for any $(a_n) \in \ell^2(\mathbb{Z})$ there exists $f \in V^2(g)$ such that $f(\lambda_n) = a_n$ and such f is unique. equivalently, a complete IS = IS + sampling.

Theorem (A.B., Yu. Belov, K. Gröchenig, 2022). Let $g(x) = e^{-ax^2}$, $a > 0$. An increasing sequence $\Lambda \subset \mathbb{R}$ is a complete IS for $V^2(g)$ iff Λ is separated and there exists its enumeration $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$, $\lambda_n = n + \delta_n$, $n \in \mathbb{Z}$, such that

(a) $(\delta_n)_{n\in\mathbb{Z}} \in \ell^\infty$;

(b) there exists $N \geq 1$ such that

$$\sup_{n\in\mathbb{Z}} \frac{1}{N}\Big|\sum_{k=n+1}^{n+N} \delta_k\Big| < \frac{1}{2}.$$

Note that $\mathbb{Z} + \delta$, $0 < \delta < 1$ are complete IS for $V^2(g)$ except $\delta = 1/2$.

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Results about the sampling and interpolation follows from our theorem. Morever:

- each separated sequence Λ with $D^-(\Lambda) > 1$ contains a complete IS;
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Relation with the small Fock space:

$$\sum_{n\in\mathbb{Z}} c_n e^{-a(z-n)^2} = e^{-az^2} \sum_{n\in\mathbb{Z}} c_n e^{-an^2} e^{2anz}.$$

Put $e^{2az} = w$, then $V^2(g)$, $g(x) = e^{-ax^2}$, is unitarily equivalent to a space of analytic functions with singularities at 0 and at infinity, which is essentially a direct sum of two copies of the space $\mathcal{F}$ with the weight $\exp(-b\log^2|z|)$.

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Secant type functions

Let $g(x) = (e^{ax} + e^{-bx})^{-1}$, $a, b \in \mathbb{C}$, $\mathrm{Re}\, a, \mathrm{Re}\, b > 0$.

Theorem (A.B., Yu. Belov, 2024).

- An increasing sequence $\Lambda \subset \mathbb{R}$ is a complete IS for $V^2(g)$ iff Λ is separated and there exists its enumeration $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$, $\lambda_n = n + \delta_n$, $n \in \mathbb{Z}$, such that
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Semi-regular frames for a secant type function

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