Soft Riemann-Hilbert problems and planar orthogonal polynomials

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Bergman kernel asymptotics and microlocal analysis

The Bergman (Hilbert) space $\mathcal{H} = A^2(\Omega, w)$ consists of the holomorphic functions $f : \Omega \to \mathbb{C}$ with

$$\|f\|_{\mathcal{H}}^2 = \int_{\mathbb{C}} |f|^2 w \mathrm{dA} < +\infty$$

where dA denotes (normalized) are measure. Here, $\Omega \subset \mathbb{C}$ and $w : \Omega \to \mathbb{R}_+$ is a continuous weight function. We could consider later on Ω as a subdomain of a Riemann surface, or a higher-dimensional complex manifold. The Bergman kernel associated with $\mathcal{H}$ a domain Ω and a weight $\omega > 0$ is the function $\mathrm{k}_{\mathcal{H}}(\cdot, \cdot)$ with

$$\forall f \in \mathcal{H} : \quad f(z) = \langle f, \mathrm{k}_{\mathcal{H}}(\cdot, z) \rangle_{\mathcal{H}}, \qquad z \in \Omega.$$

The study of Bergman kernels has a nontrivial intersection with microlocal analysis (Boutet de Monvel, Sjöstrand, Berman-Berndtsson-Sjöstrand et al). Here, we consider a nonlocal instance when the analysis takes place along a *loop* in place of a point.

Commentary on the microlocal methods

In the work of Berman-Berndtsson-Sjöstrand the following operator identity is central:

$$S\nabla = 2mM_{z-w}S, \tag{1}$$

where

$$\nabla = \partial_\theta + 2mM_{z-w}, \quad S = \exp\Big(\frac{1}{2m}\partial_w\partial_\theta\Big).$$

Here, w and θ are the variables, and we may fix wlog that $z := 0$. The identity (1) permits us to test for the *negligible amplitudes* $a = \nabla A$ by applying the diffusion operator S. In this setting, m is a positive parameter which tends to infinity, M_{z-w} is multiplication by $z - w$, and (w, θ) is a deformation of $(w, \bar{w})$ as holomorphic coordinates. This works well in the local setting of a point $z = 0$, but to handle the nonlocal setting of a loop we skip the search for S and look directly for the potential A for a given amplitude a. This search fits in with the matrix $\bar{\partial}$-problem of Its and Takhtajan [6], which explicitly asks for a potential.

The setting: the confining potential Q

Growth requirement

We require that $Q : \mathbb{C} \to \mathbb{R}$ is C^2-smooth, and that

$$Q(z) \geq (1 + \varepsilon_0) \log(1 + |z|) + \mathrm{O}(1)$$

holds in the complex plane $\mathbb{C}$, for some $\varepsilon_0 > 0$.

Subharmonic function classes

For $\tau > 0$, we let $\mathrm{Subh}_\tau(\mathbb{C})$ stand for the collection of subharmonic functions $u : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ with growth controlled by

$$u(z) \leq \tau \log(1 + |z|) + \mathrm{O}(1)$$

at infinity.

Obstacle problem

For $0 < \tau < 1 + \varepsilon_0$, let $\check{Q}_\tau$ be given pointwise by

$$\check{Q}_\tau(z) := \sup\left\{ q(z) : \ q \in \mathrm{Subh}_\tau(\mathbb{C}), \ q \leq Q \ \text{ on } \ \mathbb{C} \right\}.$$

Properties of the solution to the obstacle problem

Growth

Trivially, $\check{Q}_\tau \leq Q$ everywhere. Also,

$$\check{Q}_\tau = \tau \log(|z| + 1) + \mathrm{O}(1)$$

at infinity.

Smoothness

The function $\check{Q}_\tau$ is $C^{1,1}$-smooth on $\mathbb{C}$, and harmonic in $\mathbb{C} \setminus \mathcal{S}_\tau$, where

$$\mathcal{S}_\tau := \{z \in \mathbb{C} : \ \check{Q}_\tau(z) = Q(z)\}$$

is the *contact set*, also called the *droplet*. The droplet is a compact subset of $\mathbb{C}$.

Our setting

We will concentrate on the parameter value $\tau = 1$.

Our assumptions

The set $\mathcal{S}_1$ is diffeomorphic to the closed disk $\bar{\mathbb{D}}$, where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Moreover, the boundary $\Gamma_1 := \partial\mathcal{S}_1$ is a closed real-analytically smooth loop. In a neighborhood of Γ_1, Q is real-analytically smooth, and $\Delta Q > 0$ holds on the same neighborhood.

Remark

It is good to put these assumptions in perspective. If, e.g., we were to assume that Q is real-analytically smooth on $\mathbb{C}$, with $\Delta Q > 0$ everywhere, it would not follow from this that $\mathcal{S}_1$ is simply connected with real-analytically smooth boundary loop. Instead, a theorem of Sakai (Acta Math 1991) guarantees that under those conditions, the boundary consists of isolated points and pieces of real-analytic arcs which may meet at classified singularities (cusps or kissing points).

Exponentially varying weights

Exponentially varying weights

We consider the weights $w_{mQ} := \mathrm{e}^{-2mQ}$ and the weighted spaces $L^2_{mQ} := L^2(\mathbb{C}, w_{mQ}\mathrm{dA})$.

Then by the growth assumption on Q,

$$\int_{\mathbb{C}} (1+|z|)^{2n} \mathrm{e}^{-2mQ(z)} \mathrm{dA}(z) \leq C \int_{\mathbb{C}} (1+|z|)^{2n-2(1+\varepsilon_0)m} \mathrm{dA}(z),$$

which gives that if p is a polynomial of degree n, then $p \in L^2_{mQ}$ provided that $n < (1+\varepsilon_0)m - 1$. We recall that a polynomial is said to be *monic* if its leading coefficient equals 1.

Orthogonal polynomials in L^2_{mQ}

We let $P_0, P_1, P_2, \ldots$ denote the monic orthogonal polynomials in L^2_{mQ}. This may be a finite sequence.

The problem

Asymptotic analysis of $P = P_m$

We focus on degree $n = m$, and ask for an asymptotic formula for $P = P_m$ as $m \to +\infty$.

We notice that $m < (1 + \varepsilon_0)m - 1$ holds if and only if $\varepsilon_0 m > 1$, which is the case for big enough m.

The soft Riemann-Hilbert problem

The soft Riemann-Hilbert problem

$Y = Y(z)$ is the 1×2 matrix

$$Y := \big(P\,, \quad \Phi\big)$$

and W_{mQ} is the soft jump matrix

$$W_{mQ} := \begin{pmatrix} 0 & \mathrm{e}^{-2mQ} \\ 0 & 0 \end{pmatrix}.$$

The soft Riemann-Hilbert problem is the equation

$$\bar{\partial} Y = \bar{Y} W_{mQ} \tag{2}$$

coupled with the asymptotics at infinity

$$Y = \big(z^n + \mathrm{O}(|z|^{n-1})\,, \quad \mathrm{O}(|z|^{-n-1})\big)\,. \tag{3}$$

Analysis of the soft Riemann-Hilbert problem

We calculate

$$\bar{Y} W_{mQ} = \left(\bar{P}, \quad \bar{\Phi}\right) \begin{pmatrix} 0 & \mathrm{e}^{-2mQ} \\ 0 & 0 \end{pmatrix} = \left(0\,, \quad \bar{P}\,\mathrm{e}^{-2mQ}\right).$$

On the other hand,

$$\bar{\partial} Y = \left(\bar{\partial} P\,, \quad \bar{\partial}\Phi\right),$$

so the relation (2) amounts to

$$\bar{\partial} P = 0, \quad \bar{\partial}\Phi = \bar{P}\,\mathrm{e}^{-2mQ}.$$

In particular, P is entire. Actually, from (3) we see that P is a *monic polynomial of degree* n. The second condition in (3) asserts that

$$\Phi(z) = \mathrm{O}(|z|^{-n-1}) \quad \text{as } |z| \to +\infty.$$

We shall see that this condition expresses that P is orthogonal to all polynomials of lower degree than n in the space L^2_{mQ}.

Analysis of the soft Riemann-Hilbert problem, II

We know that P is a monic polynomial of degree n, and that

$$\bar{\partial}\Phi = \bar{P}\,\mathrm{e}^{-2mQ}.$$

This equation is solved by convolution with the fundamental solution for the $\bar{\partial}$ operator:

$$\Phi(z) = \int_{\mathbb{C}} \frac{\bar{P}(\xi)\,\mathrm{e}^{-2mQ(\xi)}}{z-\xi}\mathrm{dA}(\xi).$$

Finite geometric series expansion gives that

$$\frac{1}{z-\xi} = \frac{1}{z} + \frac{\xi}{z^2} + \cdots + \frac{\xi^n}{z^{n+1}} + \frac{\xi^{n+1}}{z^{n+1}(z-\xi)},$$

so that

$$\Phi(z) = \int_{\mathbb{C}} \frac{\bar{P}(\xi)\,\mathrm{e}^{-2mQ(\xi)}}{z-\xi}\mathrm{dA}(\xi) = \sum_{j=0}^{n} z^{-j-1}\int_{\mathbb{C}} \xi^j\,\bar{P}(\xi)\,\mathrm{e}^{-2mQ(\xi)}\mathrm{dA}(\xi) + z^{-n-1}\int_{\mathbb{C}} \frac{\xi^{n+1}\,\bar{P}(\xi)\,\mathrm{e}^{-2mQ(\xi)}}{z-\xi}\mathrm{dA}(\xi).$$

Analysis of the soft Riemann-Hilbert problem, III

Proposition

If $n \leq (1+\varepsilon_0)m - 2$, then

$$\Phi(z) = \mathrm{O}(|z|^{-n-1}) \text{ at } \infty \Longleftrightarrow \forall j = 0, \ldots, n-1: \ \langle z^j, P \rangle_{mQ} = 0.$$

In other words, under the decay condition on Φ, P is the monic degree n orthogonal polynomial in L^2_{mQ}.

Ad-hoc ansatz for P

Let $\mathcal{Q}$ be the bounded holomorphic function in $\mathbb{C} \setminus \mathcal{S}_1$ with $\mathrm{Re}\,\mathcal{Q} = Q$ on Γ_1 and $\mathrm{Im}\,\mathcal{Q}(\infty) = 0$. It extends analytically across Γ_1. Also, let $\phi : \mathbb{C} \setminus \mathcal{S}_1 \to \mathbb{D}_\mathrm{e}$ be the conformal mapping that preserves infinity with $\phi'(\infty) > 0$.

The ansatz for P

We fix $n = m$, and put

$$P = c_m \phi^m \mathrm{e}^{m\mathcal{Q}} F,$$

where $c_m := (\phi'(\infty)^{-m-1}\mathrm{e}^{-m\mathcal{Q}(\infty)} > 0$. We normalize $F(\infty) = \phi'(\infty)$.

Remark

The functions on the right-hand side are not well-defined in the entire plane, while P is. The expressions on the right-hand side are all holomorphic in $\mathbb{C} \setminus \mathcal{S}_1$ and extend across Γ_1.

The ansatz for Φ

The function $\check{Q}_1$ is harmonic in $\mathbb{C} \setminus \mathcal{S}_1$. In our setting the restriction to $\mathbb{C} \setminus \mathcal{S}_1$ possesses a harmonic extension across Γ_1, which we call $\check{Q}_1$. Then

$$\check{Q}_1 = \mathrm{Re}\, \mathcal{Q} + \log |\phi|.$$

We put $R := Q - \check{Q}_1$, which has quadratic growth around Γ_1. Let

$$\mathrm{erf}\,(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2} \mathrm{d}t$$

be the standard Gaussian error function.

The ansatz for Φ

We put

$$\Phi = c_m\, m^{-\frac{1}{2}} \phi^{-m} \mathrm{e}^{-m\mathcal{Q}} \big\{ A\, \mathrm{erf}\,(2m^{\frac{1}{2}} V) + (2\pi m)^{-\frac{1}{2}} B \chi_1\, \mathrm{e}^{-2mR} \big\},$$

where $V^2 = R$ near Γ_1. V is positive in the exterior to Γ_1, and negative in the interior. Moreover, A is holomorphic in $\mathbb{C} \setminus \mathcal{S}_1$ and across Γ_1.

Asymptotic expansions

The functions F, A, B are supposed to have asymptotic expansions in m^{-1}:

$$F = F_0 + m^{-1}F_1 + m^{-2}F_2 + \ldots,$$

where each F_j is fixed independently of m. The same for A and B of course. No convergence is assumed, however. What is meant is that for each positive integer N,

$$F = F_0 + m^{-1}F_1 + \ldots + m^{-N}F_N + \mathrm{O}(m^{-N-1}).$$

Here, each F_j is holomorphic and bounded in $\mathbb{C} \setminus \mathcal{S}_1$ and across Γ_1. Likewise, in the expansion

$$A = A_0 + m^{-1}A_1 + \ldots + m^{-N}A_N + \mathrm{O}(m^{-N-1}),$$

each A_j is holomorphic and bounded in $\mathbb{C} \setminus \mathcal{S}_1$ and across Γ_1. However, the corresponding terms B_j associated with B are only smooth in a neighborhood of Γ_1.

The basic equation

The function χ_1 is a C^∞-smooth cut-off function, which equals 1 in a fixed neighborhood of Γ_1, and vanishes off a slightly bigger neighborhood. We put $B^1 = B\chi_1$, and calculate:

$$\bar{\partial}\Phi = (2/\pi)^{\frac{1}{2}} c_m \,\phi^{-m} \mathrm{e}^{-m\mathcal{Q}} \big(A\bar{\partial}V - B^1 \bar{\partial}R + \tfrac{1}{2} m^{-1} \bar{\partial}B^1 \big) \mathrm{e}^{-2mR}$$

in a neighborhood of Γ_1. On the other hand, we observe that

$$\bar{P}\, \mathrm{e}^{-2mQ} = \bar{F} \phi^{-m} \mathrm{e}^{-m\mathcal{Q}}\, \mathrm{e}^{-2mR}.$$

Equality of these two expressions reduces to the equation (since $V^2 = R$)

$$A\bar{\partial}V - 2BV\, \bar{\partial}V + \tfrac{1}{2} m^{-1} \bar{\partial}B = (\pi/2)^{\frac{1}{2}} \bar{F} \qquad (4)$$

in a small neighborhood of Γ_1, as $B^1 = B\chi_1 = B$ there.

The first equation

We restrict to Γ_1 and use that $V = 0$ there:

$$A\bar{\partial}V + \frac{1}{2}m^{-1}\bar{\partial}B = (\pi/2)^{\frac{1}{2}}\bar{F} \quad \text{on} \quad \Gamma_1. \tag{5}$$

where it is given that

$$F(z) = \phi'(\infty) + \mathrm{O}(|z|^{-1}), \quad A(z) = \mathrm{O}(|z|^{-1}).$$

Remark

The equation (5) amounts to a Riemann-Hilbert problem in an alternative coordinate chart (where the unit circle $\mathbb{T}$ replaces Γ_1).

The second equation

For the moment we observe that the first equation involves an unknown B, but luckily it is of higher order, so it does influence the first terms A_0 and F_0 which get determined right away by the Riemann-Hilbert problem. The original equation (4) contains more information than (5) alone. Suppose for the moment we were able to solve the equation (5). Then we may solve for B using (4):

$$B = \frac{A\bar{\partial}V + \frac{1}{2}m^{-1}\bar{\partial}B - (\pi/2)^{\frac{1}{2}}\bar{F}}{2V\,\bar{\partial}V}. \tag{6}$$

Since V vanishes only to degree 1 with $\bar{\partial}V \neq 0$ along Γ_1, the division produces a smooth function, and if the numerator is real-analytic across Γ_1, so is the ratio and hence B.

The iteration

The combination of the Riemann-Hilbert "jump" problem (5) and the smooth division problem (6) supplies the full algorithm. We may liken it to the Newton algorithm for finding the zeros of polynomials: once we are in the ballpark the algorithm gets us ever closer to the solution. We use (5) with $B = 0$ to get the initial A and F. Next, we apply (6) with the previous choices of A, F, and B to get an updated choice for B. This new B is then implemented into (5) to get improved A and F. Proceeding iteratively we obtain the full asymptotic expansion.

Transfer to the unit circle

We write $\varphi = \varphi_1 := \phi_1^{-1} : \mathbb{D}_\mathrm{e} \to \mathbb{C} \setminus \mathcal{S}_1$ for the indicated conformal mapping, tacitly extended across $\mathbb{T}$, and consider

$$\mathrm{R} := R \circ \varphi, \quad \mathrm{V} := V \circ \varphi, \quad \Psi := \Phi \circ \varphi,$$

and the associated functions

$$\mathrm{A} := A \circ \varphi, \quad \mathrm{B} := B \circ \varphi, \quad \mathrm{F} := \varphi' F \circ \varphi.$$

Here, A and F are holomorphic functions in a neighborhood of $\bar{\mathbb{D}}_\mathrm{e}$ with asymptotics in accordance with the *first equation*:

$$\mathrm{F}(z) = 1 + \mathrm{O}(|z|^{-1}), \quad \mathrm{A}(z) = \mathrm{O}(|z|^{-1}),$$

as $|z| \to +\infty$. In particular, F is bounded with value 1 at infinity, while A is bounded and vanishes at infinity.

The first and second equations in new coordinates

In terms of these functions, the equation (4) reads

$$\mathrm{A}\bar{\partial}\mathrm{V} - 2\mathrm{BV}\,\bar{\partial}\mathrm{V} + \tfrac{1}{2}m^{-1}\bar{\partial}\mathrm{B} = (\pi/2)^{\frac{1}{2}}\,\bar{\mathrm{F}}.$$

Just as before, we split the equation in two separate steps:

$$\mathrm{A}\bar{\partial}\mathrm{V} + \tfrac{1}{2}m^{-1}\bar{\partial}\mathrm{B} = (\pi/2)^{\frac{1}{2}}\,\bar{\mathrm{F}} \quad \text{on} \quad \mathbb{T},$$

and

$$\mathrm{B} = \frac{\mathrm{A}\bar{\partial}\hat{\mathrm{R}} + \tfrac{1}{2}m^{-1}\bar{\partial}\mathrm{B} - (\pi/2)^{\frac{1}{2}}\,\bar{\mathrm{F}}}{2\mathrm{V}\,\bar{\partial}\mathrm{V}}.$$

Remark

We refer to these as steps I and II, which we analyze below in some detail.

The solution algorithm: preliminaries

Local analysis around the circle $\mathbb{T}$ shows that

$$\bar{\partial}\mathrm{V} = 2^{-\frac{1}{2}}(\Delta\mathrm{R})^{\frac{1}{2}}\zeta \quad \text{on } \mathbb{T}. \tag{7}$$

Here and in the sequel, ζ stands for the coordinate function $\zeta(z) = z$.
Let $\mathbf{H}_{\mathbb{D}_e}$ be the *exterior Herglotz operator*

$$\mathbf{H}_{\mathbb{D}_e} f(z) := \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta} f(\zeta)\mathrm{d}s(\zeta), \qquad z \in \mathbb{D}_e,$$

where $\mathrm{d}s$ is arclength measure, normalized so that the circle $\mathbb{T}$ gets mass 1. The characterizing property is that the real part of $\mathbf{H}_{\mathbb{D}_e} f$ has boundary value equal to f. We will also need the projection operators

$$\mathbf{P}_{H^2_-} f = \frac{1}{2}\mathbf{H}_{\mathbb{D}_e} f + \frac{1}{2}\langle f\rangle_{\mathbb{T}}, \quad \mathbf{P}_{H^2_{-,0}} f = \frac{1}{2}\mathbf{H}_{\mathbb{D}_e} f - \frac{1}{2}\langle f\rangle_{\mathbb{T}},$$

where $\langle f\rangle_{\mathbb{T}}$ is the mean of f with respect to arc length measure.

The solution algorithm: preliminaries II

We introduce the function $\mathrm{H_R}$ given by

$$\mathrm{H_R} := \pi^{-\frac{1}{4}} \exp\left(\tfrac{1}{4}\mathbf{H}_{\mathbb{D}_e}[\log \mathit{\Delta}\mathrm{R}]\right),$$

which is a bounded (and bounded away from 0) holomorphic function in $\mathbb{D}_e$ with holomorphic extension across $\mathbb{T}$. Then $|\mathrm{H_R}|^2 = \pi^{-\frac{1}{2}}(\mathit{\Delta}\mathrm{R})^{\frac{1}{2}}$ holds on $\mathbb{T}$, and the value at infinity equals

$$\mathrm{H_R}(\infty) = \pi^{-\frac{1}{4}} \exp(\tfrac{1}{4}\langle \log \mathit{\Delta}\mathrm{R}\rangle_{\mathbb{T}}) > 0.$$

It now follows that

$$\bar{\partial}\mathrm{V} = 2^{-\frac{1}{2}}\zeta(\mathit{\Delta}\mathrm{R})^{\frac{1}{2}} = (\pi/2)^{\frac{1}{2}}\zeta|\mathrm{H_R}|^2 \quad \text{on } \mathbb{T}.$$

The solution algorithm: step I

The step I equation may be written as

$$\frac{\mathrm{F}}{\mathrm{H_R}} = \overline{\zeta \mathrm{A} \mathrm{H_R}} + (2\pi)^{-\frac{1}{2}} m^{-1} \frac{\partial \bar{\mathrm{B}}}{\mathrm{H_R}} \quad \text{on } \mathbb{T}.$$

From the data, we see that $\mathrm{F}/\mathrm{H_R}$ is bounded and holomorphic in $\mathbb{D}_\mathrm{e}$, so that $\mathrm{F}/\mathrm{H_R} \in H^2_-$, whereas $\overline{\zeta \mathrm{A} \mathrm{H_R}}$ extends by Schwarz reflection to a bounded holomorphic function on $\mathbb{D}$, so that $\overline{\zeta \mathrm{A} \mathrm{H_R}} \in H^2$. This means that the step I equation is a Riemann-Hilbert problem with jump $(2\pi)^{-\frac{1}{2}} m^{-1} \partial \bar{\mathrm{B}}/\mathrm{H_R}$, a situation which can be handled.

Step I solution

We have, for a constant a_1,

$$\frac{\mathrm{F}}{\mathrm{H_R}} = a_1 + \frac{(2\pi)^{-\frac{1}{2}}}{m} \mathbf{P}_{H^2_{-,0}} \left[\frac{\partial \bar{\mathrm{B}}}{\mathrm{H_R}} \right],$$

and

$$\zeta \mathrm{A} \mathrm{H_R} = \bar{a}_1 - \frac{(2\pi)^{-\frac{1}{2}}}{m} \mathbf{P}_{H^2_-} \left[\frac{\bar{\partial} \mathrm{B}}{\bar{\mathrm{H}}_\mathrm{R}} \right].$$

The constant a_1

$$a_1 = \left\langle \frac{\mathrm{F}}{\mathrm{H}_{\mathrm{R}}} \right\rangle_{\mathbb{R}} = \frac{\mathrm{F}(\infty)}{\mathrm{H}_{\mathrm{R}}(\infty)} = \frac{1}{\mathrm{H}_{\mathrm{R}}(\infty)} = \pi^{\frac{1}{4}} \exp\left(-\tfrac{1}{4}\langle \log \Delta\mathrm{R}\rangle_{\mathbb{T}} \right) > 0,$$

since $\mathrm{F}(\infty) = 1$ is our assumed normalization and $\mathrm{H}_{\mathrm{R}}(\infty)$ is known.

The solution algorithm: step II

We write

$$\bar{\partial}\mathrm{V} = (\pi/2)^{\frac{1}{2}}\zeta|\mathrm{H_R}|^2 + 2W_\mathrm{R}\mathrm{V}\bar{\partial}\mathrm{V}, \tag{8}$$

where the expression W_R is real-analytic near $\mathbb{T}$. We let $\mathbf{U}_{\mathbb{D}_\mathrm{e}}$ stand for the harmonic extension to $\mathbb{D}_\mathrm{e}$ of the restriction to $\mathbb{T}$ of a given smooth function. We now find from Step I that

$$\zeta\mathrm{AH_R} = \mathbf{U}_{\mathbb{D}_\mathrm{e}}[\zeta\mathrm{AH_R}] = \frac{\bar{\mathrm{F}}}{\bar{\mathrm{H}}_\mathrm{R}} - (2\pi)^{-\frac{1}{2}} m^{-1}\mathbf{U}_{\mathbb{D}_\mathrm{e}}\left[\frac{\bar{\partial}\mathrm{B}}{\bar{\mathrm{H}}_\mathrm{R}}\right]$$

holds in a neighborhood of the closed exterior disk $\bar{\mathbb{D}}_\mathrm{e}$, in the sense that each term extends harmonically across $\mathbb{T}$.

Step II solution

$$\mathrm{B} = \mathrm{A}W_\mathrm{R} + \frac{1}{2m}\frac{\bar{\partial}\mathrm{B} - \bar{\mathrm{H}}_\mathrm{R}\mathbf{U}_{\mathbb{D}_\mathrm{e}}(\bar{\partial}\mathrm{B}/\bar{\mathrm{H}}_\mathrm{R})}{2\mathrm{V}\bar{\partial}\mathrm{V}}$$

Combining Steps I and II

We first apply Step I with $\mathrm{B} = 0$ to get candidates for A and F. The next step is to apply Step II with the given A to get a better B. This better B we again implement in Step I, to get new and improved A and F. Again we apply Step II to get an even better B. We intermittently apply steps I and II and zoom in on the solution.

The equation for the function B alone

It is more convenient to work with a single equation for the function B alone (get rid of A).

The equation for B

$$\mathrm{B} = a_1 \frac{W_{\mathrm{R}}}{\zeta \mathrm{H}_{\mathrm{R}}} + m^{-1} \left(\frac{\bar{\partial}\mathrm{B} - \bar{\mathrm{H}}_{\mathrm{R}} \mathbf{U}_{\mathbb{D}_{\mathrm{e}}}[\bar{\partial}\mathrm{B}/\bar{\mathrm{H}}_{\mathrm{R}}]}{4\mathrm{V}\bar{\partial}\mathrm{V}} - (2\pi)^{-\frac{1}{2}} \frac{W_{\mathrm{R}}}{\zeta \mathrm{H}_{\mathrm{R}}} \mathbf{P}_{H^2_-} \left[\frac{\bar{\partial}\mathrm{B}}{\bar{\mathrm{H}}_{\mathrm{R}}} \right] \right).$$

This is an operator equation of the type

$$\mathrm{B} = \mathrm{B}_0 + m^{-1}\mathbf{T}[\mathrm{B}], \quad \text{where} \quad \mathrm{B}_0 := a_1 \frac{W_{\mathrm{R}}}{\zeta \mathrm{H}_{\mathrm{R}}} \quad \text{and} \quad \mathbf{T}[\mathrm{B}] := \mathbf{L}[\bar{\partial}\mathrm{B}/\bar{\mathrm{H}}_{\mathrm{R}}],$$

and the operator $\mathbf{L}$ is given by

$$\mathbf{L}[f] := \bar{\mathrm{H}}_{\mathrm{R}} \frac{f - \mathbf{U}_{\mathbb{D}_{\mathrm{e}}}[f]}{4\mathrm{V}\bar{\partial}\mathrm{V}} - (2\pi)^{-\frac{1}{2}} \frac{W_{\mathrm{R}}}{\zeta \mathrm{H}_{\mathrm{R}}} \mathbf{P}_{H^2_-}[f].$$

Iterative solution for B

We may attempt to solve the equation for B by iteration, that is, by finite Neumann series approximation.

Approximate solution B

We put

$$\mathrm{B}^{\approx} = \mathrm{B}_0 + m^{-1}\mathbf{T}[\mathrm{B}_0] + \ldots + m^{-\kappa+1}\mathbf{T}^{\kappa-1}[\mathrm{B}_0],$$

for some appropriately chosen $\kappa = \kappa_*(m) \asymp \sqrt{m}$.

Then

$$\mathrm{B}^{\approx} = \mathrm{B}_0 + m^{-1}\mathbf{T}[\mathrm{B}^{\approx}] + \mathcal{E}_m, \quad \mathcal{E}_m := -m^{-\kappa}\mathbf{T}^{\kappa}[\mathrm{B}_0].$$

Since we want to minimize the error $\mathcal{E}_m$, it is natural to study the growth of the norm of $\mathbf{T}^k[\mathrm{B}_0]$ as k grows. This can be done using the methods of the Nishida-Nirenberg theorem. We obtain $\mathrm{A}^{\approx}$ and $\mathrm{F}^{\approx}$ from Step I using $\mathrm{B}^{\approx}$ in place of B.

Main result

Theorem

We put $P^{\approx} := c_m \phi^m \mathrm{e}^{m\mathcal{Q}} F^{\approx}$, where $F^{\approx} = \varphi' \mathrm{F}^{\approx} \circ \varphi$. Then each F_j extends holomorphically to a fixed neighborhood of $\mathbb{C} \setminus \operatorname{int} \mathcal{S}_1$ for each $j = 0, \ldots, \kappa_*(m)$, while $\log F_0$ is bounded and holomorphic in the same neighborhood. Then there exists a fixed C^∞-smooth cut-off function $\chi_{1,1}$ on $\mathbb{C}$, with $0 \le \chi_{1,1} \le 1$, which equals 1 in an open neighborhood of the closure of $\mathbb{C} \setminus \mathcal{S}_1$, and vanishes off a slightly larger neighborhood, such that $\chi_{1,1} P^{\approx}$ becomes globally well-defined. Moreover, it is close to the monic orthogonal polynomial P of degree $n = m$ in L^2_{mQ}:

$$\|P - \chi_{1,1} P^{\approx}\|_{mQ} = \mathrm{O}(c_m m^{\frac{1}{2}} \mathrm{e}^{-\epsilon\sqrt{m}}), \quad \text{where } \|\chi_{1,1} P^{\approx}\|_{mQ} \asymp c_m m^{-\frac{1}{4}}.$$

Here, $\epsilon > 0$ is a constant that only depends on Q. It follows that

$$P = c_m \phi^m \mathrm{e}^{m\mathcal{Q}} \big(F^{\approx} + \mathrm{O}(m\,\mathrm{e}^{-\frac{1}{2}\epsilon\sqrt{m}})\big), \quad \text{on } D_m,$$

holds in the uniform norm as $m \to +\infty$, where D_m is the union of $\mathbb{C} \setminus \mathcal{S}_1$ and a certain band of width $\asymp m^{-\frac{1}{4}}$ around the loop $\Gamma_1 = \partial \mathcal{S}_1$.

Commentary

The proof of the theorem is based on Hörmander's L^2 methods for the $\bar{\partial}$-equation using suitably chosen smooth cut-off functions, as well as the Bernstein-Walsh type pointwise growth estimate based on weighted area-integrability.

Comparison with the Hedenmalm-Wennman theorem

The first result of this type is the recent work [5]. There, no effective growth control of the asymptotic expansion for F was obtained, so the result was somewhat weaker, in particular regarding the domain where the expansion holds. In addition, the foliation method of [5] is technically more demanding. The algorithm presented here is more transparent.

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