Spatial dynamics of some 6th-order ODE from the theory of phase transitions

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10:10-10:40

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Bounded stationary (i.e. independent in time) solutions of a quasi-linear parabolic PDE

$$u_t = \frac{\partial^6 u}{\partial x^6} + A\frac{\partial^4 u}{\partial x^4} + B\frac{\partial^2 u}{\partial x^2} + u - u^3. \tag{1}$$

are studied on the whole real line. This PDE (1) was proposed as a model of a formation and moving a phase front in media where the transitions from liquid to solid phase are possible [CF86]. Its stationary solutions represent a definite interest as a possibility to find spatial structures of various including complicated types. Earlier its stationary periodic solutions in some range of wave numbers were found by variational methods [TS02].

Stationary solutions of the equation (1) are described by a nonlinear ODE of the sixth order of the Euler-Lagrange-Poisson type and therefore are transformed to the Hamiltonian system with three degrees of freedom being in addition reversible with respect two linear involutions. The system has three symmetric equilibria, two of them $P_\pm$ are hyperbolic in some region of the parameter plane (A, B) (all eigenvalues have nonzero real parts). Here we investigate and discuss, combining methods of dynamical systems theory and numerical simulations, the orbit behavior near symmetric heteroclinic connections based on these equilibria and connecting them symmetric heteroclinic orbits relative to one or another involution.

It was found that both simple (periodic) and complicated orbit behavior are possible and this depends on the type of heteroclinic connections which are formed on these two basic equilibria and on the dimension of their leading subspaces. To this end, it was applied a theorem on the global center manifold in a neighborhood of the heterocilic connection generated by two saddle type symmetric equilibria $P_\pm$ and connecting them two heteroclinic orbits [Shi+98]. The heteroclinic orbits were found by numerical methods. When the leading subspaces for $P_\pm$ are one-dimensional, the global center manifold near the heteroclinic connection is two-dimensional symplectic and all nearby compact orbits are periodic ones accumulating to the connection. But if the leading subspaces of equilibria are two-dimensional, the global center manifold is four-dimensional symplectic and we deal in fact with the connection involving two saddle-foci with the related complicated dynamics on the four-dimensional invariant manifold. A characteristic feature of this situation is the existence of infinitely many homoclinic orbits of each saddle-focus-saddle.

For the third central symmetric equilibrium O at the origin we found the region in the parameter plane (A, B), when this equilibrium is of the saddle-focus-center type (its eigenvalues are a pair of pure imaginary $\pm i\omega$ and a quadruple of complex numbers $\pm\alpha \pm i\beta$, $\alpha\beta\omega \neq 0$). We have discovered the existence of homoclinic orbits of O and nearby families of periodic orbits. Due to some Turaev's theorem, a global center manifold cannot exist near a homoclinic orbits of this equilibrium and the reduction to a lesser dimension is not possible here. Nevertheless, a complicated behavior of nearby orbits exist here when some genericity condition holds for the linear Hamiltonian system linearized at the homoclinic solution of the saddle-focus-center [KL96]. Also some numerical findings show that near a homoclinic orbit of a saddle-focus-center in levels of the Hamiltonian close to the singular level containing O there are 2-elliptic periodic orbits (all four their multipliers belong to the unit circle in the complex plane $\mathbb{C}$) that says, if some inequalities hold, about restoring a regular (quasi-periodic) KAM behavior from the irregular complicated one.

This talk is based on results obtained in the joint recent paper with N.E. Kulagin [KL24].

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